Special Lightlike Hypersurfaces of Indefinite Kaehler Manifolds

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Abstract. In this paper, we define three types of lightlike hypersurfaces of an indefinite Kaehler manifold, which are called *Hopf, recurrent* and *Lie recurrent* lightlike hypersurfaces. After that we provide several new results on such three type lightlike hypersurfaces of an indefinite Kaehler manifold or an indefinite almost complex space form.

1. Introduction

A hypersurfaces M of an almost complex manifold $\bar{M}$ has an almost contact structure (F, u, U) induced from the almost complex structure J of $\bar{M}$, where F is a (1, 1)-type tensor field, U is a vector field which is called the *structure vector field* of M, and u is a 1-form associated with U. There exist three types of hypersurfaces of an almost complex manifold. First, U is called *principal* if $AU = \alpha U$, where A is the shape operator of M and α is a smooth function. A real hypersurface of $\bar{M}$ is said to be a *Hopf hypersurface* if its structure vector field U is principal. Next, the structure tensor field F is called *recurrent* (resp. *Lie recurrent*) if, for any vector fields X and Y on M, there exists a 1-form ω (resp. a 1-form θ) on M such that

$$(\nabla_X F)Y = \omega(X)FY \qquad (\text{resp. } (\mathcal{L}_X F)Y = \theta(X)FY),$$

where ∇ and $\mathcal{L}$ denote the covariant and Lie derivative on M respectively. A real hypersurface is said to be a *recurrent* (resp. *Lie recurrent*) hypersurface if its structure tensor field F is recurrent (resp. Lie recurrent) ([2], [9]~[13]).

The theory of lightlike submanifolds is an important topic of research in differential geometry and mathematical physics. The study of such notion was initiated by Duggal and Bejancu [3] and later studied by many authors (see up-to date results in two books [4, 5]). Moreover, Sahin and Yildirim ([10]) studied both slant lightlike submanifolds and screen slant lightlike submanifolds of an indefinite Sasakian manifold. They obtained necessary and sufficient conditions for the existence of a slant lightlike submanifold. In addition, Jin and Lee studied the geometry of half lightlike submanifolds of quasi-constant curvature with some conditions ([8]). Although now we have lightlike version of a large variety of Riemannian submanifolds, the geometry of the above three types of lightlike hypersurfaces of indefinite almost complex manifolds have not been introduced as yet. The purpose of this paper is to extend and study the concepts of these three types of hypersurfaces in case that M is a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$.

2. Lightlike Hypersurfaces

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM, of rank 1, and coincides with the radical distribution $Rad(TM) = TM \cap TM^\perp$ of M. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M, by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M and by $(-.-)_i$ the i-th equation of $(-.-)$. We use same notations for any others. It is well-known [3] that, for any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$, respectively. Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM).$$

From now and in the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.1}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.2}$$
$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi, \tag{2.3}$$
$$\nabla_X \xi = -A_\xi^* X - \tau(X)\xi, \tag{2.4}$$

where ∇ and ∇^* are the liner connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, which are called the *lightlike* and *screen* second fundamental forms of M, A_N and A_ξ^* are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric on TM.

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{2.5}$$

where η is a 1-form on M such that

$$\eta(X) = \bar{g}(X, N).$$

From the fact $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we show that B is independent of the choice of $S(TM)$ and satisfies

$$B(X, \xi) = 0. \tag{2.6}$$

The above second fundamental forms are related to their shape operators by

$$B(X, Y) = g(A_\xi^* X, Y), \qquad \bar{g}(A_\xi^* X, N) = 0, \tag{2.7}$$
$$C(X, PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0. \tag{2.8}$$

From (2.7), A_ξ^* is $S(TM)$-valued self-adjoint on $\Gamma(TM)$ with respect to the induced metric g on M such that

$$A_\xi^* \xi = 0. \tag{2.9}$$

Denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$, respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss equations for M and $S(TM)$ such that

$$
\begin{aligned}
\bar{R}(X,Y)Z \; = \; & R(X,Y)Z + B(X,Z)A_{N}Y - B(Y,Z)A_{N}X \\
& + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
R(X,Y)PZ \; = \; & R^*(X,Y)PZ + C(X,PZ)A_{\xi}^*Y - C(Y,PZ)A_{\xi}^*X \\
& + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X)\}\xi.
\end{aligned}
\tag{2.11}
$$

In case $R = 0$, we say that M is *flat*.

