



The Existence of Solution for a k -dimensional System of Fractional Differential Inclusions with Anti-Periodic Boundary Value Conditions

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Abstract. We investigate the existence of solutions for a k -dimensional systems of fractional differential inclusions with anti-periodic boundary conditions. We provide two results via different conditions for obtaining solutions of the k -dimensional inclusion problem. We provide some examples to illustrate our results.

1. Introduction

The theory of fractional differential equations has a complete and interesting history. There are many published many works about the existence of solutions for many fractional differential equations by using fixed point theory (see for example, [8]-[12], [25]-[28] and the references there in). Also, some researchers tried to obtain solutions of fractional differential inclusions by using different conditions (see for example, [1]-[7], [14]-[17], [20], [25], [27]-[28], [32] and the references there in). One can find more details elementary notions on fractional differential equations in [22], [26] and [30]. Recently, Chai investigated the existence of solutions for the fractional differential equation

$${}^c D^\alpha u(t) = f(t, u(t), {}^c D^{\gamma_1} u(t), {}^c D^{\gamma_2} u(t)) \quad (1.1)$$

with the anti-periodic boundary value conditions $t^{\beta_1-1} {}^c D^{\beta_1} u(t)|_{t=0} = -t^{\beta_1-1} {}^c D^{\beta_1} u(t)|_{t=1}$, $u(0) = -u(1)$ and $t^{\beta_2-2} {}^c D^{\beta_2} u(t)|_{t=0} = -t^{\beta_2-2} {}^c D^{\beta_2} u(t)|_{t=1}$, where $t \in J = [0, 1]$, $2 < \alpha \leq 3$, $0 < \gamma_1 \leq 1$, $1 < \gamma_2 \leq 2$, $0 < \beta_1 < 1 < \beta_2 < 2$ and ${}^c D$ is the Caputo fractional differentiation ([17]). We use the basic idea of these boundary value conditions and combine it with the basic idea of [5] for investigating a k -dimensional systems of fractional differential inclusions. Also, we use Let $k \geq 2$ be a natural number. In this paper, we investigate the existence solution for a k -dimensional systems of fractional differential inclusions

$$\begin{cases} {}^c D^{\alpha_1} u_1(t) \in F_1(t, u_1(t), \dots, u_k(t), {}^c D^{\gamma_{11}^1} u_1(t), \dots, {}^c D^{\gamma_{1k}^1} u_k(t), {}^c D^{\gamma_{11}^2} u_1(t), \dots, {}^c D^{\gamma_{1k}^2} u_k(t)), \\ {}^c D^{\alpha_2} u_2(t) \in F_2(t, u_1(t), \dots, u_k(t), {}^c D^{\gamma_{21}^1} u_1(t), \dots, {}^c D^{\gamma_{2k}^1} u_k(t), {}^c D^{\gamma_{21}^2} u_1(t), \dots, {}^c D^{\gamma_{2k}^2} u_k(t)), \\ \cdot \\ \cdot \\ \cdot \\ {}^c D^{\alpha_k} u_k(t) \in F_k(t, u_1(t), \dots, u_k(t), {}^c D^{\gamma_{k1}^1} u_1(t), \dots, {}^c D^{\gamma_{kk}^1} u_k(t), {}^c D^{\gamma_{k1}^2} u_1(t), \dots, {}^c D^{\gamma_{kk}^2} u_k(t)), \end{cases} \quad (1.2)$$

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with the anti-periodic boundary value conditions $t^{\beta_{i1}-1} {}^c D^{\beta_{i1}} u_i(t)|_{t=0} = -t^{\beta_{i1}-1} {}^c D^{\beta_{i1}} u_i(t)|_{t=1}$, $u_i(0) = -u_i(1)$ and $t^{\beta_{i2}-2} {}^c D^{\beta_{i2}} u_i(t)|_{t=0} = -t^{\beta_{i2}-2} {}^c D^{\beta_{i2}} u_i(t)|_{t=1}$ for $i = 1, \dots, k$, where $t \in J$, $2 < \alpha_i \leq 3$, $0 < \gamma_{ij}^1 \leq 1$, $1 < \gamma_{ij}^2 \leq 2$, $0 < \beta_{i1} < 1 < \beta_{i2} < 2$ and $F_i : J \times \mathbb{R}^{3k} \rightarrow 2^{\mathbb{R}}$ is a multifunction for all $1 \leq i, j \leq k$.

