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Generalized Hyers–Ulam Stability of Cubic Functional Inequality

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Abstract. In this article, we investigate the generalized Hyers–Ulam stability of a cubic functional inequality in Banach spaces and in non-Archimedean Banach spaces by using fixed point method and direct method, respectively.

1. Introduction

The first stability problem was raised by S.M. Ulam [27] during his talk at the University of Wisconsin in 1940.

Let G be a group and G' a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a number $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In the next year, D.H. Hyers [14] have presented a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias' theorem was obtained by P. Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach. The Hyers–Ulam stability of the quadratic functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ was first proved by F. Skof [26] for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. P. W. Cholewa [5] noticed that Skof's theorem is also valid if E_1 is replaced by an abelian group. In 1992, S. Czerwik [6] proved the generalized Hyers–Ulam stability of quadratic functional equation in the spirit of Rassias approach.

A. Gilányi [10] and J. Rätz [25] proved that for a function $f : G \to E$ mapping from an abelian group G divisible by 2 into an inner product space E, the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\| \qquad (x, y \in G) \tag{1}$$

implies the Jordan–von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}) \qquad (x, y \in G).$$

W. Fechner [8] and A. Gilányi [11] have investigated the generalized Hyers–Ulam stability of the functional inequality (1). Park et al.[23] have proved the generalized Hyers–Ulam stability of functional

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inequalities associated with Jordan–von Neumann type additive functional equations, and Y. Cho and H. Kim [3] have proved the generalized Hyers–Ulam stability of functional inequalities with Cauchy–Jensen additive mappings. Recently, H. Kim, K. Jun and E. Son [19] have established the generalized Hyers–Ulam stability of quadratic functional inequality in Banach spaces and in non-Archimedean Banach spaces.

In 2002, K. Jun and H. Kim [15] established the general solution and the Hyers–Ulam stability of the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{2}$$

in linear spaces and every solution of the cubic functional equation (2) is said to be a cubic mapping. If f is cubic, then we see from [15] and [16] that (2) implies the following functional equation

$$\begin{aligned} f(x+y+z)+f(x+y-z)+2f(x)+2f(y)-2f(x+y)-f(x+z)-f(x-z)& \\ -f(y+z)-f(y-z)=0& \end{aligned}$$

and so

$$\begin{aligned} f(2x+y+z)+16f(x)+2f(y)-2f(2x+y)-f(2x+z)-f(2x-z)& \\ -f(y+z)-f(y-z)=-f(2x+y-z).& \end{aligned}$$

Thus, we now consider the following functional inequality

$$\begin{aligned} \|f(2x+y+z)+16f(x)+2f(y)-2f(2x+y)-f(2x+z)-f(2x-z)& \\ -f(y+z)-f(y-z)\| \le \|f(2x+y-z)\|& \end{aligned} \tag{3}$$

in normed linear spaces. As a result, we can easily prove that if a mapping $f : X \to Y$ satisfies the functional inequality (3), then f is cubic. In this paper, we make an attempt to establish the generalized Hyers–Ulam stability of cubic functional inequality (3). In Section 2, we investigate the generalized Hyers–Ulam stability of the functional inequality (3) in Banach spaces by using fixed point method. In Section 3, we establish the generalized Hyers–Ulam stability of the functional inequality (3) in Banach spaces by using direct method. In Section 4, we prove the generalized Hyers–Ulam stability of the functional inequality (3) in non-Archimedean Banach spaces by using fixed point method. In Section 5, we investigate the generalized Hyers–Ulam stability of the functional inequality (3) in non-Archimedean Banach spaces by using direct method.

2. Stability of (3) by Fixed Point Method

In this part, assume that X is a normed space and that Y is a Banach space. Now, we are going to investigate the stability of the functional inequality (3) in Banach space by using fixed point method. We recall that a fixed point theory investigates the existence and uniqueness of fixed points under some conditions for operators on abstract spaces. A classical Banach contraction principle is one of the fundamental results in the fixed point theory, and then several authors have studied to generalize and improve the fixed point theory by applying new contractive conditions for operators and by replacing complete metric spaces with various abstract spaces (see e.g. [1],[7],[18],[22]).