Definition 2.1. *A lightlike hypersurface M of a semi-Riemannian manifold $\bar{M}$ is said to be*

(1) *totally umbilical [3] if there exist a smooth function ρ on a coordinate neighborhood $\mathcal{U}$ such that $A_{\xi}^*X = \rho PX$, or equivalently,*

$$B(X,PY) = \rho g(X,Y).$$

In case $\rho = 0$ on $\mathcal{U}$, we say that M is totally geodesic.

(2) *screen totally umbilical [3] if there exist a smooth function λ on a coordinate neighborhood $\mathcal{U}$ such that $A_{N}X = \lambda PX$, or equivalently,*

$$C(X,PY) = \lambda g(X,Y).$$

In case $\lambda = 0$ on $\mathcal{U}$, we say that M is screen totally geodesic.

(3) *screen conformal [1] if there exist a non-vanishing smooth function φ on a coordinate neighborhood $\mathcal{U}$ such that $A_{N} = \varphi A_{\xi}^*$, or equivalently,*

$$C(X,PY) = \varphi B(X,Y).$$

3. Recurrent Lightlike Hypersurfaces

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be a real $2m$-dimensional indedinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$ $(0 < v < m)$ and J is an indefinite almost complex structure on $\bar{M}$ satisfying

$$
J^2 = -I, \quad \bar{g}(JX,JY) = \bar{g}(X,Y), \quad (\bar{\nabla}_X J)Y = 0,
\tag{3.1}
$$

for any vector fields X and Y of $\bar{M}$ ([3, 5, 7]).

An *indefinite complex space form*, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$
\bar{R}(X,Y)Z = \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\},
\tag{3.2}
$$

for any vector fields X, Y and Z on $\bar{M}$.

Suppose that M is a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then its screen distribution $S(TM)$ splits as follows [3, 7]:

If ξ and N are local sections of $TM^{\perp}$ and $tr(TM)$ respectively, then

$$
\bar{g}(J\xi, \xi) = \bar{g}(J\xi, N) = \bar{g}(JN, \xi) = \bar{g}(JN, N) = 0, \quad \bar{g}(J\xi, JN) = 1.
$$

It follow that the vector fields $J\xi$ and JN belong to $S(TM)$. Thus $J(TM^\perp)$ and $J(tr(TM))$ are distributions on M of rank 1 such that $TM^\perp \cap J(TM^\perp) = \{0\}$ and $TM^\perp \cap J(tr(TM)) = \{0\}$. Hence $J(TM^\perp) \oplus J(tr(TM))$ is a subbundle of $S(TM)$, of rank 2. Therefore, there exists a non-degenerate almost complex distribution D_o on M with respect to J, i.e., $J(D_o) = D_o$, such that

$$TM = TM^\perp \oplus_{orth} \{J(TM^\perp) \oplus J(tr(TM)) \oplus_{orth} D_o\}.$$

Consider the 2-lightlike almost complex distribution D such that

$$D = \{TM^\perp \oplus_{orth} J(TM^\perp)\} \oplus_{orth} D_o, \qquad TM = D \oplus J(tr(TM)). \tag{3.3}$$

Consider the local lightlike vector fields U and V such that

$$U = -JN, \qquad V = -J\xi. \tag{3.4}$$

Denote by S the projection morphism of TM on D with respect to the decomposition $(3.3)_2$. Then any vector field X on M is expressed as

$$X = SX + u(X)U,$$

where u and v are 1-forms locally defined on M by

$$u(X) = g(X, V), \qquad v(X) = g(X, U). \tag{3.5}$$

Using (3.4), the action JX of any vector field X on M by J is expressed as

$$JX = FX + u(X)N, \tag{3.6}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$.
   Applying J to (3.6) and using (3.1) and (3.4), we have

$$F^2X = -X + u(X)U, \qquad FU = 0, \qquad u(U) = 1. \tag{3.7}$$

Therefore, the structure set (F, u, U) defines an indefinite almost contact structure on M. The vector field U is called the *structure vector field* of M.
   Applying $\bar{\nabla}_X$ to (3.4), (3.5) and (3.6) by turns, and using (2.1)~(2.8), (3.1), (3.4), (3.5) and (3.6), we have

$$B(X, U) = C(X, V), \tag{3.8}$$
$$\nabla_X U = F(A_N X) + \tau(X)U, \tag{3.9}$$
$$\nabla_X V = F(A_\xi^* X) - \tau(X)V, \tag{3.10}$$
$$(\nabla_X F)(Y) = u(Y)A_N X - B(X, Y)U, \tag{3.11}$$
$$(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X, FY), \tag{3.12}$$
$$(\nabla_X v)(Y) = v(Y)\tau(X) - g(A_N X, FY). \tag{3.13}$$