2. Preliminaries

Here, we give some needed notions. Let (X, d) be a metric space. It is well known that the Pompeiu-Hausdorff metric $H : 2^X \times 2^X \rightarrow [0, \infty)$ is defined by

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ and 2^X is the set of all nonempty subsets of X ([13]). Then $(CB(X), H)$ is a metric space and $(C(X), H)$ is a generalized metric space, where $CB(X)$ is the set of closed and bounded subsets of X and $C(X)$ is the set of closed subsets of X ([23]). Denote the set of compact and convex subsets of X by $P_{cp,cv}(X)$. Let $T : X \rightarrow 2^X$ be a multifunction. An element $x \in X$ is called a fixed point of T whenever $x \in Tx$ ([21]). A multifunction $T : X \rightarrow C(X)$ is called a contraction whenever there exists $\gamma \in (0, 1)$ such that $H(N(x), N(y)) \leq \gamma d(x, y)$ for all $x, y \in X$ ([21]). A multifunction $T : X \rightarrow 2^X$ is called lower semi-continuous whenever for each open set A of X , the set $T^{-1}(A) := \{x \in X : Tx \cap A \neq \emptyset\}$ is open in X ([21]). We say that T is upper semi-continuous whenever for each open set A of X , the set $\{x \in X : Tx \subset A\}$ is open in X ([21]). Also, $T : X \rightarrow 2^X$ is called compact whenever for each bounded subsets S of X , \overline{TS} is a compact set of X ([21]). A multifunction $T : X \rightarrow 2^X$ is said to be completely continuous whenever for each bounded subset B of X , $T(B)$ is relatively compact ([21]). A multifunction $T : J \rightarrow 2^{\mathbb{R}}$ is said to be measurable whenever the function $t \mapsto d(y, T(t))$ is measurable for all $y \in \mathbb{R}$, where $J = [0, 1]$ ([19]). We say that $F : J \times \mathbb{R}^{3k} \rightarrow 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever the map $t \mapsto F(t, x_1, \dots, x_{3k})$ is measurable for all $x_1, \dots, x_{3k} \in \mathbb{R}$ and the map $(x_1, \dots, x_{3k}) \mapsto F(t, x_1, \dots, x_{3k})$ is upper semi-continuous for almost all $t \in J$ (see [7], [19] and [23]). Also, a Caratheodory multifunction $F : J \times \mathbb{R}^{3k} \rightarrow 2^{\mathbb{R}}$ is called L^1 -Caratheodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x_1, \dots, x_{3k})\| = \sup\{|v| : v \in F(t, x_1, \dots, x_{3k})\} \leq \phi_\rho(t)$$

for all $x_1, \dots, x_{3k} \in \mathbb{R}$ with $|x_1|, \dots, |x_{3k}| \leq \rho$ and almost all $t \in J$ (see [7], [19] and [23]).

Define the space $X_i = \{x : x, {}^c D^{\gamma_{i1}^1} x, {}^c D^{\gamma_{i2}^2} x \in C(J, \mathbb{R})\}$ endowed with the norm

$$\|x\|_i = \sup_{t \in J} |x(t)| + \sup_{t \in J} |{}^c D^{\gamma_{i1}^1} x(t)| + \sup_{t \in J} |{}^c D^{\gamma_{i2}^2} x(t)|$$

for all $i \in \{1, \dots, k\}$. Also, define the product space $X = X_1 \times \dots \times X_k$ endowed with the norm $\|(u_1, \dots, u_k)\| = \sum_{i=1}^k \|u_i\|_i$. Then, $(X, \|\cdot\|)$ is a Banach space ([31]). By using the idea of the papers [4], [6], [27] and [32], we define the set of the selections of F_i at x by

$$S_{F_i, x} = \left\{ v \in L^1[0, 1] : v(t) \in F_i(t, x_1(t), \dots, x_k(t), {}^c D^{\gamma_{i1}^1} x_1(t), \dots, {}^c D^{\gamma_{i1}^1} x_k(t), {}^c D^{\gamma_{i2}^2} x_1(t), \dots, {}^c D^{\gamma_{i2}^2} x_k(t)) \text{ for almost all } t \in J \right\}$$

for all $x = (x_1, \dots, x_k) \in X$ and $1 \leq i \leq k$. We need the following results in the sequel. One can find the Ascoli-Arzelà theorem in [19] and [21].

