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Bicyclic Graphs with Maximum Degree Resistance Distance

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Abstract. Graph invariants, based on the distances between the vertices of a graph, are widely used in theoretical chemistry. Recently, Gutman, Feng and Yu (*Transactions on Combinatorics*, 01 (2012) 27-40) introduced the *degree resistance distance* of a graph G, which is defined as $D_R(G) = \sum_{\{u,v\}\subseteq V(G)}[d_G(u) + d_G(v)]R_G(u,v)$, where $d_G(u)$ is the degree of vertex u of the graph G, and $R_G(u,v)$ denotes the resistance distance between the vertices u and v of the graph G. Further, they characterized n-vertex unicyclic graphs having minimum and second minimum degree resistance distance. In this paper, we characterize n-vertex bicyclic graphs having maximum degree resistance distance.

1. Introduction

The graphs considered in this paper are finite, loopless and contain no multiple edges. Given a graph G, let $V(G)$ and $E(G)$ be the vertex and edge sets of G, respectively. The ordinary distance $d_G(u,v)$ between the vertices u and v of the graph G is the length of the shortest path between u and v.

The famous *Wiener index* $W(G)$ is the sum of ordinary distances between all pairs of vertices, that is, $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$. The Winner index is the oldest and one of the most popular molecular structure descriptors [9, 10], well correlated with many physical and chemical properties of a variety of classes of chemical compounds.

A modified version of the Wiener index is the *degree distance* defined as $D(G) = \sum_{\{u,v\}\subseteq V(G)}(d_G(u) + d_G(v))d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u of the graph G. The degree distance was also widely studied [4, 5, 11, 15, 19]. Tomescu [15] determined the unicyclic and bicyclic graphs with minimum degree distance. Yuan and An [19] determined the unicyclic graphs with maximum degree distance.

Sharpe [14] introduced a distance function named *resistance distance*, based on the theory of electrical networks. They viewed G as an electric network N by replacing each edge of G with a unit resistor. The resistance distance between the vertices u and v of the graph G, denoted by $R_G(u,v)$, is then defined to be the effective resistance between the nodes u and v in N. This kind of distance between vertices of a graph was eventually studied in detail [1–3, 12, 13, 16, 21].

If the ordinary distance is replaced by resistance distance in the expression for the Wiener index, we can arrive at the *Kirchhoff index* $Kf(G) = \sum_{\{u,v\}\subseteq V(G)} R_G(u,v)$, which also has been widely studied [6, 7, 17, 18, 20–22].

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Similarly, if the ordinary distance is replaced by resistance distance in the expression for the degree distance, Gutman, Feng and Yu [8] introduced the *degree resistance distance*:

$$D_R(G) = \sum_{\{u,v\}\subseteq V(G)}(d_G(u) + d_G(v))R_G(u,v).$$

They gave some properties of degree resistance distance and determined the unicyclic graphs with minimum and second minimum degree resistance distance.

Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one. In this paper we determine the bicyclic graphs having maximum degree resistance distance.

2. Preliminaries

It is important that $R_G(u,v) = R_G(v,u)$, $R_G(u,u) = 0$ and that $d_G(u,v) \geqslant R_G(u,v)$ with equality if and only if there is a unique path linking the vertices u and v. For a vertex v in G, we define $Kf_v(G) = \sum_{u\in G} R_G(u,v)$ and $D_v(G) = \sum_{u\in G} d_G(u)R_G(u,v)$.

By the definition of $D_R(G)$, we also have $D_R(G) = \sum_{\{u,v\}\subseteq V(G)}[d_G(u)+d_G(v)]R_G(u,v) = \sum_{v\in G} d_G(v)\sum_{u\in G} R_G(u,v)$.

Lemma 2.1. *[12] Let G be a graph, x be a cut vertex of G and let u, v be vertices belonging to different components which arise upon deletion of x. Then $R_G(u,v) = R_G(u,x) + R_G(x,v)$.*

For a graph G and its vertex v, let $G - v$ be the graph obtained by removing v and all edges incident to v from G.