Definition 3.1. *The structure tensor field F of M is said to be recurrent if there exists a 1-form ω on M such that*

$$(\nabla_X F)Y = \omega(X)FY. \tag{3.14}$$

A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called recurrent if it admits a recurrent structure tensor field F.

   In the sequel, we shall denote σ the 1-form defined by

$$\sigma(X) = B(X, U) = C(X, V), \quad \text{and let} \quad \alpha = \sigma(V), \quad \beta = \sigma(U).$$

Proposition 3.2. *Let M be a recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then F is parallel with respect to ∇, and*

$$A^*_\xi X = \sigma(X)V, \qquad A_N X = \sigma(X)U. \tag{3.15}$$

Moreover, if M is screen conformal, then it is totally and screen totally geodesic.

Proof. Assume that M is recurrent. From (3.11) and (3.14), we get

$$\omega(X)FY = u(Y)A_N X - B(X, Y)U. \tag{3.16}$$

Replacing Y by ξ and using (2.6), (3.5) and the fact that $F\xi = -V$, we get $\omega(X)V = 0$. Taking the scalar product with U to this, we obtain $\omega = 0$. It follows that $\nabla_X F = 0$. Therefore, F is parallel with respect to ∇.

Replacing Y by U to (3.16) such that $\omega = 0$, we get $A_N X = \sigma(X)U$. Taking the scalar product with V to (3.16), we have $B(X, Y) = u(Y)\sigma(X)$, that is,

$$g(A^*_\xi X, Y) = g(\sigma(X)V, Y).$$

As $S(TM)$ is non-degenerate, we get $A^*_\xi X = \sigma(X)V$. Thus we obtain (3.15).

If M is screen conformal, then, from the two equations of (3.15), we have

$$\sigma(X)U = \varphi\sigma(X)V.$$

Taking the scalar product with V to this, we have $\sigma = 0$. Thus, by (3.15), we get $A^*_\xi = 0$ and $A_N = 0$. Therefore, M is totally and screen totally geodesic. $\square$

Definition 3.3. *Let $\nabla^\perp_X N = \pi(\bar{\nabla}_X N)$, where π is the projection morphism of $T\bar{M}$ on $tr(TM)$. Then $\nabla^\perp$ is a linear connection on $tr(TM)$. We say that $\nabla^\perp$ is the transversal connection. We define the curvature tensor $R^\perp$ of $tr(TM)$ by*

$$R^\perp(X, Y)N = \nabla^\perp_X \nabla^\perp_Y N - \nabla^\perp_Y \nabla^\perp_X N - \nabla^\perp_{[X,Y]}N.$$

The transversal connection $\nabla^\perp$ is called flat if $R^\perp$ vanishes identically [6].

As $\nabla^\perp_X N = \tau(X)N$, we show [6] that the transversal connection is flat if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.

Theorem 3.4. *Let M be a recurrent lightlike hypersurfaces of an indefinite complex space form $\bar{M}(c)$. Then $c = 0$, i.e., $\bar{M}(c)$ is flat ; M is also flat, and the transversal connection of M is flat.*

Proof. Comparing the tangential parts of (2.10) and (3.2), we get

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + \bar{g}(JY, Z)FX - \bar{g}(JX, Z)FY + 2\bar{g}(X, JY)FZ\} \tag{3.17}$$
$$+ B(Y, Z)A_N X - B(X, Z)A_N Y.$$

Taking the scalar product with N to (2.10) and (2.11), we have

$$\bar{g}(\bar{R}(X, Y)PZ, N) = \bar{g}(R(X, Y)PZ, N),$$
$$\bar{g}(R(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X).$$