Lemma 2.1. *Let X be compact metric space and A a closed, equi-continuous and bounded subset of $C(X)$. Then A is relatively compact.*

In 1970, Covitz and Nadler proved next result ([18]).

Lemma 2.2. Let (X, d) be a complete metric space. If $N : X \rightarrow C(X)$ is a contractive multifunction, then N has a fixed point.

Lemma 2.3. ([24]) Let X be a Banach space, $F : J \times X \rightarrow P_{cp,cv}(X)$ an L^1 -Caratheodory multifunction and Θ a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator $\Theta \circ S_F : C(J, X) \rightarrow P_{cp,cv}(C(J), X)$ defined by $(\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.4. ([21]) Let E be a Banach space, C a closed convex subset of E , U an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \rightarrow P_{cp,cv}(C)$ is a upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C . Then either F has a fixed point in \bar{U} or there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.

Lemma 2.5. ([19]) Let X and Y be Banach spaces and $F : X \rightarrow P(Y)$ a completely continuous multifunction via nonempty compact values. Then F is upper semi-continuous if and only if F has a closed graph, that is, $x_n \rightarrow x_0$, $y_n \rightarrow y_0$ and $y_n \in F(x_n)$ for all n imply that $y_0 \in F(x_0)$.

Lemma 2.6. ([17]) Let $v \in C(J, \mathbb{R})$, $\alpha \in (2, 3]$ and $0 < \beta_1 < 1 < \beta_2 < 2$. Then a function $u \in C(J, \mathbb{R})$ is a solution of the fractional boundary value problem ${}^c D^\alpha u(t) = v(t)$ with the boundary value conditions $u(0) = -u(1)$, $t^{\beta_1-1} {}^c D^{\beta_1} u(t)|_{t \rightarrow 0} = -t^{\beta_1-1} {}^c D^{\beta_1} u(t)|_{t=1}$ and $t^{\beta_2-2} {}^c D^{\beta_2} u(t)|_{t \rightarrow 0} = -t^{\beta_2-2} {}^c D^{\beta_2} u(t)|_{t=1}$ if and only if u is a solution of the integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha)} (1-s)^{\alpha-1} + \frac{(1-2t)\Gamma(2-\beta_1)}{4\Gamma(\alpha-\beta_1)} (1-s)^{\alpha-\beta_1-1} + \frac{1-\beta_1+2t-2(2-\beta_1)t^2}{8(2-\beta_1)} \frac{\Gamma(3-\beta_2)}{\Gamma(\alpha-\beta_2)} (1-s)^{\alpha-\beta_2-1} \right] v(s) ds.$$

3. Main Results

Now, we are ready to provide our results about the existence of solution for the k -dimensional system of fractional differential inclusions (1.2). We say that a function $(u_1, u_2, \dots, u_k) \in X$ is a solution for the k -dimensional system of inclusions (1.2) whenever there exists functions v_1, \dots, v_k in $L^1[0, 1]$ such that

$$v_i(t) \in F_i(t, u_1(t), \dots, u_k(t), {}^c D^{\gamma_{i1}} u_1(t), \dots, {}^c D^{\gamma_{ik}} u_k(t), {}^c D^{\gamma_{i1}} u_1(t), \dots, {}^c D^{\gamma_{ik}} u_k(t))$$

for almost all $t \in J$,

$$u_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i(s) ds$$

and $t^{\beta_{1i}-1} {}^c D^{\beta_{1i}} u_i(t)|_{t \rightarrow 0} = -t^{\beta_{1i}-1} {}^c D^{\beta_{1i}} u_i(t)|_{t=1}$, $t^{\beta_{2i}-2} {}^c D^{\beta_{2i}} u_i(t)|_{t \rightarrow 0} = -t^{\beta_{2i}-2} {}^c D^{\beta_{2i}} u_i(t)|_{t=1}$ and $u_i(0) = -u_i(1)$ for all $i = 1, \dots, k$.