Theorem 2.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists $0 < L < 1$ with*

$$\varphi(2x,2y,2z) \le 8L\varphi(x,y,z), \quad \Big(\varphi(\frac{x}{2},\frac{y}{2},\frac{z}{2}) \le \frac{L}{8}\varphi(x,y,z), \text{ respectively}\Big) \tag{4}$$

for all $x,y,z \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality

$$\begin{aligned} \|f(2x+y+z)+16f(x)+2f(y)-2f(2x+y)-f(2x+z)-f(2x-z)& \\ -f(y+z)-f(y-z)\| \le \|f(2x+y-z)\|+\varphi(x,y,z)& \end{aligned} \tag{5}$$

for all $x, y, z \in X$. Then, there exists a unique cubic mapping $T : X \to Y$ given by $T(x) = \lim_{m\to\infty} \frac{1}{8^m} f(2^m x)$ $\Big(T(x) = \lim_{m\to\infty} 8^m f(\frac{x}{2^m})$, resp.$\Big)$ such that

$$\|f(x) - T(x)\| \le \frac{1}{16 - 16L}\varphi(x, -2x, 0), \tag{6}$$
$$\Big(\|f(x) - T(x)\| \le \frac{L}{16 - 16L}\varphi(x, -2x, 0), \ resp.\Big)$$

for all $x \in X$.

Proof. Letting $y = -2x$ and $z = 0$ in (5), we obtain

$$\|16f(x) - 2f(2x)\| \le \varphi(x, -2x, 0) \tag{7}$$

for all $x \in X$. Dividing by 16 in (7), we obtain

$$\|f(x) - \frac{1}{8}f(2x)\| \le \frac{1}{16}\varphi(x, -2x, 0) \tag{8}$$

all $x \in X$.

Consider the set

$$S := \{h : X \to Y | h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in [0, \infty) : \|g(x) - h(x)\| \le \mu\varphi(x, -2x, 0), \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is generalized complete metric space (see the proof of Theorem 3.1 of [17]). Now we consider an operator $J : S \to S$ defined as

$$Jg(x) := \frac{1}{8}g(2x), \qquad g \in S,$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) \le \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x, -2x, 0)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| \le \|\frac{1}{4}g(2x) - \frac{1}{4}h(2x)\| \le \varepsilon L\varphi(x, -2x, 0)$$

for all $x \in X$. So $d(g, h) \le \epsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$. Thus J is a strictly contractive mapping with Lipschitz constant L, and it follows from (8) that $d(f, Jf) \le \frac{1}{16}$. By the fixed point theorem [20], there exists a mapping $T : X \to Y$ satisfying the followings (1),(2) and (3):

(1) T is a fixed point of J, i.e.,

$$T(2x) = 8T(x) \tag{9}$$

for all $x \in X$. The mapping T is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that T is a unique fixed point of J such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - T(x)\| \le \mu\varphi(x, -2x, 0)$$

for all $x \in X$;

(2) $d(J^m f, T) \to 0$ as $m \to \infty$. This implies the equality $T(x) = \lim_{m\to\infty} \frac{1}{8^m} f(2^m x)$ for all $x \in X$;

(3) $d(f,T) \le \frac{1}{1-L} d(f,Jf) \le \frac{1}{16-16L}$, which implies that the inequality (6) holds.
Now, we show that the mapping T is cubic. It follows from (4) and (5) that

$$\begin{aligned}
&\|T(2x+y+z)+16T(x)+2T(y)-2T(2x+y)-T(2x+z)-T(2x-z)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -T(y+z)-T(y-z)\|\\
&\le \|T(2x+y-z)\| + \lim_{m\to\infty}\frac{1}{8^m}\varphi(2^m x, 2^m y, 2^m z)\\
&\le \|T(2x+y-z)\| + \lim_{m\to\infty} L^m \varphi(x,y,z)\\
&= \|T(2x+y-z)\|
\end{aligned}$$

for all $x, y, z \in X$. Therefore, the mapping $T : X \to Y$ is cubic, as desired. □

We obtain the following corollary concerning the stability for approximate cubic mappings controlled by a sum of the same powers of norms.

