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A Recurrence Formula for the q-Beta Integral and its Applications

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Dedicated to Professor Hari M. Srivastava on his 75th Birth Anniversary

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Abstract. In this paper we derive a recurrence formula for the q-beta integral using the q-Chu-Vandermonde formula and show some special cases and applications.

Throughout this paper we suppose $|q| < 1$. The q-shifted factorial are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1, \tag{1}$$

$$(a;q)_\infty = \lim_{n\to\infty}\prod_{k=0}^{n-1}(1-aq^k) = \prod_{k=0}^{\infty}(1-aq^k). \tag{2}$$

Clearly,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{3}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_m;q)_\infty$$

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The Generalized hypergeometric series ${}_r\Phi_s$ are defined by (see [16, p. 347 et seq.], or [5, 8, 15])

$$ {}_r\Phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r; \\ b_1, \ldots, b_s; \end{matrix} q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q, b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \tag{4} $$

Letting $r = s + 1$ in (4), we have

$$ {}_{s+1}\Phi_s\left(\begin{matrix} a_1, a_2, \cdots, a_{s+1}; \\ b_1, b_2, \cdots, b_s; \end{matrix} q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{s+1};q)_n}{(q, b_1, b_2, \cdots, b_s;q)_n} x^n. \tag{5} $$

It is not difficult, we get the following identity from (1) and (3).

$$ (aq^{-n};q)_n = (q/a;q)_n (-a/q)^n q^{-\binom{n}{2}}. \tag{6} $$

Jackson below defined the q-integral from 0 to b and from a to b (see [8], or [9])

$$ \int_0^b f(t) d_q t = b(1-q) \sum_{n=0}^{\infty} f(bq^n) q^n, \tag{7} $$

and

$$ \int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \tag{8} $$

He also defined the q-integral on $(0, \infty)$

$$ \int_0^{\infty} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n, \tag{9} $$

and the bilateral q-integral

$$ \int_{-\infty}^{\infty} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)] q^n, \tag{10} $$

provided the sums converge absolutely.

The q-beta integral plays an important role in the basic hypergeometric series. Askey obtained an elegant q-beta integral formula (see [4]):

$$ \int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega;q)_\infty} d_q\omega = \frac{2(1-q)(q^2;q^2)_\infty^2 (de, q/de, a/e, -a/d, b/e, -b/d;q)_\infty}{(q;q)_\infty (d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty (-ab/deq;q)_\infty}. \tag{11} $$

provided that $|q| < 1$, $|ab/deq| < 1$ and there are no zero factors in the denominator of the integrals.

Andrews and Askey gave another q-beta integral formula for q-integral from c to d in series of q-Gamma functions $\Gamma_q(x)$ (see [3]). Al-Salam and Verma gave more general q-beta integral formula that can be written as a well-poised ${}_8\Phi_7$ (see [2]). Wang researched the Askey's q-beta integral formula (see [18–20]). In [20] Wang extended Askey's q-beta integral formula (11) as follows:

$$ \begin{aligned} &\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty (s\omega;q)_n (t\omega;q)_m}{(-d\omega, e\omega;q)_\infty} d_q\omega \\ &= 2(1-q)^n q^{m^2+mn+n^2} \frac{t^m s^n (q^2;q^2)_\infty^2 (de, q/de, a/eq^{m+n}, -a/dq^{m+n}, b/e, -b/d;q)_\infty}{a^{m+n} (q;q)_\infty (d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty (-ab/deq^{m+n+1};q)_\infty} \\ &\qquad \times \sum_{k=0}^{n} \frac{(q^{-m}, a/sq^n, -ab/deq^{m+n+1};q)_k q^{k(1-m)}}{(q, a/eq^{m+n}, -a/dq^{m+n};q)_k} {}_3\Phi_2\left(\begin{matrix} q^{-n}, a/tq^{m+n-k}, -ab/deq^{m+n-k+1}; \\ a/eq^{m+n-k}, -a/dq^{m+n-k}; \end{matrix} q, q\right). \end{aligned} \tag{12} $$

provided that $|q| < 1, |ab/deq| < 1$ and $0 \le n + m < \frac{\log |ab/deq|}{\log |q|}$, and there are no zero factors in the denominator of the integrals.