Lemma 2.2. *[6] Let G be a connected graph of order n, v be a pendant vertex of G and w be its neighbor. Then $Kf_v(G) = Kf_w(G - v) + n - 1$.*

Lemma 2.3. *Let G be a bicyclic graph of order n, v be a pendant vertex of G and w be its neighbor. Then $D_v(G) = D_w(G - v) + 2n + 1$.*

Proof. From the definition, we have

$$\begin{aligned}
D_v(G) &= \sum_{u\in G} d_G(u)R_G(u,v) \\
&= \sum_{u\in G-v} d_G(u)R_G(u,v) + d_G(v)R_G(v,v) \\
&= \sum_{u\in G-v} d_G(u)[R_G(u,w) + R_G(w,v)] \ \ (\text{By } R_G(v,v) = 0 \text{ and Lemma 2.1}) \\
&= \sum_{u\in G-v} d_G(u)R_G(u,w) + \sum_{u\in G-v} d_G(u) \ \ (\text{By } R_G(w,v) = 1) \\
&= \sum_{u\in G-v} d_{G-v}(u)R_{G-v}(u,w) + 2(n+1) - 1 \ \ (\text{By } E(G) = n+1 \text{ and } d_G(v) = 1) \\
&= D_w(G - v) + 2n + 1.
\end{aligned}$$

This proof is complete. □

Lemma 2.4. *[6] Let G be a bicyclic graph of order n and $v \in V(G)$. Then $Kf_v(G) \leq \frac{n^2}{2} - \frac{n}{2} - \frac{15}{4}$.*

The base of a bicyclic graph G, denoted by $\hat{G}$, is the unique bicyclic subgraph of G containing no pendent vertices, while G can be obtained from $\hat{G}$ by attaching trees to some vertices of $\hat{G}$.

Definition 2.5. *Let G be a bicyclic graph and v be a vertex in $\hat{G}$. Let G_i, $i \in I$, be the components of $G - v$ such that $G[V(G_i)\bigcup\{v\}]$ contains no cycles. The tree $T_v(G) = G[(\bigcup_{i\in I} V(G_i))\bigcup\{v\}]$, rooted at v, is called the tree suspended at v. Note that, if the index set I is empty, $T_v(G)$ consists of only the single vertex v.*

Lemma 2.6. *Let G be a bicyclic graph of order n and $v \in V(G)$. Then $D_v(G) \leq n^2 + 2n - \frac{73}{4}$.*

Proof. The only bicyclic graph G with 4 vertices is $K_4 - e$, and a straightforward calculation shows that for any vertex $v \in V(G)$, $D_v(G) \leq 23/4 = 4^2 + 2 \times 4 - 73/4$.

We now distinguish the following two cases.

Case 1. The vertex v is a pendant vertex. Let w be its neighbor. We prove this case by induction on n. Clearly $G - v$ satisfies the induction hypothesis. By Lemma 2.3 we have

$$\begin{aligned} D_v(G) &= D_w(G-v) + 2n + 1 \\ &\leq ((n-1)^2 + 2(n-1) - \frac{73}{4}) + 2n + 1 \\ &= n^2 + 2n - \frac{73}{4}. \end{aligned}$$

Case 2. The vertex v isn't a pendant vertex. We consider the following two subcases.

Subcase 1. The vertex v is not in any cycle of G.

In this subcase, $G - v$ has at least two components. Let the components of $G - v$ be $A_1, A_2, \cdots, A_k$, $k \geqslant 2$. Since v is not in any cycle, v has only one adjacent vertex, say u_i, in each component A_i, $1 \leqslant i \leqslant k$.

Now, we construct a new bicyclic graph $G_1 = G - vu_1 + u_1u_2$. We will prove that $D_v(G) = \sum_{w \in G} d_G(w) R_G(w, v) < D_v(G_1) = \sum_{w \in G_1} d_{G_1}(w) R_{G_1}(w, v)$. Firstly, if $w \in \bigcup_{i=2}^{k} V(A_i) \setminus \{u_2\}$, then $d_{G_1}(w) = d_G(w)$ and $R_{G_1}(w, v) = R_G(w, v)$. Secondly, if $w = u_2$, then $d_{G_1}(w) = d_G(w) + 1$ and $R_{G_1}(w, v) = R_G(w, v) = 1$. Finally, if $w \in V(A_1)$, then $d_{G_1}(w) = d_G(w)$ and $R_{G_1}(w, v) = R_G(w, v) + 1$. Thus, we have that $D_v(G_1) > D_v(G)$.