Corollary 2.2. *Let $\theta \ge 0$ be a real number and p a positive real number with $p \ne 3$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}
&\|f(2x+y+z)+16f(x)+2f(y)-2f(2x+y)-f(2x+z)-f(2x-z)\\
&\qquad -f(y+z)-f(y-z)\| \le \|f(2x+y-z)\| + \theta(\|x\|^p+\|y\|^p+\|z\|^p)
\end{aligned}$$

for all $x, y, z \in X$, then we can find a unique cubic mapping $T : X \to Y$ satisfying the inequality

$$\|f(x)-T(x)\| \le \frac{\theta}{2|8-2^p|}(1+2^p)\|x\|^p$$

for all $x \in X$.

3. Stability of (3) by Direct Method

We prove stability problem of the cubic functional inequality (3) with perturbed control function φ. In this section, let X be a normed space and Y a Banach space.

Theorem 3.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\sum_{i=0}^{\infty}\frac{1}{8^i}\varphi(2^i x, 2^i y, 2^i z) < \infty, \quad \Big(\sum_{i=1}^{\infty} 8^i \varphi\Big(\frac{x}{2^i},\frac{y}{2^i},\frac{z}{2^i}\Big) < \infty, \ resp.\Big) \tag{10}$$

for all $x, y, z \in X$. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality

$$\begin{aligned}
&\|f(2x+y+z)+16f(x)+2f(y)-2f(2x+y)-f(2x+z)-f(2x-z)\\
&\qquad\qquad -f(y+z)-f(y-z)\| \le \|f(2x+y-z)\| + \varphi(x,y,z)
\end{aligned} \tag{11}$$

for all $x, y, z \in X$. Then, there exists a unique cubic mapping $T : X \to Y$ given by $T(x) = \lim_{m\to\infty}\frac{1}{8^m}f(2^m x)$ $\Big(T(x) = \lim_{m\to\infty} 8^m f(\frac{x}{2^m}),\ resp.\Big)$ such that

$$\|f(x) - \frac{f(0)}{14} - T(x)\| \le \frac{1}{16}\sum_{i=0}^{\infty}\frac{1}{8^i}\varphi(2^i x, -2^{i+1}x, 0), \tag{12}$$

$$\Big(\|f(x)-T(x)\| \le \frac{1}{16}\sum_{i=1}^{\infty} 8^i\varphi\Big(\frac{x}{2^i},\frac{-2x}{2^i},0\Big),\ resp.\Big)$$

for all $x \in X$.

Proof. Letting $y = -2x$ and $z = 0$ in (11), we obtain

$$\|16f(x) - 2f(2x) - f(0)\| \leq \varphi(x, -2x, 0) \tag{13}$$

for all $x \in X$. If we put $\tilde{f}(x) = f(x) - \frac{f(0)}{14}$, then it follows from (13) that

$$\|16\tilde{f}(x) - 2\tilde{f}(2x)\| \leq \varphi(x, -2x, 0) \tag{14}$$

for all $x \in X$. Dividing by 16 in (14), we obtain

$$\|\tilde{f}(x) - \frac{1}{8}\tilde{f}(2x)\| \leq \frac{1}{16}\varphi(x, -2x, 0) \tag{15}$$

for all $x \in X$. Therefore we prove from inequality (15) that for any integers m, l with $m > l \geq 0$

$$\begin{aligned} \left\|\frac{\tilde{f}(2^l x)}{8^l} - \frac{\tilde{f}(2^m x)}{8^m}\right\| &\leq \sum_{i=l}^{m-1} \left\|\frac{1}{8^i}\tilde{f}(2^i x) - \frac{1}{8^{i+1}}\tilde{f}(2^{i+1}x)\right\| \\ &\leq \sum_{i=l}^{m-1} \frac{1}{8^i} \left\|\tilde{f}(2^i x) - \frac{1}{8}\tilde{f}(2^{i+1}x)\right\| \\ &\leq \frac{1}{16}\sum_{i=l}^{m-1} \frac{1}{8^i}\varphi(2^i x, -2^{i+1}x, 0) \end{aligned} \tag{16}$$