From these two equations and (3.2), we see that

$$(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ) \tag{3.18}$$
$$= \frac{c}{4}\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y) + \bar{g}(JY, PZ)v(X) - \bar{g}(JX, PZ)v(Y) + 2\bar{g}(X, JY)v(PZ)\}.$$

As F is recurrent, taking the scalar product with U to $(3.15)_2$, we have

$$C(Y, U) = 0, \quad \forall Y \in \Gamma(TM).$$

Applying ∇_X to this and using (3.9), (3.15)$_2$ and the result: $FU = 0$, we have

$$(\nabla_X C)(Y, U) = 0.$$

Replacing PZ by U to (3.18) and using the last two equations, we obtain

$$\frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$ to this equation, we see that $c = 0$.

As $c = 0$, substituting (3.15)$_{1,2}$ into (3.17), we get

$$R(X, Y)Z = \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}u(Z)U = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Therefore $R = 0$ and M is a flat manifold.

From (3.9), (3.15)$_2$ and the fact that $FU = 0$, we get

$$\nabla_X U = \tau(X)U.$$

Substituting this into $\nabla_X\nabla_Y U - \nabla_Y\nabla_X U - \nabla_{[X,Y]}U = 0$, we get $d\tau = 0$. $\square$

4. Lie Recurrent Lightlike Hypersurfaces

Definition 4.1. *The structure tensor field F of M is said to be Lie recurrent if there exists a 1-form θ on M such that*

$$(\mathcal{L}_x F)Y = \theta(X)FY, \tag{4.1}$$

here $\mathcal{L}_x$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_x F)Y = [X, FY] - F[X, Y] = (\nabla_X F)Y - \nabla_{FY}X + F\nabla_Y X. \tag{4.2}$$

The structure tensor field F is called Lie parallel if $\mathcal{L}_x F = 0$.

A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called Lie recurrent if it admits a Lie recurrent structure tensor field F.

Proposition 4.2. *Let M be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. Then the structure tensor field F is Lie parallel ; and the shape operators A_ξ^* and A_N and the 1-form σ are satisfied*

$$A_\xi^* V = 0, \qquad A_\xi^* U = A_N V = 0, \qquad \sigma = 0.$$

Proof. As F is Lie recurrent, from (3.11), (4.1) and (4.2) we get

$$\theta(X)FY = u(Y)A_N X - B(X, Y)U - \nabla_{FY}X + F\nabla_Y X. \tag{4.3}$$

Replacing Y by ξ to (4.3) and using (2.6), (3.5) and $F\xi = -V$, we have

$$-\theta(X)V = \nabla_V X + F\nabla_\xi X. \tag{4.4}$$

As $g(F\nabla_\xi X, V) = 0$, taking the scalar product with V to (4.4), we get

$$u(\nabla_V X) = g(\nabla_V X, V) = 0. \tag{4.5}$$

On the other hand, taking $Y = V$ to (4.3) and using (2.6) and (3.5), we have

$$\theta(X)\xi = -B(X, V)U - \nabla_\xi X + F\nabla_V X.$$

Applying F to this and using $(3.7)_1$, (4.5) and the fact that $FU = 0$, we have

$$\theta(X)V = \nabla_V X + F\nabla_\xi X.$$

From this equation and (4.4), we obtain $\theta = 0$. Therefore, F is Lie parallel.

Taking $X = U$ to (4.3) and using $(3.7)_1$, (3.8), (3.9) and $FU = 0$, we get

$$u(Y)A_N U - F(A_N FY) - \tau(FY)U - A_N Y = 0. \tag{4.6}$$

Taking $Y = V$ to (4.6) and using (3.8) and the fact that $FV = \xi$, we obtain

$$F(A_N \xi) + \tau(\xi)U + A_N V = 0.$$

Taking the scalar product with V and U to this equation by turns, we get

$$\alpha = C(V, V) = -\tau(\xi), \qquad C(V, U) = 0. \tag{4.7}$$

On the other hand, taking the scalar product with V to (4.6), we have

$$B(Y, U) = -\tau(FY) + u(Y)C(U, V). \tag{4.8}$$

Replacing X by V to (4.3) and using (3.10), $(3.7)_1$ and $FV = \xi$, we obtain

$$u(Y)A_N V - F(A_\xi^* FY) + \tau(FY)V - A_\xi^* Y - \tau(Y)\xi = 0. \tag{4.9}$$

Taking $Y = \xi$ to (4.9) and using (2.9) and the fact that $\tau(V) = 0$, we obtain

$$F(A_\xi^* V) - \tau(V)V - \tau(\xi)\xi = 0.$$

Taking the scalar product with N and U to this equation by turns, we have

$$\alpha = C(V, V) = \tau(\xi), \qquad \tau(V) = 0.$$

Comparing this and (4.7), we obtain $\alpha = \tau(\xi) = 0$.