Recursively, we construct the bicyclic graph $G_i = G_{i-1} - vu_i + u_iu_{i+1}$, $2 \leqslant i \leqslant k-1$. Similarly, we can prove that $D_v(G_1) < D_v(G_2) < \cdots < D_v(G_{k-1})$. In G_{k-1}, v is a pendant vertex, thus by Case 1 $D_v(G) < D_v(G_{k-1}) = n^2 + 2n - \frac{73}{4}$.

Subcase 2. The vertex v is a vertex in a cycle of G.

Let $W = V(\hat{G})$, where $\hat{G}$ is the base of G. We need first to prove the following two claims:

Claim 1. For a vertex $w \in W$, suppose that the tree $T_w(G)$ suspended at w contains $k \geqslant 2$ vertices and is not a path. The bicyclic graph G' with n vertices is obtained by deleting all vertices in $V(T_w(G)) \setminus \{w\}$ from G and attaching one pendant path of order $k-1$ to the vertex w. Then $D_v(G) < D_v(G')$.

Proof. Suppose that P is a longest path starting at w in $T_w(G)$ and P ends at vertex w_1. Since $T_w(G)$ is not a path, there exists another pendant vertex, say w_2. And let w_3 be the neighbor of w_2. Construct a new bicyclic graph $G_1 = G - w_3w_2 + w_1w_2$, then

$$\begin{aligned} D_v(G_1) - D_v(G) &= \sum_{i=1}^{3} (d_{G_1}(w_i) R_{G_1}(w_i, v) - d_G(w_i) R_G(w_i, v)) \\ &= R_G(w_1, v) + (R_{G_1}(w_2, v) - R_G(w_2, v)) + (-R_G(w_3, v)) \\ &= (R_G(w_1, w) - R_G(w_3, w)) + (R_{G_1}(w_2, w) - R_G(w_2, w)) \\ &> 0. \end{aligned}$$

For each pendant vertex which is not in the longest path starting at w of the current graph, we consecutively use the process. At last, we obtain the bicyclic graph G' and $D_v(G) < D_v(G')$. $\square$

Without loss of generality, let u be a vertex in W so that $R_G(u, v) = \max_{w \in W} R_G(w, v)$.

Claim 2. For a vertex $w \in W$, suppose that the tree $T_w(G)$ suspended at w contains $k \geqslant 2$ vertices and is a path $w_1(w = w_1)w_2w_3 \cdots w_k$. Construct a bicyclic graph $G' = G - ww_2 + uw_2$, then $D_v(G) \leqslant D_v(G')$.

Proof.

$$\begin{aligned}
D_v(G') - D_v(G) &= \sum_{i=1}^{k}(d_{G'}(w_i)R_{G'}(w_i, v) - d_G(w_i)R_G(w_i, v)) \\
&\quad +(d_{G'}(u)R_{G'}(u, v) - d_G(u)R_G(u, v)) \\
&= (-R_G(w_1, v)) + \sum_{i=2}^{k} d_G(w_i)(R_{G'}(w_i, v) - R_G(w_i, v)) + R_G(u, v) \\
&= (R_G(u, v) - R_G(w, v)) + \sum_{i=2}^{k} d_G(w_i)(R_G(u, v) - R_G(w, v)) \\
&\geqslant 0. \quad \square
\end{aligned}$$

Suppose that a bicyclic graph G'' with n vertices is obtained from $\hat{G}$ by attaching a pendant path to the vertex u. By consecutive application of Claim 1 and Claim 2, $D_v(G) \leqslant D_v(G'')$.

Let G_1 be the bicyclic graphs with n vertices obtained from $K_4 - e$ by attaching a pendant path to a vertex of degree 2 in $K_4 - e$. Let s_1 be the unique pendant vertex of G_1. If the graph G'' has three vertices of degree 3, then it is easy to see that $D_v(G'') \leqslant D_{s_1}(G_1)$. And

$$\begin{aligned}
D_{s_1}(G_1) &= 2 \times (1 + 2 + \cdots + (n - 5)) + 3 \times (n - 4) \\
&\quad +2 \times 3 \times (n - 4 + \frac{5}{8}) + 2 \times (n - 3) \\
&= n^2 + 2n - \frac{73}{4},
\end{aligned}$$

thus $D_v(G) \leqslant n^2 + 2n - \frac{73}{4}$.