for all $x \in X$. Since the right-hand side of (16) tends to zero as $l \to \infty$, we obtain that the sequence $\{\frac{\tilde{f}(2^m x)}{8^m}\}$ is Cauchy for all $x \in X$. Because of the fact that Y is complete, it follows that the sequence $\{\frac{\tilde{f}(2^m x)}{8^m}\}$ converges in Y. Therefore, we can define a mapping $T : X \to Y$ as

$$T(x) = \lim_{m\to\infty} \frac{\tilde{f}(2^m x)}{8^m} = \lim_{m\to\infty} \frac{f(2^m x)}{8^m}, \quad x \in X.$$

Moreover, letting $l = 0$ and taking $m \to \infty$ in (16), we get the desired inequality (12). It follows from (10) and (11) that

$$\begin{aligned} &\|T(2x + y + z) + 16T(x) + 2T(y) - 2T(2x + y) - T(2x + z) - T(2x - z) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - T(y + z) - T(y - z)\| \\ &\leq \|T(2x + y - z)\| + \lim_{m\to\infty} \frac{1}{8^m}\varphi(2^m x, 2^m y, 2^m z) \\ &= \|T(2x + y - z)\| \end{aligned}$$

for all $x, y, z \in X$. Therefore, the mapping $T : X \to Y$ is cubic.

Next, let $T' : X \to Y$ be another cubic mapping satisfying (12). Then, we have

$$\begin{aligned} \|T(x) - T'(x)\| &= \left\|\frac{1}{8^k}T(2^k x) - \frac{1}{8^k}T'(2^k x)\right\| \\ &\leq \frac{1}{8^k}\{\|T(2^k x) - f(2^k x) + \frac{f(0)}{14}\| + \|f(2^k x) - \frac{f(0)}{14} - T'(2^k x)\|\} \\ &\leq 2\Big[\frac{1}{16}\sum_{i=0}^{\infty} \frac{1}{8^{i+k}}\varphi(2^{i+k}x, -2^{i+k+1}x, 0)\}\Big] \\ &= \frac{1}{8}\sum_{i=k}^{\infty} \frac{1}{8^i}\varphi(2^i x, -2^{i+1}x, 0) \end{aligned}$$

for all $k \in \mathbb{N}$ and all $x \in X$. Taking the limit as $k \to \infty$, we conclude that $T(x) = T'(x)$ for all $x \in X$. $\square$

We obtain the following corollary concerning the stability for approximate cubic mappings controlled by a sum of different powers of norms.

Corollary 3.2. *Let $\theta_i \geq 0$ be a real number and p_i a positive real number with $p_i < 3$ or $p_i > 3$ for all $i = 1, 2, 3$. If a mapping $f : X \to Y$ satisfies the inequality*

$$\begin{aligned}\|f(2x+y+z) + 16f(x) + 2f(y) - 2f(2x+y) - f(2x+z) - f(2x-z)\\ -f(y+z) - f(y-z)\| \leq \|f(2x+y-z)\| + \theta_1\|x\|^{p_1} + \theta_2\|y\|^{p_2} + \theta_3\|z\|^{p_3}\end{aligned}$$

for all $x, y, z \in X$, then we can find a unique cubic mapping $T : X \to Y$ satisfying the inequality

$$\|f(x) - \frac{f(0)}{14} - T(x)\| \leq \frac{\theta_1}{2|8-2^{p_1}|}\|x\|^{p_1} + \frac{2^{p_2}\theta_2}{2|8-2^{p_2}|}\|x\|^{p_2}$$

for all $x \in X$.

