Filomat 30:7 (2016), 1717–1721
DOI 10.2298/FIL1607717K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Symmetric Identities for the Higher-Order q-Euler Polynomials Related to Symmetry Group S_3 Arising from p-Adic q-Fermionic Integrals on $\mathbb{Z}_p$

Dae San Kim[a], Taekyun Kim[b]

[a]*Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*
[b]*Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*

Abstract. The purpose of this paper is to give some new symmetric identities for the higher-order q-Euler polynomials of the first kind related to symmetry group S_3 arising from p-adic q-fermionic integrals on $\mathbb{Z}_p$.

1. Introduction

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks about a q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. The q-number of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the p-adic q-fermionic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-1)^x q^x, \quad (\text{see } [6, 13]). \tag{1}$$

Note that

$$\lim_{q\to 1} \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) \tag{2}$$

$$= \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad (\text{see } [1\text{–}23]).$$

2010 *Mathematics Subject Classification*. 11B68; 11S80
Keywords. symmetric identities, higher-order q-Euler polynomial, p-adic q-fermionic integral
Received: 07 September 2014; Accepted: 27 November 2014
Communicated by Hari M. Srivastava
Email addresses: dskim@sogang.ac.kr (Dae San Kim), tkkim@kw.ac.kr (Taekyun Kim)

As is well known, the higher-order Euler polynomials are defined by

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \quad (r \in \mathbb{N}). \tag{3}$$

Thus, by (2) and (3), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x+x_1+\cdots+x_r)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ =&\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see [1–23]}). \end{aligned} \tag{4}$$

From (4), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (x+x_1+\cdots+x_r)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = E_n^{(r)}(x), \quad (n \geq 0). \tag{5}$$

In view of (5), we consider the higher-order q-Euler polynomials which are given by the p-adic q-fermionic integral on $\mathbb{Z}_p$

$$\begin{aligned} &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{[x,+x_1+\cdots+x_r]_q t} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\ =&\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!}. \end{aligned} \tag{6}$$

Thus, by (6), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} [x+x_1+\cdots+x_r]_q^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = E_{n,q}^{(r)}(x), \quad (n \geq 0). \tag{7}$$

From (7), we have

$$E_{n,q}^{(r)}(x) = \frac{[2]_q^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(\frac{1}{1+q^{l+1}}\right)^r = [2]_q^r\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m q^m [m+x]_q^n. \tag{8}$$

By (8), we see that the generating function of $E_{n,q}^{(r)}(x)$ is given by

$$\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!} = [2]_q^r\sum_{m=0}^{\infty}\binom{r+m-1}{m}(-1)^m q^m e^{[m+x]_q t}. \tag{9}$$

From (9), we note that $\lim_{q\to 1} E_{n,q}^{(r)}(x) = E_n^{(r)}(x)$.

The purpose of this paper is to give some new symmetric identities for the higher-order q-Euler polynomials related to symmetric group S_3 arising from p-adic q-fermionic integrals on $\mathbb{Z}_p$.

Recently, several researchers have studied the q-extension of Euler polynomials in the various areas (see [1–23]).

2. Some Identities for Higher-Order q-Euler Polynomials

Let w_1, w_2, w_3 be odd natural numbers. Then we observe that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left[w_1w_2w_3x+w_2w_3\sum_{l=1}^r x_l+w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l\right]_q t} d\mu_{-q^{w_2w_3}}(x_1)\cdots d\mu_{-q^{w_2w_3}}(x_r) \tag{10}$$
$$= \lim_{N\to\infty}\left(\frac{1}{[w_1p^N]_{-q^{w_2w_3}}}\right)^r \sum_{k_1,\ldots,k_r=0}^{w_1-1}\sum_{x_1,\ldots,x_r=0}^{p^N-1} e^{\left[w_2w_3\sum_{l=1}^r(k_l+w_1x_l)+w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l+w_1w_2w_3x\right]_q t}$$
$$\times q^{w_2w_3\sum_{l=1}^r(k_l+w_1x_l)}(-1)^{\sum_{l=1}^r(k_l+x_l)}.$$

By (10), we get

$$\left(\frac{[2]_q}{[2]_{q^{w_2w_3}}}\right)^r \sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{j_1,\ldots,j_r=0}^{w_3-1}(-1)^{\sum_{l=1}^r(i_l+j_l)} q^{w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l} \tag{11}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left[w_2w_3\sum_{l=1}^r x_l+w_1w_2w_3x+w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l\right]_q t} d\mu_{-q^{w_2w_3}}(x_1)\cdots d\mu_{-q^{w_2w_3}}(x_r)$$
$$= \lim_{N\to\infty}\frac{1}{[w_1w_2w_3p^N]_{-q}^r}\sum_{i_1,\ldots,i_r=0}^{w_2-1}\sum_{j_1,\ldots,j_r=0}^{w_3-1}\sum_{k_1,\ldots,k_r=0}^{w_1-1}\sum_{x_1,\ldots,x_r=0}^{p^N-1} q^{w_2w_3\sum_{l=1}^r k_l+w_1w_3\sum_{l=0}^r i_l+w_1w_2\sum_{l=1}^r j_l}$$
$$\times(-1)^{\sum_{l=1}^r(i_l+j_l+k_l)}(-1)^{\sum_{l=1}^r x_l} q^{w_1w_2w_3\sum_{l=1}^r x_l} e^{\left[w_2w_3\sum_{k=1}^r k_l+w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l+w_1w_2w_3(x+x_l)\right]_q t}.$$