3.1. The Caratheodory case

Theorem 3.1. Suppose that $F_1, \dots, F_k : J \times \mathbb{R}^{3k} \rightarrow P_{cp,cv}(\mathbb{R})$ are Caratheodory multifunctions, there exist a non-decreasing, bounded and continuous map $\psi : [0, \infty) \rightarrow (0, \infty)$ and continuous functions $p_1, \dots, p_k : J \rightarrow (0, \infty)$ such that

$$\|F_i(t, u_1(t), \dots, u_k(t), {}^c D^{\gamma_{i1}} u_1(t), \dots, {}^c D^{\gamma_{ik}} u_k(t), {}^c D^{\gamma_{i1}} u_1(t), \dots, {}^c D^{\gamma_{ik}} u_k(t))\| =$$

$$\sup\{|y| : y \in F_i(t, u_1(t), \dots, u_k(t), {}^c D^{\gamma_{i1}^1} u_1(t), \dots, {}^c D^{\gamma_{ik}^1} u_k(t), {}^c D^{\gamma_{i1}^2} u_1(t), \dots, {}^c D^{\gamma_{ik}^2} u_k(t))\} \leq p_i(t)\psi(\|(u_1, \dots, u_k)\|)$$

for all $1 \leq i \leq k$, $(u_1, \dots, u_k) \in \mathcal{X}$ and almost all $t \in J$. Assume that there exist constants $L_i > 0$ such that $\frac{L_i}{(\Lambda_1^i + \Lambda_2^i + \Lambda_3^i)\|p_i\|_\infty \psi(\|(u_1, u_2, \dots, u_k)\|)} > 1$ for all $(u_1, \dots, u_k) \in \mathcal{X}$, where

$$\Lambda_1^i = \frac{3}{\Gamma(\alpha_i + 1)} + \frac{3\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i} + 1)} + \frac{5\Gamma(3 - \beta_{2i})}{4\Gamma(\alpha_i - \beta_{2i} + 1)},$$

$$\Lambda_2^i = \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^1 + 1)} + \frac{\Gamma(2 - \beta_{1i})}{2\Gamma(\alpha_i - \beta_{1i} + 1)\Gamma(2 - \gamma_{ii}^1)} + \frac{\Gamma(3 - \beta_{2i})}{4\Gamma(\alpha_i - \beta_{1i} + 1)\Gamma(2 - \gamma_{ii}^1)(2 - \beta_{1i})} + \frac{\Gamma(3 - \beta_{2i})}{2\Gamma(\alpha_i - \beta_{2i} + 1)\Gamma(3 - \gamma_{ii}^1)},$$

$\Lambda_3^i = \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^2 + 1)} + \frac{\Gamma(3 - \beta_{2i})}{2\Gamma(\alpha_i - \beta_{2i} + 1)\Gamma(3 - \gamma_{ii}^2)}$ and $\|p_i\|_\infty = \sup_{t \in J} |p_i(t)|$ for all $i = 1, \dots, k$. Then the k -dimensional system of inclusions with the boundary value conditions (1.2) has at least one solution.

Proof. Define the operator $N : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ by $N(u_1, \dots, u_k) = \begin{pmatrix} N_1(u_1, \dots, u_k) \\ N_2(u_1, \dots, u_k) \\ \vdots \\ N_k(u_1, \dots, u_k) \end{pmatrix}$, where

$$N_i(u_1, \dots, u_k) = \left\{ h \in \mathcal{X}_i : \text{there exists } v \in S_{F_i(u_1, \dots, u_k)} \text{ such that} \right.$$

$$h(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v(s) ds \Big\}.$$

Now, we show that the operator N has a fixed point. First, we show that $N(u_1, \dots, u_k)$ is convex for all $(u_1, \dots, u_k) \in \mathcal{X}$. Let $(h_1, \dots, h_k), (h'_1, \dots, h'_k) \in N(u_1, \dots, u_k)$. Choose $v_i, v'_i \in S_{F_i(u_1, \dots, u_k)}$ such that

$$h_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i(s) ds$$

and

$$h'_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v'_i(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v'_i(s) ds$$

for all $t \in J$ and $1 \leq i \leq k$. Let $0 \leq w \leq 1$. Then, we have

$$[wh_i + (1-w)h'_i](t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} [wv_i(s) + (1-w)v'_i(s)] ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] [wv_i(s) + (1-w)v'_i(s)] ds$$

$$\frac{\Gamma(3 - \beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})}(1 - s)^{\alpha_i - \beta_{2i} - 1} \Big] [wv_i(s) + (1 - w)v'_i(s)] ds.$$