4. Stability of (3) in Non-Archimedean Spaces by Fixed Point Method

We recall that a non-Archimedean valuation in a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to [0, \infty)$ with
(i) $|r| = 0$ if and only if $r = 0$;
(ii) $|rs| = |r||s|$ for all $r, s \in \mathbb{K}$;
(iii) $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$.
Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field; in any such field we have $|1| = |-1| = 1$ and $|n| \leq 1$ for all nonzero integers $n \in \mathbb{Z}$.

Definition 4.1. *Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:*
(i) *$\|x\| = 0$ if and only if $x = 0$;*
(ii) *$\|rx\| = |r|\|x\|$, for all $r \in \mathbb{K}$ and $x \in X$;*
(iii) *$\|x+y\| \leq max\{\|x\|, \|y\|\}$, for all $x, y \in X$.*
Then $(X, \|\cdot\|)$ is called a non-Archimedean space.

Definition 4.2. *Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.*
(1) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a Cauchy sequence if the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.
(2) The sequence $\{x_n\}$ is said to be convergent if, for any $\varepsilon > 0$, there are a positive integer N and $x \in X$ such that

$$\|x_n - x\| < \varepsilon, \qquad \forall n \geq N.$$

Then the point $x \in X$ is called the limit of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.
(3) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

In 2007, M.S. Moslehian and Th.M. Rassias [21] proved the generalized Hyers–Ulam stability of the Cauchy and quadratic functional equations in non-Archimedean normed spaces. Some papers ([4],[12],[13]) on the stability of various functional equations and inequalities in non-Archimedean normed spaces have been published after their stability results.

In this section, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Now, we are going to investigate the stability of the functional inequality (3) in non-Archimedean Banach space by using fixed point method.

Theorem 4.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists $0 < L < 1$ with*

$$\varphi(2x, 2y, 2z) \leq |8|L\varphi(x, y, z), \quad \Big(\varphi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) \leq \frac{L}{|8|}\varphi(x, y, z), \ resp.\Big) \tag{17}$$

for all $x, y, z \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality

$$\|f(2x + y + z) + 16f(x) + 2f(y) - 2f(2x + y) - f(2x + z) - f(2x - z) \\ - f(y + z) - f(y - z)\| \le \|f(2x + y - z)\| + \varphi(x, y, z) \tag{18}$$

for all $x, y, z \in X$. Then, there exists a unique cubic mapping $T : X \to Y$ given by $T(x) = \lim_{m\to\infty} \frac{1}{8^m} f(2^m x)$ $\Big(T(x) = \lim_{m\to\infty} 8^m f(\frac{x}{2^m})$, resp.$\Big)$ such that

$$\|f(x) - T(x)\| \le \frac{1}{|16| - |16|L}\varphi(x, -2x, 0), \tag{19}$$
$$\Big(\|f(x) - T(x)\| \le \frac{L}{|16| - |16|L}\varphi(x, -2x, 0), \text{ resp.}\Big)$$

for all $x \in X$.

Proof. Letting $y = -2x$ and $z = 0$ in (18), we obtain

$$\|16f(x) - 2f(2x)\| \le \varphi(x, -2x, 0) \tag{20}$$

for all $x \in X$. Dividing by $|16|$ in (20), we obtain

$$\|f(x) - \frac{1}{8}f(2x)\| \le \frac{1}{|16|}\varphi(x, -2x, 0) \tag{21}$$

all $x \in X$. Applying the similar argument to the corresponding proof of Theorem 2.1 on the complete generalized metric space (S, d), we get the desired result. □

We obtain the following corollary concerning the stability for approximate cubic mappings controlled by a sum of the same powers of norms.

Corollary 4.4. *Let $\theta \ge 0$ be a real number and p a positive real number with $p \ne 3$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|f(2x + y + z) + 16f(x) + 2f(y) - 2f(2x + y) - f(2x + z) - f(2x - z) \\ - f(y + z) - f(y - z)\| \le \|f(2x + y - z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then we can find a unique cubic mapping $T : X \to Y$ satisfying the inequality

$$\|f(x) - T(x)\| \le \frac{\theta}{|2||8 - 2^p|}(1 + |2|^p)\|x\|^p$$

for all $x \in X$.