As this expression is invariant under any permutation w_1, w_2, w_3, we have the following theorem.

Theorem 2.1. *Let w_1, w_2, w_3 be odd natural numbers. Then the following expressions*

$$\left(\frac{[2]_q}{[2]_{q^{w_{\sigma(2)}w_{\sigma(3)}}}}\right)^r \sum_{i_1,\ldots,i_r=0}^{w_{\sigma(2)}-1}\sum_{j_1,\ldots,j_r=0}^{w_{\sigma(3)}-1}(-1)^{\sum_{l=1}^r(i_l+j_l)} q^{w_{\sigma(1)}w_{\sigma(3)}\sum_{l=1}^r i_l}$$
$$\times q^{w_{\sigma(1)}w_{\sigma(2)}\sum_{l=1}^r j_l}$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left[w_{\sigma(2)}w_{\sigma(3)}\sum_{l=1}^r x_l+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}x+w_{\sigma(1)}w_{\sigma(3)}\sum_{l=1}^r i_l+w_{\sigma(1)}w_{\sigma(2)}\sum_{l=1}^r j_l\right]_q t}$$
$$\times d\mu_{-q^{w_{\sigma(2)}w_{\sigma(3)}}}(x_1)\cdots d\mu_{-q^{w_{\sigma(2)}w_{\sigma(3)}}}(x_r)$$

are the same for any $\sigma\in S_3$.

Now we observe that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left[w_2w_3\sum_{l=1}^r x_l+w_1w_2w_3x+w_1w_3\sum_{l=1}^r i_l+w_1w_2\sum_{l=1}^r j_l\right]_q t} d\mu_{-q^{w_2w_3}}(x_1)\cdots d\mu_{-q^{w_2w_3}}(x_r) \tag{12}$$
$$= \sum_{n=0}^\infty [w_2w_3]_q^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\sum_{l=1}^r x_l+w_1x+\frac{w_1}{w_2}\sum_{l=1}^r i_l+\frac{w_1}{w_3}\sum_{l=1}^r j_l\right]_{q^{w_2w_3}}^n$$
$$\times d\mu_{-q^{w_2w_3}}(x_1)\cdots d\mu_{-q^{w_2w_3}}(x_r)\frac{t^n}{n!}$$
$$= \sum_{n=0}^\infty [w_2w_3]_q^n E_{n,q^{w_2w_3}}^{(r)}\left(w_1x+\frac{w_1}{w_2}\sum_{l=1}^r i_l+\frac{w_1}{w_3}\sum_{l=1}^r j_l\right)\frac{t^n}{n!}.$$

Therefore, by Theorem 2.1 and (12), we obtain the following theorem.

Theorem 2.2. *For $w_1, w_2, w_3 \in \mathbb{N}$ with $w_1 \equiv 1$ (mod 2), $w_2 \equiv 1$ (mod 2), $w_3 \equiv 1$ (mod 2) and $n \in \mathbb{N} \cup \{0\}$, the following expressions*

$$\left(\frac{[2]_q}{[2]_{q^{w_{\sigma(2)}w_{\sigma(3)}}}}\right)^r \left[w_{\sigma(2)}w_{\sigma(3)}\right]_q^n \sum_{i_1,\ldots,i_r=0}^{w_{\sigma(2)}-1} \sum_{j_1,\ldots,j_r=0}^{w_{\sigma(3)}-1} (-1)^{\sum_{l=1}^r (i_l+j_l)} q^{w_{\sigma(1)}w_{\sigma(3)} \sum_{l=1}^r i_l + w_{\sigma(1)}w_{\sigma(2)} \sum_{l=1}^r j_l}$$
$$\times E_{n,q^{w_{\sigma(2)}w_{\sigma(3)}}}^{(r)} \left(w_{\sigma(1)}x + \frac{w_{\sigma(1)}}{w_{\sigma(2)}} \sum_{l=1}^r i_l + \frac{w_{\sigma(1)}}{w_{\sigma(3)}} \sum_{l=1}^r j_l\right)$$

are the same for any $\sigma \in S_3$.