Replacing X by U to (4.9) and using (3.9), $(3.7)_1$ and $FU = 0$, we obtain

$$A_N V - A_\xi^* U - \tau(U)\xi = 0.$$

Taking the scalar product with N to this result, we have $\tau(U) = 0$. Thus

$$A_N V = A_\xi^* U. \tag{4.10}$$

Taking the scalar product with X to (4.10), we have $B(X, U) = C(V, X)$. From this and (3.8), we see that $C(X, V) = C(V, X)$. Replacing X by U to this, we get $\gamma = C(U, V) = C(V, U) = 0$. If follows from (4.8) that

$$B(X, U) = -\tau(FX).$$

Taking the scalar product with U and V by turns to (4.9), we have

$$B(Y, U) = \tau(FY), \qquad B(Y, V) = 0.$$

Comparing the last two equations, we get $\sigma(X) = B(X, U) = 0$ and $\tau(FX) = 0$. From the facts that $B(V, X) = B(U, X) = 0$, we obtain

$$g(A_\xi^* V, X) = 0, \qquad g(A_\xi^* U, X) = 0. \tag{4.11}$$

As $S(TM)$ is non-degenerate, from (4.10) and (4.11), we have our assertion. $\square$

Definition 4.3. *The Jacobi operator on M with respect to the vector field X is defined by $R(\cdot, X)X$. In case $X = U$, the Jacobi operator is called structure Jacobi operator and is denoted by $\phi = R(\cdot, U)U$.*

Theorem 4.4. *Let M be a Lie recurrent lightlike hypersurfaces of an indefinite complex space form $\bar{M}(c)$. Then $c = 0$, i.e., $\bar{M}(c)$ is flat, and the structure Jacobi operator ϕ is satisfied $\phi = 0$.*

Proof. Comparing the transversal parts of (2.10) and (3.2), we get

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + B(Y, Z)\tau(X) - B(X, Z)\tau(Y)$$
$$= \frac{c}{4}\{\bar{g}(JY, Z)u(X) - \bar{g}(JX, Z)u(Y) + 2\bar{g}(X, JY)u(Z)\}.$$

Taking $Z = U$ to this equation and using (3.1) and (3.4), we obtain

$$(\nabla_X B)(Y, U) - (\nabla_Y B)(X, U) + B(Y, U)\tau(X) - B(X, U)\tau(Y) \tag{4.12}$$
$$= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.$$

Assume that F is Lie recurrent. Taking the scalar product with X to $A_\xi^* U = 0$, we have

$$B(X, U) = 0, \quad \forall X \in \Gamma(TM). \tag{4.13}$$

Substituting this equation into (4.12) and using (3.9), we obtain

$$B(X, F(A_N Y)) - B(Y, F(A_N X)) = \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.$$

Taking $X = \xi$ and $Y = U$ to this equation, we get $c = 0$.
   Taking $Y = Z = U$ to (3.17) we obtain

$$\phi X = B(U, U)A_N X - B(X, U)A_N U. \tag{4.14}$$

Substituting (4.13) into (4.14), we see that $\phi = 0$. $\quad\square$

5. Hopf Lightlike Hypersurfaces

In the classical geometry of non-degenerate submanifolds, each submanifold has only one type of fundamental forms with their one type of respective shape operators. It is known that the second fundamental forms and their respective shape operators of a non-degenerate submanifolds are related by means of the metric tensor. Contrary to this, each lightlike submanifold has two types of fundamental forms with their two types of respective shape operators. We see from (2.7) and (2.8) that there are interrelations between the lightlike and screen second fundamental forms and their respective shape operators. Due to this reason, we define Hopf lightlike hypersurfaces M of an indefinite almost complex manifold $\bar{M}$ as follow:

Definition 5.1. *The structure vector field U on a lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called principal, with respect to the shape operator A_ξ^*, if there exists a smooth function α such that*

$$A_\xi^* U = \alpha U. \tag{5.1}$$

A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called a Hopf lightlike hypersurface if it admits a principal structure vector field U, with respect to the shape operator A_ξ^*.