Let G_2 be the bicyclic graphs with n vertices obtained from $K_4 - e$ by attaching a pendant path to a vertex of degree 3 in $K_4 - e$. Let s_2 be the unique pendant vertex of G_2. If there are one vertex of degree 4 and one vertex of degree 3 in G'', then it is easy to see that $D_v(G'') \leqslant D_{s_2}(G_2)$. And

$$\begin{aligned}
D_{s_2}(G_2) &= 2 \times (1 + 2 + \cdots + (n - 5)) + 4 \times (n - 4) \\
&\quad +2 \times 2 \times (n - 4 + \frac{5}{8}) + 3 \times (n - 4 + \frac{1}{2}) \\
&= n^2 + 2n - 20,
\end{aligned}$$

thus $D_v(G) \leqslant n^2 + 2n - 20 < n^2 + 2n - \frac{73}{4}$.

The proof is complete. $\square$

Lemma 2.7. *Let G be a bicyclic graph, v be a pendant vertex of G and w be its neighbor. Then $D_R(G) = D_R(G - v) + D_w(G - v) + 2Kf_w(G - v) + 3n$.*

Proof. From the definition, we have

$$\begin{aligned}
D_R(G) &= \sum_{u\in G} d_G(u) \sum_{x\in G} R_G(u,x) \\
&= \sum_{u\in G-v} d_G(u) \sum_{x\in G} R_G(u,x) + d_G(v) \sum_{x\in G} R_G(v,x) \\
&= \sum_{u\in G-v} d_G(u)(\sum_{x\in G-v} R_G(u,x) + R_G(u,v)) + Kf_v(G) \\
&= \sum_{u\in G-v} d_G(u)(\sum_{x\in G-v} R_{G-v}(u,x) + R_G(u,v)) + Kf_v(G) \\
&= \sum_{u\in G-v} d_G(u) \sum_{x\in G-v} R_{G-v}(u,x) + \sum_{u\in G-v} d_G(u) R_G(u,v) + Kf_v(G) \\
&= \sum_{u\in G-v} d_{G-v}(u) \sum_{x\in G-v} R_{G-v}(u,x) + \sum_{x\in G-v} R_{G-v}(w,x) + \sum_{u\in G-v} d_G(u) R_G(u,v) + Kf_v(G) \\
&= D_R(G-v) + Kf_w(G-v) + \sum_{u\in G-v} d_G(u) R_G(u,v) + Kf_v(G) \\
&= D_R(G-v) + Kf_w(G-v) + \sum_{u\in G-v} d_G(u)(R_G(u,w) + R_G(w,v)) + Kf_v(G) \\
&= D_R(G-v) + Kf_w(G-v) + \sum_{u\in G-v} d_{G-v}(u) R_{G-v}(u,w) + \sum_{u\in G-v} d_G(u) + Kf_v(G) \\
&= D_R(G-v) + Kf_w(G-v) + D_w(G-v) + 2(n+1) - 1 + Kf_w(G-v) + n - 1 \\
&= D_R(G-v) + D_w(G-v) + 2Kf_w(G-v) + 3n.
\end{aligned}$$

This proves the result. □

Lemma 2.8. *[8] Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $V(G_1) \bigcup V(G_2) = V(G)$. Let $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$, $m_1 = |E(G_1)|$ and $m_2 = |E(G_2)|$. Then*

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2 Kf_v(G_1) + 2m_1 Kf_v(G_2) + (n_2 - 1)D_v(G_1) + (n_1 - 1)D_v(G_2).$$

Lemma 2.9. *[8] For the cycle C_k and $v \in C_k$, $Kf(C_k) = \frac{k^3-k}{12}$, $D_R(C_k) = \frac{k^3-k}{3}$, $Kf_v(C_k) = \frac{k^2-1}{6}$ and $D_v(C_k) = \frac{k^2-1}{3}$.*

Lemma 2.10. *Let H be connected graph of order $h > 2$ and C_k be a cycle of order $k \geq 4$. Let F be the graph of order k obtained from C_3 by attaching one pendant path of order $k-3$ to one vertex of C_3. Further suppose G_1 is the graph obtained from H and C_k by identifying one vertex in H and one vertex in C_k; G_2 is the graph obtained from H and F by identifying one vertex in H and the pendant vertex in F. Then we have $D_R(G_1) < D_R(G_2)$.*