It is easy to show that

$$\left[\sum_{l=1}^r x_l + w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r i_l + \frac{w_1}{w_3} \sum_{l=1}^r j_l\right]_{q^{w_2 w_3}} \tag{13}$$
$$= \frac{[w_1]_q}{[w_2 w_3]_q} \left[w_3 \sum_{l=1}^r i_l + w_2 \sum_{l=1}^r j_l\right]_{q^{w_1}} + q^{w_1 w_3 \sum_{l=1}^r i_l + w_1 w_2 \sum_{l=1}^r j_l} \left[\sum_{l=1}^r x_l + w_1 x\right]_{q^{w_2 w_3}}.$$

Thus, by (13), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\sum_{l=1}^r x_l + w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r i_l + \frac{w_1}{w_3} \sum_{l=1}^r j_l\right]_{q^{w_2 w_3}}^n d\mu_{-q^{w_2 w_3}}(x_1) \cdots d\mu_{-q^{w_2 w_3}}(x_r) \tag{14}$$
$$= \sum_{k=0}^n \binom{n}{k} \left(\frac{[w_1]_q}{[w_2 w_3]_q}\right)^{n-k} \left[w_3 \sum_{l=1}^r i_l + w_2 \sum_{l=1}^r j_l\right]_{q^{w_1}}^{n-k} q^{k\left(w_1 w_3 \sum_{l=1}^r i_l + w_1 w_2 \sum_{l=1}^r j_l\right)} E_{k,q^{w_2 w_3}}^{(r)}(w_1 x).$$

From (12), Theorem 2.2 and (14), we have

$$\left(\frac{[2]_q}{[2]_{q^{w_2 w_3}}}\right)^r [w_2 w_3]_q^n \sum_{i_1,\ldots,i_r=0}^{w_2-1} \sum_{j_1,\ldots,j_r=0}^{w_3-1} (-1)^{\sum_{l=1}^r (i_l+j_l)} q^{w_1 w_3 \sum_{l=1}^r i_l + w_1 w_2 \sum_{l=1}^r j_l} \tag{15}$$
$$\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\sum_{l=1}^r x_l + w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r i_l + \frac{w_1}{w_3} \sum_{l=1}^r j_l\right]_{q^{w_2 w_3}}^n d\mu_{-q^{w_2 w_3}}(x_1) \cdots d\mu_{-q^{w_2 w_3}}(x_r)$$
$$= \left(\frac{[2]_q}{[2]_{q^{w_2 w_3}}}\right)^r \sum_{k=0}^n \binom{n}{k} [w_2 w_3]_q^k [w_1]_q^{n-k} E_{k,q^{w_2 w_3}}^{(r)}(w_1 x)$$
$$\times \sum_{i_1,\ldots,i_r=0}^{w_2-1} \sum_{j_1,\ldots,j_r=0}^{w_3-1} (-1)^{\sum_{l=1}^r (i_l+j_l)} q^{\left(w_1 w_3 \sum_{l=1}^r i_l + w_1 w_2 \sum_{l=1}^r j_l\right)(k+1)} \left[w_3 \sum_{l=1}^r i_l + w_2 \sum_{l=1}^r j_l\right]_{q^{w_1}}^{n-k}$$
$$= \left(\frac{[2]_q}{[2]_{q^{w_2 w_3}}}\right)^r \sum_{k=0}^n \binom{n}{k} [w_2 w_3]_q^k [w_1]_q^{n-k} E_{k,q^{w_2 w_3}}^{(r)}(w_1 x)\, \widetilde{T}_{n,q^{w_1}}^{(r)}(w_2, w_3|k),$$

where

$$\widetilde{T}_{n,q}^{(r)}(w_1, w_2|k) \tag{16}$$
$$= \sum_{i_1,\ldots,i_r=0}^{w_1-1} \sum_{j_1,\ldots,j_r=0}^{w_2-1} q^{(k+1)\left(w_2 \sum_{l=1}^r i_l + w_1 \sum_{l=1}^r j_l\right)} (-1)^{\sum_{l=1}^r (i_l+j_l)} \left[w_2 \sum_{l=1}^r i_l + w_1 \sum_{l=1}^r j_l\right]_q^{n-k}.$$

As this expression is invariant under any permutation of w_1, w_2, w_3, we have the following theorem.

Theorem 2.3. *Let w_1, w_2, w_3 be odd natural numbers and let n be a nonnegative integer. Then the following expressions*

$$\left(\frac{[2]_q}{[2]_{q^{w_{\sigma(2)}w_{\sigma(3)}}}}\right)^r \sum_{k=0}^{n} \binom{n}{k} \left[w_{\sigma(2)} w_{\sigma 3}\right]_q^k \left[w_{\sigma(1)}\right]_q^{n-k} E^{(r)}_{k,q^{w_{\sigma(2)}w_{\sigma(3)}}} \left(w_{\sigma(1)} x\right) \widetilde{T}^{(r)}_{n,q^{w_{\sigma(1)}}} \left(w_{\sigma(2)}, w_{\sigma(3)} | k\right)$$

are the same for any $\sigma \in S_3$.