Taking the scalar product with X to (5.1) and using (2.7) and (3.5), we get

$$B(X, U) = \alpha v(X). \tag{5.2}$$

Using (3.8) and (5.2), for all $X \in \Gamma(TM)$, we see that

$$C(X, V) = \alpha v(X). \tag{5.3}$$

Example 5.2. *Every totally umbilical lightlike hypersurfaces M of indefinite almost complex manifolds $\bar{M}$ is a Hopf lightlike hypersutface of $\bar{M}$.*

Theorem 5.3. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. If V is parallel with respect to the induced connection ∇ on M, then M is a Hopf lightlike hypersurface of $\bar{M}$ such that $\sigma = 0$, and $\tau = 0$.*

Proof. If V is parallel with respect to ∇, then, from (3.6) and (3.10), we have

$$J(A_\xi^* X) - u(A_\xi^* X)N - \tau(X)V = 0, \quad \forall X \in \Gamma(TM).$$

Applying J to this equation and using (3.1) and (3.4), we obtain

$$A_\xi^* X - u(A_\xi^* X)U + \tau(X)\xi = 0.$$

Taking the scalar product with N to this equation, we get $\tau = 0$. Therefore,

$$A_\xi^* X = u(A_\xi^* X)U, \quad \forall X \in \Gamma(TM). \tag{5.4}$$

This implies that $A_\xi^* U = \alpha U$. Thus M is a Hopf lightlike hypersurface. Taking the scalar product with U to (5.4), we get $\sigma(X) = B(X, U) = 0$. $\square$

Theorem 5.4. *Let M be a Hopf lightlike hypersurfaces of an indefinite complex space form $\bar{M}(c)$. Then $c = 0$, i.e., $\bar{M}(c)$ is flat.*

Proof. Applying ∇_Y to (5.2) and using (3.1), (3.6), (3.9) and (3.13), we have

$$(\nabla_X B)(Y, U) = (X\alpha)v(Y) - \alpha g(A_N X, FY) + g(A_N X, F(A_\xi^* Y)).$$

Substituting this equation and (5.2) into (4.12), we have

$$\begin{aligned}
&(X\alpha)v(Y) - (Y\alpha)v(X) + g(A_N X, F(A_\xi^* Y)) - g(A_N Y, F(A_\xi^* X)) \\
&+ \alpha\{g(A_N Y, FX) - g(A_N X, FY) + \tau(X)v(Y) - \tau(Y)v(X)\} \\
&= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.
\end{aligned}$$

Taking $X = \xi$ and $Y = U$ to this equation and using (2.9), (3.5), (5.1), (5.3) and the facts that $FU = 0$ and $F\xi = -V$, we obtain $c = 0$. $\square$

Theorem 5.5. *Let M be a Hopf lightlike hypersurfaces of an indefinite Kaehler manifold $\bar{M}$. If (1) M is screen totally umbilical or (2) F is parallel with respect to ∇, then $\sigma = 0$ and M is screen totally geodesic.*

Proof. (1) If M is screen totally umbilical, then, by replacing PY by V to $C(X, PY) = \lambda g(X, PY)$ and using (3.5) and (5.3), we have $\alpha v(X) = \lambda u(X)$. Taking $X = U$ to this, we have $\lambda = 0$. As $\lambda = 0$, we have $A_N = C = 0$ and $\sigma(X) = g(A_N X, V) = 0$. Therefore, M is screen totally geodesic and $\sigma = 0$.