5. Stability of (3) in Non-Archimedean Spaces by Direct Method

We prove the stability of the functional inequality (3) in non-Archimedean Banach space by direct method. In this section, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space.

Theorem 5.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{1}{|8|^m}\varphi(2^m x, 2^m y, 2^m z) = 0, \quad \Big(\lim_{m\to\infty} |8|^m \varphi\Big(\frac{x}{2^m}, \frac{y}{2^m}, \frac{z}{2^m}\Big) = 0, \text{ resp.}\Big) \tag{22}$$

for all $x, y, z \in X$ and

$$\tilde{\varphi}(x) = \lim_{m\to\infty} \max\Big\{\frac{1}{|8|^k}\varphi(2^k x, -2^{k+1}x, 0) : 0 \le k < m\Big\}, \tag{23}$$
$$\Big(\tilde{\varphi}(x) = \lim_{m\to\infty} \max\Big\{|8|^k \varphi\Big(\frac{x}{2^k}, \frac{-x}{2^{k-1}}, 0\Big) : 1 \le k < m + 1\Big\}, \text{ resp.}\Big)$$

exists for all $x \in X$. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality

$$\begin{aligned}&\|f(2x+y+z)+16f(x)+2f(y)-2f(2x+y)-f(2x+z)-f(2x-z)\\&\qquad -f(y+z)-f(y-z)\| \le \|f(2x+y-z)\| + \varphi(x,y,z)\end{aligned} \tag{24}$$

for all $x, y, z \in X$. Then there exists a cubic mapping $T : X \to Y$ defined as $T(x) = \lim_{m\to\infty} \frac{1}{8^m} f(2^m x)$ $\big(T(x) = \lim_{m\to\infty} 8^m f(\frac{x}{2^m})$, resp.$\big)$ such that

$$\|f(x) - T(x)\| \le \frac{1}{|16|}\tilde{\varphi}(x), \tag{25}$$

$$\Big(\|f(x) - \frac{f(0)}{14} - T(x)\| \le \frac{1}{|16|}\tilde{\varphi}(x),\ resp.\Big)$$

for all $x \in X$. Moreover, if

$$\lim_{l\to\infty}\lim_{m\to\infty}\max\Big\{\frac{1}{|8|^k}\varphi(2^k x, -2^{k+1}x, 0) : l \le k < m+l\Big\} = 0, \tag{26}$$

$$\Big(\lim_{l\to\infty}\lim_{m\to\infty}\max\Big\{|8|^k\varphi\Big(\frac{x}{2^k}, \frac{-x}{2^{k-1}}, 0\Big) : l+1 \le k < m+l+1\Big\} = 0,\ resp.\Big)$$

for all $x \in X$, then T is a unique cubic mapping satisfying (25).

Proof. In this case, $f(0) = 0$ since $\varphi(0,0,0) = 0$ by (22). Replacing x by $2^{m-1}x$ and dividing by $|8|^{m-1}$ in (21), we have

$$\left\|\frac{f(2^{m-1}x)}{8^{m-1}} - \frac{f(2^m x)}{8^m}\right\| \le \frac{1}{|16|}\frac{1}{|8|^{m-1}}\varphi(2^{m-1}x, -2^m x, 0), \tag{27}$$

for all $x \in X$. It follows from (22) and (27) that the sequence $\{\frac{f(2^m x)}{8^m}\}$ is Cauchy for all $x \in X$. Because of the fact that Y is complete, it follows that the sequence $\{\frac{f(2^m x)}{8^m}\}$ converges in Y. Therefore, we can define a mapping $T : X \to Y$ as

$$T(x) = \lim_{m\to\infty}\frac{f(2^m x)}{8^m}, \quad x \in X.$$

It follows from (21) and (27) that

$$\left\|f(x) - \frac{f(2^m x)}{8^m}\right\| \le \frac{1}{|16|}\max\Big\{\frac{1}{|8|^k}\varphi(2^k x, -2^{k+1}x, 0) : 0 \le k < m\Big\} \tag{28}$$

for all $m \in \mathbb{N}$ and all $x \in X$. Applying the similar argument to the corresponding proof of Theorem 3.1 in non-Archimedean spaces, we get the required result. □