Recently, Srivastava [17] gave some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials to deal with the method of q-analysis. More q-series and q-analysis and related to the topics see [1, 6, 7, 11–13].

In the present paper we obtain a recurrence formula for the q-beta integral. Some special cases and interesting identities of ${}_3\Phi_2$ be also shown. In particular, we obtain the terminating Sears' transformation formula and the evaluation of q-integral $\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega;q)_\infty} \omega^n d_q\omega$.

Below we state and prove our main result by using the q-Chu-Vandermonde formula.

Theorem 1. *For m and $n_i (i = 1, 2, \ldots, m+1)$ are the nonnegative integers, $|q| < 1, |ab/deq| < 1$ and $0 \le n_1 + n_2 + \cdots + n_{m+1} < \frac{\log |ab/deq|}{\log |q|}$ and there are no zero factors in the denominator of the integrals there are no zero factors in the denominator of the integrals, we have*

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega;q)_\infty} \prod_{i=1}^{m+1} P_{n_i}(\omega, d_i;q) d_q\omega \\
&\quad = \frac{(ed_{m+1};q)_{n_{m+1}}}{e^{n_{m+1}}} \sum_{k=0}^{n_{m+1}} \frac{(q^{-n_{m+1}};q)_k q^k}{(q, ed_{m+1};q)_k} \int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega q^k;q)_\infty} \prod_{i=1}^{m} P_{n_i}(\omega, d_i;q) d_q\omega.
\end{aligned} \tag{13}$$

where

$$P_0(a, b;q) = 1, \qquad P_n(a, b;q) = (a-b)(a-bq)\cdots(a-bq^{n-1}), \qquad n \ge 1.$$

Proof. First we recall the q-Chu-Vandermonde convolution formula (see [8, p. 14, Eq. (1.5.3)])

$${}_2\Phi_1\left(\begin{matrix} q^{-n}, a; \\ c; \end{matrix} q, q\right) = \sum_{k=0}^{n} \frac{(q^{-n}, a;q)_k}{(q, c;q)_k} q^k = \frac{a^n (c/a;q)_n}{(c;q)_n}. \tag{14}$$

By (3), q-Chu-Vandermonde convolution formula (14) can be written as

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q, c;q)_k} \frac{1}{(aq^k;q)_\infty} = \frac{a^n}{(c;q)_n} \cdot \frac{(c/a;q)_n}{(a;q)_\infty}. \tag{15}$$

Let $a \longmapsto a\omega$ in (15) and multiply the factor

$$\frac{(b\omega, e\omega;q)_\infty}{(-d\omega;q)_\infty} \prod_{i=1}^{m} P_{n_i}(\omega, d_i;q)$$

on both sides of (15), then we obtain

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q, c;q)_k} \cdot \frac{(b\omega, e\omega;q)_\infty}{(-d\omega, aq^k\omega;q)_\infty} \prod_{i=1}^{m} P_{n_i}(\omega, d_i;q) \\
&\quad = \frac{a^n}{(c;q)_n} \cdot \frac{(b\omega, e\omega;q)_\infty}{(-d\omega, a\omega;q)_\infty} P_n(\omega, c/a;q) \prod_{i=1}^{m} P_{n_i}(\omega, d_i;q).
\end{aligned} \tag{16}$$

Now taking the q-integral on both sides of (16) with respect to the variable ω, we get