Proof. Suppose $V(H) \bigcap V(C_k) = V(H) \bigcap V(F) = v$, $|E(H)| = m$. By Lemma 2.8 we have

$$D_R(G_1) = D_R(H) + D_R(C_k) + 2kKf_v(H) + 2mKf_v(C_k) + (k-1)D_v(H) + (h-1)D_v(C_k).$$

$$D_R(G_2) = D_R(H) + D_R(F) + 2kKf_v(H) + 2mKf_v(F) + (k-1)D_v(H) + (h-1)D_v(F).$$

Therefore,

$$D_R(G_1) - D_R(G_2) = D_R(C_k) - D_R(F) + 2m[Kf_v(C_k) - Kf_v(F)] + (h-1)[D_v(C_k) - D_v(F)].$$

We can get the following results by straightforward calculating:

$$\begin{aligned}
Kf(F) &= \frac{1}{6}(k^3 - 11k + 18). \\
D_R(F) &= 4Kf(F) + [\frac{2}{3} + \frac{2}{3} + 1 + 2 + \cdots + (k-3)] \\
&\quad -[1 + 2 + \cdots + (k-3) + (k-3+\frac{2}{3}) + (k-3+\frac{2}{3})] \\
&= \frac{2}{3}(k^3 - 14k + 27). \\
Kf_v(F) &= \frac{1}{6}(3k^2 - 3k - 10). \\
D_v(F) &= 2Kf(F) + k - 3 = \frac{1}{3}(3k^2 - 2k - 19).
\end{aligned}$$

Then,

$$\begin{aligned}
D_R(C_k) - D_R(F) &= \frac{k^3-k}{3} - \frac{2}{3}(k^3 - 14k + 27) = -\frac{1}{3}k^3 + 9k - 18 < 0. \\
Kf_v(C_k) - Kf_v(F) &= \frac{k^2-1}{6} - \frac{1}{6}(3k^2 - 3k - 10) = -\frac{1}{3}k^2 + \frac{1}{2}k + \frac{3}{2} < 0. \\
D_v(C_k) - D_v(F) &= \frac{k^2-1}{3} - \frac{1}{3}(3k^2 - 2k - 19) = -\frac{2}{3}k^2 + \frac{2}{3}k + 6 < 0.
\end{aligned}$$

Therefore $D_R(G_1) - D_R(G_2) < 0$. We get finally that $D_R(G_1) < D_R(G_2)$. □

3. Main Results

In this section, we give the bicyclic graphs of order at least 6 with maximum degree resistance distance.

Let G be a bicyclic graph. We consider first the base $\hat{G}$ of G. It is well known that there are the following three kinds of bicyclic graphs containing no pendant vertices:

Let $B(p,q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by identifying vertices u of C_p and v of C_q.

Let $B(p,l,q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by joining vertices u of C_p and v of C_q by a new path $uu_1u_2\cdots u_{l-1}v$ with length l ($l \geq 1$).

Let $B(P_k, P_l, P_m)$, $1 \leq m \leq \min\{k,l\}$ be the bicyclic graph obtained from three pairwise internal disjoint paths from a vertex x to a vertex y. These three paths are $xv_1v_2\cdots v_{k-1}y$ with length k, $xu_1u_2\cdots u_{l-1}y$ with length l, and $xw_1w_2\cdots w_{m-1}y$ with length m.

Lemma 3.1. *Let B_n be the bicyclic graph of order n obtained from two vertex-disjoint triangles C_3^1 and C_3^2 by joining vertices u of C_3^1 and v of C_3^2 by a path $uu_1u_2\cdots u_{n-6}v$, i.e., $B_n \cong B(3,n-5,3)$. Then $D_R(B_n) = \frac{1}{3}(2n^3+3n^2-57n+88)$.*