Corollary 5.2. *Let $\rho : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$(i)\ \rho(|2|t) = \rho(|2|)\rho(t) \quad \text{for all} \quad t \ge 0 \qquad \text{and} \qquad (ii)\ \rho(|2|) < |2|^3.$$

Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\begin{aligned}&\|f(2x+y+z)+16f(x)+2f(y)-2f(2x+y)-f(2x+z)-f(2x-z)\\&\qquad -f(y+z)-f(y-z)\| \le \|f(2x+y-z)\| + \varepsilon\{\rho(\|x\|)+\rho(\|y\|)+\rho(\|z\|)\}\end{aligned}$$

for all $x, y, z \in X$ and for some $\varepsilon > 0$. Then there exists a unique cubic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{\varepsilon}{|16|}(1 + \rho(|2|))\rho(\|x\|)$$

for all $x \in X$.i

Proof. Letting $\varphi(x,y,z) = \varepsilon\{\rho(\|x\|) + \rho(\|y\|) + \rho(\|z\|)\}$, we obtain

$$\lim_{m\to\infty} \frac{1}{|8|^m}\varphi(2^m x, 2^m y, 2^m z) = \Big(\frac{\rho(|2|)}{|8|}\Big)^m \varphi(x,y,z) = 0$$

for all $x, y, z \in X$ and also

$$\begin{aligned}\varphi(x,-2x,0) &= \varepsilon\{\rho(\|x\|) + \rho(\|2x\|)\}\\ &= \varepsilon\{\rho(\|x\|) + \rho(|2|)\rho(\|x\|)\}\\ &= \varepsilon(1+\rho(|2|))\rho(\|x\|)\end{aligned}$$

for all $x \in X$. By direct calculation,

$$\tilde{\varphi}(x) = \lim_{m\to\infty} \max\Big\{\frac{1}{|8|^k}\varphi(2^k x, -2^{k+1}x, 0) : 0 \le k < m\Big\} = \varphi(x,-2x,0)$$

exists and

$$\lim_{l\to\infty}\lim_{m\to\infty} \max\Big\{\frac{1}{|8|^k}\varphi(2^k x, -2^{k+1}x, 0) : l \le k < m+l\Big\} = \lim_{l\to\infty}\frac{1}{|8|^l}\varphi(2^l x, -2^{l+1}x, 0) = 0$$

holds for all $x \in X$. Applying Theorem **??**, we conclude that

$$\|f(x) - T(x)\| \le \frac{1}{|16|}\tilde{\varphi}(x) = \frac{1}{|16|}\varphi(x,-2x,0) = \frac{\varepsilon}{|16|}(1+\rho(|2|))\rho(\|x\|)$$

for all $x \in X$. □

Corollary 5.3. *Let $\rho : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$(i)\ \rho\Big(\frac{t}{|2|}\Big) = \rho\Big(\frac{1}{|2|}\Big)\rho(t) \quad \text{for all} \quad t \ge 0 \qquad \text{and} \qquad (ii)\ \rho\Big(\frac{1}{|2|}\Big) < |2|^{-3}.$$

Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\begin{aligned}&\|f(2x+y+z) + 16f(x) + 2f(y) - 2f(2x+y) - f(2x+z) - f(2x-z)\\ &\quad -f(y+z) - f(y-z)\| \le \|f(2x+y-z)\| + \varepsilon\{\rho(\|x\|) + \rho(\|y\|) + \rho(\|z\|)\}\end{aligned}$$

for all $x, y, z \in X$ and for some $\varepsilon > 0$. Then there exists a unique cubic mapping $T : X \to Y$ such that

$$\|f(x) - \frac{f(0)}{14} - T(x)\| \le \frac{\varepsilon}{|2|}\Big(1 + \rho\Big(\frac{1}{|2|}\Big)\Big)\rho(\|x\|)$$

for all $x \in X$.

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