Since F_i is convex valued for all $1 \leq i \leq k$, $[wh_i + (1 - w)h'_i] \in N_i(u_1, \dots, u_k)$. Thus,

$$\begin{aligned} & w(h_1, \dots, h_k) + (1 - w)(h'_1, \dots, h'_k) \\ &= (wh_1 + (1 - w)h'_1, \dots, wh_k + (1 - w)h'_k) \in N(u_1, \dots, u_k). \end{aligned}$$

Now, we show that N maps bounded sets of \mathcal{X} into bounded sets. Let $r > 0$, $B_r = \{(u_1, \dots, u_k) \in \mathcal{X} : \|(u_1, \dots, u_k)\| \leq r\}$, $(u_1, \dots, u_k) \in B_r$ and $(h_1, \dots, h_k) \in N(u_1, \dots, u_k)$. Choose $(v_1, \dots, v_k) \in S_{F_1(u_1, \dots, u_k)} \times \dots \times S_{F_k(u_1, \dots, u_k)}$ such that

$$\begin{aligned} h_i(t) = & \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} v_i(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1 - s)^{\alpha_i - 1} + \frac{(1 - 2t)\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})} (1 - s)^{\alpha_i - \beta_{1i} - 1} \right. \\ & \left. + \frac{1 - \beta_{1i} + 2t - 2(2 - \beta_{1i})t^2}{8(2 - \beta_{1i})} \frac{\Gamma(3 - \beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})} (1 - s)^{\alpha_i - \beta_{2i} - 1} \right] v_i(s) ds \end{aligned}$$

for all $t \in J$ and $1 \leq i \leq k$. Hence, we get

$$\begin{aligned} {}^c D^{\gamma_{ii}^1} h_i(t) = & \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^1)} \int_0^t (t - s)^{\alpha_i - \gamma_{ii}^1 - 1} v_i(s) ds \\ & - \frac{t^{1 - \gamma_{ii}^1} \Gamma(2 - \beta_{1i})}{2\Gamma(2 - \gamma_{ii}^1) \Gamma(\alpha_i - \beta_{1i})} \int_0^1 (1 - s)^{\alpha_i - \beta_{1i} - 1} v_i(s) ds \\ & + \frac{t^{1 - \gamma_{ii}^1} \Gamma(3 - \beta_{2i})}{4\Gamma(2 - \gamma_{ii}^1) \Gamma(\alpha_i - \beta_{1i})(2 - \beta_{1i})} \int_0^1 (1 - s)^{\alpha_i - \beta_{2i} - 1} v_i(s) ds \\ & - \frac{t^{2 - \gamma_{ii}^1} \Gamma(3 - \beta_{2i})}{2\Gamma(3 - \gamma_{ii}^1) \Gamma(\alpha_i - \beta_{2i})} \int_0^1 (1 - s)^{\alpha_i - \beta_{2i} - 1} v_i(s) ds \end{aligned}$$

and

$$\begin{aligned} {}^c D^{\gamma_{ii}^2} h_i(t) = & \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^2)} \int_0^t (t - s)^{\alpha_i - \gamma_{ii}^2 - 1} v_i(s) ds \\ & - \frac{t^{2 - \gamma_{ii}^2} \Gamma(3 - \beta_{2i})}{2\Gamma(3 - \gamma_{ii}^2) \Gamma(\alpha_i - \beta_{2i})} \int_0^1 (1 - s)^{\alpha_i - \beta_{2i} - 1} v_i(s) ds. \end{aligned}$$