References

[1] S. Araci and M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 3, 399–406. MR 2976598

[2] A. Bayad and J. Chikhi, *Apostol-Euler polynomials and asymptotics for negative binomial reciprocals*, Adv. Stud. Contemp. Math. (Kyungshang) **24** (2014), no. 1, 33–37. MR 3157406

[3] D. Ding and J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 1, 7–21. MR 2597988 (2011k:05030)

[4] S. Gaboury, R. Tremblay, and B.-J. Fugère, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Proc. Jangjeon Math. Soc. **17** (2014), no. 1, 115–123. MR 3184467

[5] J.-H. Jeong, J. H. Jin, J.-W. Park, and S.-H. Rim, *On the twisted weak q-Euler numbers and polynomials with weight* 0, Proc. Jangjeon Math. Soc. **16** (2013), no. 2, 157–163. MR 3097729

[6] D. S. Kim, T. Kim, S.-H. Rim, and J.-J. Seo, *A note on symmetric properties of the multiple q-Euler zeta functions and higher-order q-Euler polynomials*, Appl. Math. Sci. (Ruse) **8** (2014), no. 29-32, 1585–1591. MR 3200156

[7] D. S. Kim and T. K. Kim, *q-Bernoulli polynomials and q-umbral calculus*, Sci. China Math. **57** (2014), no. 9, 1867–1874. MR 3249396

[8] D. S. Kim, N. Lee, J. Na, and K. H. Park, *Identities of symmetry for higher-order Euler polynomials in three variables (I)*, Adv. Stud. Contemp. Math. (Kyungshang) **22** (2012), no. 1, 51–74. MR 2931605

[9] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **16** (2009), no. 1, 93–96. MR 2486809 (2010c:11028)

[10] ———, *New approach to q-Euler polynomials of higher order*, Russ. J. Math. Phys. **17** (2010), no. 2, 218–225. MR 2660920 (2011f:33026)

[11] ———, *Symmetry properties of the generalized higher-order Euler polynomials*, Proc. Jangjeon Math. Soc. **13** (2010), no. 1, 13–16. MR 2656437 (2011d:11039)

[12] ———, *A study on the q-Euler numbers and the fermionic q-integral of the product of several type q-Bernstein polynomials on $\mathbb{Z}_p$*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 1, 5–11. MR 3059313

[13] T. Kim, J. Y. Choi, and J. Y. Sug, *Extended q-Euler numbers and polynomials associated with fermionic p-adic q-integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **14** (2007), no. 2, 160–163. MR 2318827 (2008e:33041)

[14] T. Kim, D. S. Kim, D. V. Dolgy, and S. H. Rim, *Some identities on the Euler numbers arising from Euler basis polynomials*, Ars Combin. **109** (2013), 433–446. MR 3087232

[15] Y.-H. Kim and K.-W. Hwang, *Symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **18** (2009), no. 2, 127–133. MR 2508978 (2010b:11027)

[16] H. Y. Lee, N. S. Jung, J. Y. Kang, and C. S. Ryoo, *Extension of the higher-order twisted q-Euler numbers and polynomials with multi weight β_i and higher h_i*, Proc. Jangjeon Math. Soc. **16** (2013), no. 2, 203–213. MR 3097734

[17] Q.-M. Luo and F. Qi, *Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **7** (2003), no. 1, 11–18. MR 1981601

[18] N. I. Mahmudov and M. Momenzadeh, *On a class of q-Bernoulli, q-Euler, and q-Genocchi polynomials*, Abstr. Appl. Anal. (2014), Art. ID 696454, 10. MR 3206813

[19] H. Ozden, I. N. Cangul, and Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **18** (2009), no. 1, 41–48. MR 2479746 (2009k:11037)

[20] J.-W. Park, *New approach to q-Bernoulli polynomials with weight or weak weight*, Adv. Stud. Contemp. Math. (Kyungshang) **24** (2014), no. 1, 39–44. MR 3157407

[21] S.-H. Rim, J. Joung, J.-H. Jin, and S.-J. Lee, *A note on the weighted Carlitz's type q-Euler numbers and q-Bernstein polynomials*, Proc. Jangjeon Math. Soc. **15** (2012), no. 2, 195–201. MR 2954142

[22] C. S. Ryoo, *A note on the weighted q-Euler numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **21** (2011), no. 1, 47–54. MR 2984863

[23] Z. Zhang and H. Yang, *Some closed formulas for generalized Bernoulli-Euler numbers and polynomials*, Proc. Jangjeon Math. Soc. **11** (2008), no. 2, 191–198. MR 2482602 (2010a:11036)