(2) If F is parallel with respect to ∇, then, from (3.11), we have

$$u(Y)A_N X = B(X, Y)U. \tag{5.5}$$

Replacing Y by U to this and using the fact that $\sigma(X) = B(X, U)$, we have

$$A_N X = \sigma(X)U. \tag{5.6}$$

Assume that M is Hopf lightlike hypersurface. From (5.2) and (5.6), we get

$$A_N X = \alpha v(X)U. \tag{5.7}$$

On the other hand, taking the scalar product with V to (5.5), we have

$$B(X, Y) = u(Y)u(A_N X).$$

Replacing X by U to this and using the fact that $u(A_N U) = 0$, we obtain

$$\sigma(X) = B(X, U) = 0.$$

It follows from (5.6) that $A_N = 0$. Therefore, M is screen totally geodesic. $\square$

Theorem 5.6. *Let M be a Hopf lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. If U is parallel with respect to the induced connection ∇ on M, then $S(TM)$ is an integrable distribution and $\tau = 0$.*

Proof. If U is parallel with respect to ∇, then, from (3.9), we have

$$J(A_N X) - u(A_N X)N + \tau(X)U = 0, \quad \forall X \in \Gamma(TM).$$

Applying J to this equation and using (3.1) and (3.4), we obtain

$$A_N X - u(A_N X)U - \tau(X)N = 0.$$

Taking the scalar product with ξ to this equation, we get $\tau = 0$ and

$$A_N X = \sigma(X)U. \tag{5.8}$$

As M is Hopf lightlike hypersurface, from (5.3) and (5.8), we obtain

$$A_N X = \alpha v(X)U. \tag{5.9}$$

Taking the scalar product with Y to the last equation, we see that

$$g(A_N X, Y) = \alpha v(X)v(Y).$$

It follow that A_N is self-adjoint operator with respect to g. Consequently, C is symmetric on $S(TM)$ due to (2.8). By using (2.3) we obtain

$$\eta([X, Y]) = C(X, Y) - C(Y, X) = 0,$$

which implies that $[X, Y] \in \Gamma(S(TM))$ for any $X, Y \in \Gamma(S(TM))$. Thus $S(TM)$ is an integrable distribution. $\square$

Theorem 5.7. *Let M be a Hopf lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$. If F or U is parallel with respect to the induced connection ∇ on M, then the structure Jacobi operator ϕ is satisfied $\phi = 0$.*

Proof. Assume that M is a Hopf lightlike hypersurface. Then $c = 0$ by Theorem 5.4. Substituting (5.2), (5.7) and (5.9) into (4.14), we see that $\phi = 0$. $\square$

Definition 5.8. *A lightlike hypersurface M of an indefinite almost complex manifold $\bar{M}$ is called a quasi-Hopf lightlike hypersurface if it admits a principal structure vector field U, with respect to the shape operator A_N, that is,*

$$A_N U = \beta U. \tag{5.10}$$

Taking the scalar product with PX to (5.10) and using (2.8), we get

$$C(U, PX) = \beta v(X). \tag{5.11}$$

Remark 5.9. *Let M be a screen conformal lightlike hypersurface of $\bar{M}$. Then, from (5.10) and the fact that $A_N = \varphi A_\xi^*$, we see that the definitions of Hopf and quasi-Hopf lightlike hypersurfaces are equivalent to each other.*

Theorem 5.10. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. If F or U is parallel with respect to ∇, then M is a quasi-Hopf lightlike hypersurface of $\bar{M}$. Moreover if the ambient manifold $\bar{M}$ is an indefinite almost complex space form $\bar{M}(c)$, then $c = 0$.*

Proof. Assume that F or U is parallel with respect to ∇. Then, from (5.6) and (5.8), we obtain $A_N U = \beta U$. Thus M is a quasi-Hopf lightlike hypersurface.

By taking the scalar product with U to (5.6) and (5.8), we get

$$C(X, U) = 0, \quad \forall X \in \Gamma(TM).$$

Replacing PZ by U to (3.18) and using (3.9), (5.6) and (5.8), we obtain

$$\frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$ to this equation, we get $c = 0$. $\square$

Theorem 5.11. *Let M be a screen conformal lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$. If F or U is parallel with respect to ∇, then M is a Hopf lightlike hypersurface of $\bar{M}$ such that $\alpha = \beta = 0$. Moreover if $\bar{M}$ is an indefinite almost complex space form $\bar{M}(c)$, then $c = 0$.*