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(q^{-n};q)_k q^k}{(q, c;q)_k} \int_{-\infty}^{\infty} \frac{(b\omega, e\omega;q)_\infty}{(-d\omega, aq^k\omega;q)_\infty} \prod_{i=1}^{m} P_{n_i}(\omega, d_i;q) d_q\omega \\
&\quad = \frac{a^n}{(c;q)_n} \int_{-\infty}^{\infty} \frac{(b\omega, e\omega;q)_\infty}{(-d\omega, a\omega;q)_\infty} P_n(\omega, c/a;q) \prod_{i=1}^{m} P_{n_i}(\omega, d_i;q) d_q\omega.
\end{aligned} \tag{17}$$

Setting $n = n_{m+1}, c = ad_{m+1}$ in (17), we have

$$\sum_{k=0}^{n_{m+1}} \frac{(q^{-n_{m+1}};q)_k q^k}{(q, ad_{m+1};q)_k} \int_{-\infty}^{\infty} \frac{(b\omega, e\omega;q)_\infty}{(-d\omega, a\omega q^k;q)_\infty} \prod_{i=1}^{m} P_{n_i}(\omega, d_i;q) d_q\omega$$
$$= \frac{a^{n_{m+1}}}{(ad_{m+1};q)_{n_{m+1}}} \int_{-\infty}^{\infty} \frac{(b\omega, e\omega;q)_\infty}{(-d\omega, a\omega;q)_\infty} \prod_{i=1}^{m+1} P_{n_i}(\omega, d_i;q) d_q\omega. \quad (18)$$

Interchanging a and e in (18), we obtain (13) immediately. This proof is complete. □

Remark 2. *We definte an empty product $\prod_{i=1}^{m} = 1$ for $m = 0$ and $m = -1$. We also say that $n_0 = 0$ when $m = -1$. Therefore the equation* (13) *is true when $m = -1$.*

It follows we give some special cases and applications of Theorem 1.

Theorem 3. *For n_1 is the nonnegative, $|q| < 1, |ab/deq| < 1$, and $0 \le n_1 < \frac{\log|ab/deq|}{\log|q|}$ and there are no zero factors in the denominator of the integrals there are no zero factors in the denominator of the integrals, we have*

$$\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega;q)_\infty} P_{n_1}(\omega, d_1;q) d_q\omega$$
$$= \frac{2(1-q)(d_1e;q)_{n_1}(q^2;q^2)_\infty^2 (de, q/de, a/e, -a/d, b/e, -b/d;q)_\infty}{e^{n_1}(q;q)_\infty (d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty (-ab/deq;q)_\infty} {}_3\Phi_2\left(\begin{matrix} q^{-n_1}, qe/a, qe/b; \\ d_1e, -q^2de/ab; \end{matrix} q, q\right). \quad (19)$$

Proof. Letting $m = 0$ in (13) and $e = eq^k$ in (11) and noting that (6), we obtain (19) immediately. □

Corollary 4. *For n is the nonnegative integers, $|q| < 1, |ab/deq| < 1$, and $0 \le n < \frac{\log|ab/deq|}{\log|q|}$, we have*

$$\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty (s\omega;q)_n}{(-d\omega, e\omega;q)_\infty} d_q\omega$$
$$= \frac{2(1-q)(s/e;q)_n(q^2;q^2)_\infty^2 (de, q/de, a/e, -a/d, b/e, -b/d;q)_\infty}{(q;q)_\infty (d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty (-ab/deq;q)_\infty} {}_3\Phi_2\left(\begin{matrix} q^{-n}, qe/a, qe/b; \\ e/sq^{n-1}, -q^2de/ab; \end{matrix} q, q\right). \quad (20)$$

Proof. Setting $n_1 = n$ in (19) and using the relation

$$P_n(\omega, d_1;q) = (-d_1)^n q^{\binom{n}{2}} (\omega/d_1q^{n-1};q)_n, \quad (21)$$

and

$$(-1)^n q^{\binom{n}{2}} \frac{(e/sq^{n-1};q)_n}{(e/s)^n} = (s/e;q)_n \quad (22)$$

we easily get (20). □

Remark 5. *The formula* (20) *of is just an analogue of Wang's result [20, p. 656, Corollary 3.1]:*

$$\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty (s\omega;q)_n}{(-d\omega, e\omega;q)_\infty} d_q\omega$$
$$= \frac{2s^n q^{n^2}(1-q)(q^2;q^2)_\infty^2 (de, q/de, a/eq^n, -a/dq^n, b/e, -b/d;q)_\infty}{a^n(q;q)_\infty (d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty (-ab/deq^{n+1};q)_\infty} {}_3\Phi_2\left(\begin{matrix} q^{-n}, a/sq^n, -ab/deq^{n+1}; \\ a/eq^n, -a/dq^n; \end{matrix} q, q\right). \quad (23)$$