Proof. It is known that [20] $Kf(B_n) = \sum_{\{x,y\}\subseteq V(B_n)} R_{B_n}(x,y) = \frac{1}{6}(n^3 - 21n + 36)$. Any vertex in $V(B_n) \setminus \{u,v\}$ has degree 2 and $d_{B_n}(u) = d_{B_n}(v) = 3$. So we have

$$\begin{aligned} D_R(B_n) &= \sum_{\{x,y\}\subseteq V(B_n)} (d_{B_n}(x) + d_{B_n}(y))R_{B_n}(x,y) \\ &= 4Kf(B_n) + \sum_{x\in B_n} R_{B_n}(x,u) + \sum_{x\in B_n} R_{B_n}(x,v) \\ &= 4\times\frac{1}{6}(n^3 - 21n + 36) + 2\times[\frac{2}{3} + \frac{2}{3} + 1 + 2 + \cdots + n - 5 \\ &\quad +(n - 5 + \frac{2}{3}) + (n - 5 + \frac{2}{3})] \\ &= \frac{2}{3}(n^3 - 21n + 36) + 2\times[\frac{4}{3} + \frac{(1 + n - 5)(n - 5)}{2} + 2n - 10 + \frac{4}{3}] \\ &= \frac{1}{3}(2n^3 + 3n^2 - 57n + 88). \end{aligned}$$

This proves the result. □

Theorem 3.2. *Let G be a bicyclic graph of order $n \geqslant 6$. Then $D_R(G) \leqslant \frac{1}{3}(2n^3 + 3n^2 - 57n + 88)$. The equality holds if and only if $G \cong B_n$.*

Proof. A straightforward calculation shows that for all bicyclic graphs G with 6 vertices, $D_R(G) \leqslant 286/3 = \frac{1}{3}(2\times 6^3 + 3\times 6^2 - 57\times 6 + 88)$, and equality holds if and only if $G \cong B_6$.

Now, we distinguish the following cases.

Case 1. G has a pendant vertex. We prove this case by induction. Let v be a pendant vertex of G and w be its neighbor. Clearly $G - v$ satisfies the induction hypothesis, and

$$\begin{aligned} D_R(G) &= D_R(G - v) + D_w(G - v) + 2Kf_w(G - v) + 3n \text{ (By Lemma 2.7)} \\ &\leq \frac{1}{3}(2(n-1)^3 + 3(n-1)^2 - 57(n-1) + 88) + ((n-1)^2 + 2(n-1) - \frac{73}{4}) \\ &\quad +2(\frac{(n-1)^2}{2} - \frac{(n-1)}{2} - \frac{15}{4}) + 3n \text{ (By Lemma 2.6 and Lemma 2.4)} \\ &= \frac{1}{3}(2n^3 + 3n^2 - 57n + 71\frac{3}{4}) \\ &< \frac{1}{3}(2n^3 + 3n^2 - 57n + 88) = D_R(B_n) \text{ (By Lemma 3.1).} \end{aligned}$$

Case 2. G has no pendant vertex. There are only three types of bicyclic graphs with no pendant vertices, and we consider the following three subcases.

Subcase 1. G is of form $B(p,q)$. By Lemma 2.10, we have that $D_R(G) \leqslant D_R(B_n)$. The equality holds if and only if $G \cong B_n$.

Subcase 2. G is of form $B(p,l,q)$. By Lemma 2.10, we have that $D_R(G) \leqslant D_R(B_n)$. The equality holds if and only if $G \cong B_n$.

Subcase 3. G is of form $B(P_k, P_l, P_m)$, i.e., G can be obtained from three pairwise internal disjoint paths from a vertex x to a vertex y. G has n vertices; any vertex in $V(G)\setminus\{x,y\}$ has degree 2 and $d_G(x) = d_G(y) = 3$. It is well known that [6] $Kf(G) \leq \frac{1}{8}n^3$, then

$$\begin{aligned} D_R(G) &= \sum_{\{u,v\}\subseteq V(G)} [d_G(u) + d_G(v)]R_G(u,v) \\ &= 4Kf(G) + \sum_{w\in G} R_G(w,x) + \sum_{w\in G} R_G(w,y) \\ &\leq 4\cdot\frac{1}{8}n^3 + 2(\frac{n^2}{2} - \frac{n}{2} - \frac{15}{4}) \text{ (By Lemma 2.4)} \\ &= \frac{1}{2}(n^3 + 2n^2 - 2n - 15). \end{aligned}$$

If $n \geq 10$, then $\frac{1}{2}(n^3 + 2n^2 - 2n - 15) < D_R(B_n)$. We have calculated all $D_R(G)$ which G are of form $B(P_k, P_l, P_m)$ when $n = 6, 7, 8, 9$, and found that $D_R(G) < \frac{1}{3}(2n^3 + 3n^2 - 57n + 88)$ for all $n = 6, 7, 8, 9$.

The proof is complete. $\square$

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