Proof. As M is screen conformal, by Remark 5.9 and Theorem 5.10 if F or U is parallel with respect to ∇, then M is a Hopf lightlike hypersurface of $\bar{M}$. Moreover if $\bar{M}$ is an indefinite almost complex space form $\bar{M}(c)$, then $c = 0$. As M is Hopf lightlike hypersurface, from (5.3) we obtain

$$C(U, V) = 0.$$

Since M is quasi-Hopf lightlike hypersurface, taking $PX = V$ to (5.11) we get $\beta = C(U, V) = 0$. As M is screen conformal, we see that

$$0 = C(U, V) = \varphi B(U, V) = \varphi C(V, V) = \alpha\varphi.$$

As $\varphi \neq 0$, we see that $\alpha = 0$. This completes the proof of the theorem. $\square$

Theorem 5.12. *Let M be a Hopf and quasi-Hopf lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$. If ϕ is non-vanishing and parallel with respect to ∇, then M is screen totally geodesic.*

Proof. Assume that M is Hopf lightlike hypersurface. Then $c = 0$ by Theorem 5.4. Substituting (5.2) and (5.10) into (4.14), we have

$$\phi X = -\alpha\beta v(X)U. \tag{5.12}$$

Applying ∇_Y to (5.12) and using (3.9), (3.13) and (5.12), we obtain

$$\{X[\alpha\beta] + 2\alpha\beta\tau(X)\}v(Y)U = \alpha\beta\{g(A_N X, FY)U - v(Y)F(A_N X))\},$$

as ϕ is parallel with respect to ∇. Taking the scalar product with V to this equation, we have

$$\{X[\alpha\beta] + 2\alpha\beta\tau(X)\}v(Y) = \alpha\beta g(A_N X, FY).$$

Replacing Y by V to this, we get $X[\alpha\beta] + 2\alpha\beta\tau(X) = 0$. As ϕ is non-vanishing, we see from (5.12) that $\alpha\beta \neq 0$. Thus, from the last two equations, we obtain

$$g(A_N X, FY) = 0, \qquad F(A_N X) = 0. \tag{5.13}$$

Replacing Y by FZ to $(5.13)_1$ and using $(3.7)_1$, we have

$$g(A_N X, Z) = g(C(X, U)V, Z).$$

As $S(TM)$ is non-degenerate, it follows that

$$A_N X = C(X, U)V.$$

On the other hand, applying F to $(5.13)_2$ and using $(3.7)_1$, we have

$$A_N X = \sigma(X)U.$$

Comparing the last two equations, we see that

$$\sigma(X)U = C(X, U)V.$$

This implies that $\sigma(X) = 0$ and $C(X, U) = 0$. Therefore, we have $A_N = 0$ and M is screen totally geodesic. □

References

[1] C. Atindogbe, K.L. Duggal, *Conformal screen on lightlike hypersurfaces*, International J. of Pure and Applied Math., 11(4) (2004) 421-442.
[2] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. 395 (1989) 132-141.
[3] K.L. Duggal, A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.
[4] K.L. Duggal, D.H. Jin, *Null curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.
[5] K.L. Duggal, B. Sahin, *Differential geometry of lightlike submanifolds*, Frontiers in Mathematics, Birkhäuser, 2010.
[6] D.H. Jin, *Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold*, Indian J. of Pure and Applied Math. 41(4) (2010) 569-581.
[7] D.H. Jin, *Screen conformal lightlike real hypersurfaces of an indefinite complex space form*, Bull. Korean Math. Soc. 47(2) (2010) 341-353.
[8] D.H. Jin and J.W. Lee, *Half lightlike submanifolds of a semi-Riemannian manifold of quasi-constant curvature*, accepted to Filomat, 2014.
[9] G. Kaimakamis, K. Panagiotidou, *Real hypersurfaces in a non-flat complex space form with Lie recurrent structure Jacobi operator*, Bull. Korean Math. Soc. 50(6) (2013) 2089-2101.
[10] B. Sahin, C. Yildirim, Slant lightlike submanifolds of indefinite Sasakian manifolds, Filomat, 26:2 (2012), 71-81.
[11] J. Saito, *Real hypersurfaces in a complex hyperbolic space with three constant principal curvatures*, Tsukuba J. Math. 23 (1999) 353-367.
[12] Q.M. Wang, *Real hypersurfaces with constant principal curvatures in complex projective spaces (I)*, Sci. Sinica Ser. A 26 (1983) 1017-1024.
[13] Z. Xu, *Real hypersurfaces with constant principal curvatures in complex hyperbolic spaces*, J. Math. Wuhan 16 (1996) 490-496.