Comparing (20) and (23) and noting

$$(aq^{-n};q)_n = \frac{(aq^{-n};q)_\infty}{(a;q)_\infty}, \tag{24}$$

we get directly the following transformation formula for ${}_3\Phi_2$:

Corollary 6. *For n is the nonnegative integers, $|q|<1, |ab/deq|<1$, and $0 \le n < \frac{\log|ab/deq|}{\log|q|}$, we have*

$$ {}_3\Phi_2\left(\begin{matrix} q^{-n}, qe/a, qe/b; \\ e/sq^{n-1}, -q^2de/ab; \end{matrix} q, q\right) = \frac{s^n q^{n^2}(a/eq^n, -a/dq^n;q)_n}{a^n(s/e, -ab/deq^{n+1};q)_n} {}_3\Phi_2\left(\begin{matrix} q^{-n}, a/sq^n, -ab/deq^{n+1}; \\ a/eq^n, -a/dq^n; \end{matrix} q, q\right). \tag{25}$$

Corollary 7 (The terminating Sears' ${}_3\Phi_2$ transformation formula).

$$ {}_3\Phi_2\left(\begin{matrix} q^{-n}, a_1, a_2; \\ b_1, b_2; \end{matrix} q, q\right) = (a_1a_2/b_1)^n \frac{(b_1b_2/a_1a_2;q)_n}{(b_2;q)_n} {}_3\Phi_2\left(\begin{matrix} q^{-n}, b_1/a_1, b_1/a_2; \\ b_1, b_1b_2/a_1a_2; \end{matrix} q, q\right). \tag{26}$$

Proof. Letting $e \longleftrightarrow -d$ in (19), we have

$$ {}_3\Phi_2\left(\begin{matrix} q^{-n_1}, qe/a, qe/b; \\ d_1e, -deq^2/ab; \end{matrix} q, q\right) = (-e/d)^{n_1} \frac{(-d_1d;q)_{n_1}}{(d_1e;q)_{n_1}} {}_3\Phi_2\left(\begin{matrix} q^{-n_1}, -dq/a, -dq/b; \\ -d_1d, -q^2de/ab; \end{matrix} q, q\right). \tag{27}$$

Setting $eq/a = a_1, eq/b = a_2, d_1e = b_2, -deq^2/ab = b_1, n_1 = n$ in (27), we obtain (26). □

Remark 8. *The formula* (26) *can be found in [10], which is used there to prove Sears' ${}_4\Phi_3$ transformation formula in [14].*

Letting $m = 1$ in (13) and applying the (3), (6) and (19), we obtain

Theorem 9. *For n_1 and n_2 are the nonnegative integers, $|q|<1, |ab/deq|<1$, and $0 \le n_1+n_2 < \frac{\log|ab/deq|}{\log|q|}$ and there are no zero factors in the denominator of the integrals there are no zero factors in the denominator of the integrals, we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega;q)_\infty} P_{n_1}(\omega, d_1;q) P_{n_2}(\omega, d_2;q) d_q\omega \\ &= \frac{2(1-q)(d_2e;q)_{n_2}(q^2;q^2)_\infty^2 (de, q/de, a/e, -a/d, b/e, -b/d;q)_\infty}{e^{n_1+n_2}(q;q)_\infty (d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty (-ab/deq;q)_\infty} \\ &\qquad \times \sum_{k=0}^{n_2} \frac{(q^{-n_2}, qe/a, qe/b;q)_k (d_1eq^k;q)_{n_1} q^{k(1-n_1)}}{(q, d_2e, -deq^2/ab;q)_k} {}_3\Phi_2\left(\begin{matrix} q^{-n_1}, eq^{k+1}/a, eq^{k+1}/b; \\ d_1eq^k, -deq^{k+2}/ab; \end{matrix} q, q\right). \end{aligned} \tag{28}$$