Hence,

$$\begin{aligned} |h_i(t)| \leq & \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i - 1} |v_i(s)| ds + \int_0^1 \left| \left[\frac{-1}{2\Gamma(\alpha_i)} (1 - s)^{\alpha_i - 1} \right. \right. \\ & \left. \left. + \frac{(1 - 2t)\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})} (1 - s)^{\alpha_i - \beta_{1i} - 1} + \frac{1 - \beta_{1i} + 2t - 2(2 - \beta_{1i})t^2}{8(2 - \beta_{1i})} \right. \right. \\ & \left. \left. \frac{\Gamma(3 - \beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})} (1 - s)^{\alpha_i - \beta_{2i} - 1} \right] v_i(s) \right| ds \leq \|p_i\|_{\infty} \psi(\|(u_1, \dots, u_k)\|) \Lambda_1^i, \\ |{}^c D^{\gamma_{ii}^1} h_i(t)| \leq & \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^1)} \int_0^t (t - s)^{\alpha_i - \gamma_{ii}^1 - 1} |v_i(s)| ds \\ & + \frac{t^{1 - \gamma_{ii}^1} \Gamma(2 - \beta_{1i})}{2\Gamma(2 - \gamma_{ii}^1) \Gamma(\alpha_i - \beta_{1i})} \int_0^1 (1 - s)^{\alpha_i - \beta_{1i} - 1} |v_i(s)| ds \end{aligned}$$

$$\begin{aligned} & \frac{t^{1-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{4\Gamma(2-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{1i})(2-\beta_{1i})} \int_0^1 (1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)|ds \\ & + \frac{t^{2-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{2i})} \int_0^1 (1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)|ds \leq \|p_i\|_\infty \psi(\|(u_1, \dots, u_k)\|)\Lambda_2^i \end{aligned}$$

and

$$\begin{aligned} |{}^c D^{\gamma_{ii}^2} h_i(t)| & \leq \frac{1}{\Gamma(\alpha_i-\gamma_{ii}^2)} \int_0^t (t-s)^{\alpha_i-\gamma_{ii}^2-1}|v_i(s)|ds \\ & \frac{t^{2-\gamma_{ii}^2}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^2)\Gamma(\alpha_i-\beta_{2i})} \int_0^1 (1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)|ds \leq \|p_i\|_\infty \psi(\|(u_1, \dots, u_k)\|)\Lambda_3^i \end{aligned}$$

for all $t \in J$ and $1 \leq i \leq k$. Thus,

$$\|h_i\|_i \leq (\Lambda_1^i + \Lambda_2^i + \Lambda_3^i)\|p_i\|_\infty \psi(r)$$

and so $\|(h_1, \dots, h_k)\| = \sum_{i=1}^k \|h_i\|_i \leq \psi(r) \sum_{i=1}^k (\Lambda_1^i + \Lambda_2^i + \Lambda_3^i)\|p_i\|_\infty$. Now, we show that N maps bounded sets to equi-continuous subsets of \mathcal{X} . Let $(u_1, \dots, u_k) \in B_r$, $t_1, t_2 \in J$ with $t_1 < t_2$ and $(h_1, \dots, h_k) \in N(u_1, \dots, u_k)$. Then, we have