Corollary 10. *For n, N are any nonnegative integers, $|q|<1, |ab/deq|<1$, and $0 \le n+N < \frac{\log|ab/deq|}{\log|q|}$, we have*

$$\sum_{k=0}^{N} \frac{(q^{-N}, qe/a, qe/b;q)_k q^{k(1-n)}}{(q, d_1e, -deq^2/ab;q)_k} {}_3\Phi_2\left(\begin{matrix} q^{-n}, eq^{k+1}/a, eq^{k+1}/b; \\ d_1eq^k, -deq^{k+2}/ab; \end{matrix} q, q\right) = {}_3\Phi_2\left(\begin{matrix} q^{-(n+N)}, qe/a, qe/b; \\ d_1e, -deq^2/ab; \end{matrix} q, q\right). \tag{29}$$

Proof. Letting $n_1 = n, n_2 = N, d_2 = d_1q^n$ in (28), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{(a\omega, b\omega;q)_\infty}{(-d\omega, e\omega;q)_\infty} P_n(\omega, d_1;q) P_N(\omega, d_1q^n;q) d_q\omega \\ &= \frac{2(1-q)(d_1eq^n;q)_N(q^2;q^2)_\infty^2 (de, q/de, a/e, -a/d, b/e, -b/d;q)_\infty}{e^{n+N}(q;q)_\infty (d^2, e^2, q^2/d^2, q^2/e^2;q^2)_\infty (-ab/deq;q)_\infty} \\ &\qquad \times \sum_{k=0}^{N} \frac{(q^{-N}, qe/a, qe/b;q)_k (d_1eq^k;q)_n q^{k(1-n)}}{(q, d_1eq^n, -deq^2/ab;q)_k} {}_3\Phi_2\left(\begin{matrix} q^{-n}, eq^{k+1}/a, eq^{k+1}/b; \\ d_1eq^k, -deq^{k+2}/ab; \end{matrix} q, q\right). \end{aligned} \tag{30}$$

On the other hand, from $P_n(a,b;q)=(a-b)(a-bq)\cdots(a-bq^{n-1})$ we see easily that

$$P_n(a,b;q)P_N(a,bq^n;q)=P_{n+N}(a,b;q),$$

and noting that (19), we find that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{(a\omega,b\omega;q)_\infty}{(-d\omega,e\omega;q)_\infty}P_n(\omega,d_1;q)P_N(\omega,d_1q^n;q)d_q\omega&=\int_{-\infty}^{\infty}\frac{(a\omega,b\omega;q)_\infty}{(-d\omega,e\omega;q)_\infty}P_{n+N}(\omega,d_1;q)d_q\omega\\&=\frac{2(1-q)(d_1e;q)_{n+N}(q^2;q^2)^2_\infty(de,q/de,a/e,-a/d,b/e,-b/d;q)_\infty}{e^{n+N}(q;q)_\infty(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(-ab/deq;q)_\infty}{}_3\Phi_2\left(\begin{matrix}q^{-(n+N)},qe/a,qe/b;\\d_1e,-q^2de/ab;\end{matrix}q,q\right).\end{aligned}\tag{31}$$

Combining (31) and (30), and using $(aq^n;q)_k=\frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}$, we obtain (29) immediately. □

Corollary 11. *For n,N are any nonnegative integers, $|q|<1,|ab/deq|<1$, and $0\le n+N<\frac{\log|ab/deq|}{\log|q|}$, we have*

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{(a\omega,b\omega;q)_\infty(s\omega;q)_n(t\omega;q)_N}{(-d\omega,e\omega;q)_\infty}d_q\omega\\&=\frac{2(-s)^nq^{\binom{n}{2}}(t/e;q)_N(1-q)(q^2;q^2)^2_\infty(de,q/de,a/e,-a/d,b/e,-b/d;q)_\infty}{e^n(q;q)_\infty(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(-ab/deq;q)_\infty}\\&\qquad\times\sum_{k=0}^{N}\frac{(q^{-N},qe/a,qe/b;q)_k(eq^{k-n+1}/s;q)_nq^{k(1-n)}}{(q,e/tq^{N-1},-deq^2/ab;q)_k}{}_3\Phi_2\left(\begin{matrix}q^{-n},eq^{k+1}/a,eq^{k+1}/b;\\eq^{k-n+1}/s,-deq^{k+2}/ab;\end{matrix}q,q\right).\end{aligned}\tag{32}$$

Proof. Letting $n_1=n,n_2=N$ in (28). Using (22), we obtain (32). □

Remark 12. *The is just an analogue to Wang's main result [20, p. 653, Theorem 1.1].*

Theorem 13. *For n,p are any nonnegative integers, $|q|<1,|ab/deq|<1$, and $0\le n+p<\frac{\log|ab/deq|}{\log|q|}$ and there are no zero factors in the denominator of the integrals there are no zero factors in the denominator of the integrals, we have*

$$\int_{-\infty}^{\infty}\frac{(a\omega,b\omega;q)_\infty}{(-d\omega,e\omega;q)_\infty}\omega^{n+p}d_q\omega=\frac{1}{e^n}\sum_{k=0}^{n}\frac{(q^{-n};q)_kq^k}{(q;q)_k}\int_{-\infty}^{\infty}\frac{(a\omega,b\omega;q)_\infty}{(-d\omega,e\omega q^k;q)_\infty}\omega^pd_q\omega.\tag{33}$$

Proof. Putting $d_i=0$ for $i=1,2,\dots,m+1$, we have $\prod_{i=1}^{m+1}P_{n_i}(\omega,d_i;q)=\omega^{n_1+\cdots+n_m+n_{m+1}}$ and $\prod_{i=1}^{m}P_{n_i}(\omega,d_i;q)=\omega^{n_1+n_2+\cdots+n_m}$, setting $n_1+n_2+\cdots+n_m=p,\ n_{m+1}=n$, it follows (33). □

Below we deduce an interesting q-integral formula from the above recurrence formula.

Corollary 14. *For n is any nonnegative integers, $|q|<1,|ab/deq|<1$, and $0\le n<\frac{\log|ab/deq|}{\log|q|}$ and there are no zero factors in the denominator of the integrals there are no zero factors in the denominator of the integrals, we have*

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{(a\omega,b\omega;q)_\infty}{(-d\omega,e\omega;q)_\infty}\omega^nd_q\omega\\&\quad=\frac{2(1-q)(q^2;q^2)^2_\infty(de,q/de,a/e,-a/d,b/e,-b/d;q)_\infty}{e^n(q;q)_\infty(d^2,e^2,q^2/d^2,q^2/e^2;q^2)_\infty(-ab/deq;q)_\infty}{}_3\Phi_2\left(\begin{matrix}q^{-n},qe/a,qe/b;\\0,-deq^2/ab;\end{matrix}q,q\right).\end{aligned}\tag{34}$$

Proof. Setting $p=0$ in (33) and then letting $e\longmapsto eq^k$) in (11), and using the formulas

$$(a^2;q^2)_n=(a;q)_n(-a;q)_n\quad\text{and}\quad(a;q)_{-n}=\frac{(-q/a)^nq^{\binom{n}{2}}}{(q/a;q)_n},$$

via some simple computation, we get (34). □

Remark 15. *If taking $n=0$ in* (34)*, we directly obtain Askey's formula* (11)*, i.e., our formula* (34) *is another extension of Askey's formula* (11)*.*

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