$$\begin{aligned} |h_i(t_2) - h_i(t_1)| & = \left| \frac{1}{\Gamma(\alpha_i)} \int_0^{t_2} (t_2-s)^{\alpha_i-1} v_i(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} \right. \right. \\ & \left. \left. + \frac{(1-2t_2)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t_2-2(2-\beta_{1i})t_2^2}{8(2-\beta_{1i})} \right. \right. \\ & \left. \left. \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i(s) ds - \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} (t_1-s)^{\alpha_i-1} v_i(s) ds \right. \\ & \left. + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t_1)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} \right. \right. \\ & \left. \left. + \frac{1-\beta_{1i}+2t_1-2(2-\beta_{1i})t_1^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i(s) ds \right| \\ & \leq \|p_i\|_\infty \psi(\|(u_1, \dots, u_k)\|) \left[\frac{t_2^{\alpha_i} - t_1^{\alpha_i}}{\Gamma(\alpha_i+1)} - \frac{2(t_2-t_1)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i}+1)} \right. \\ & \left. + \frac{2(t_2-t_1)\Gamma(3-\beta^2-i)}{8(2-\beta_{1i})\Gamma(\alpha_i-\beta_{2i}+1)} \right] \\ |{}^c D^{\gamma_{ii}^1} h_i(t_2) - {}^c D^{\gamma_{ii}^1} h_i(t_1)| & \leq \|p_i\|_\infty \psi(\|(u_1, \dots, u_k)\|) \left[\frac{t_2^{\alpha_i-\gamma_{ii}^1} - t_1^{\alpha_i-\gamma_{ii}^1}}{\Gamma(\alpha_i-\gamma_{ii}^1+1)} \right. \\ & \left. + \frac{(t_2^{1-\gamma_{ii}^1} - t_1^{1-\gamma_{ii}^1})\Gamma(2-\beta_{1i})}{2\Gamma(\alpha_i-\beta_{1i}+1)\Gamma(2-\gamma_{ii}^1)} + \frac{(t_2^{1-\gamma_{ii}^1} - t_1^{1-\gamma_{ii}^1})\Gamma(3-\beta_{2i})}{4\Gamma(\alpha_i-\beta_{1i}+1)\Gamma(2-\gamma_{ii}^1)(2-\beta_{1i})} \right. \\ & \left. + \frac{(t_2^{2-\gamma_{ii}^1} - t_1^{2-\gamma_{ii}^1})\Gamma(3-\beta_{2i})}{2\Gamma(\alpha_i-\beta_{2i}+1)\Gamma(3-\gamma_{ii}^1)} \right] \end{aligned}$$

and

$$\begin{aligned} & |{}^c D^{\gamma_{ii}^2} h_i(t_2) - {}^c D^{\gamma_{ii}^2} h_i(t_1)| \\ & \leq \|p_i\|_\infty \psi(\|(u_1, \dots, u_k)\|) \left[\frac{t_2^{\alpha_i-\gamma_{ii}^2} - t_1^{\alpha_i-\gamma_{ii}^2}}{\Gamma(\alpha_i-\gamma_{ii}^2+1)} + \frac{(t_2^{2-\gamma_{ii}^2} - t_1^{2-\gamma_{ii}^2})\Gamma(3-\beta_{2i})}{2\Gamma(\alpha_i-\beta_{2i}+1)\Gamma(3-\gamma_{ii}^2)} \right] \end{aligned}$$

for all $1 \leq i \leq k$. This implies that $\lim_{t_2 \rightarrow t_1} |(h_1(t_2) - h_1(t_1), \dots, h_k(t_2) - h_k(t_1))| = 0$, $\lim_{t_2 \rightarrow t_1} |({}^c D^{\gamma_{ii}^1} h_1(t_2) - {}^c D^{\gamma_{ii}^1} h_1(t_1), \dots, {}^c D^{\gamma_{ii}^1} h_k(t_2) - {}^c D^{\gamma_{ii}^1} h_k(t_1))| = 0$ and

$$\lim_{t_2 \rightarrow t_1} |({}^c D^{\gamma_{ii}^2} h_1(t_2) - {}^c D^{\gamma_{ii}^2} h_1(t_1), \dots, {}^c D^{\gamma_{ii}^2} h_k(t_2) - {}^c D^{\gamma_{ii}^2} h_k(t_1))| = 0.$$

Hence by using the Arzela-Ascoli theorem for each bounded subset B of \mathcal{X} , $N(B)$ is relatively compact. Thus, N is completely continuous. Now, we show that N has a closed graph. Let $(u_1^n, \dots, u_k^n) \in \mathcal{X}$ and $(h_1^n, \dots, h_k^n) \in N(u_1^n, \dots, u_k^n)$ ($n \geq 1$) with $(u_1^n, \dots, u_k^n) \rightarrow (u_1^0, \dots, u_k^0)$ and $(h_1^n, \dots, h_k^n) \rightarrow (h_1^0, \dots, h_k^0)$. We show that $(h_1^0, \dots, h_k^0) \in N(u_1^0, \dots, u_k^0)$. For each natural number n , choose $(v_1^n, \dots, v_k^n) \in S_{F_1, (u_1^n, \dots, u_k^n)} \times \dots \times S_{F_k, (u_1^n, \dots, u_k^n)}$ such that

$$h_i^n(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i^n(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i^n(s) ds$$

for all $t \in J$ and $1 \leq i \leq k$. Define the continuous linear operator $\theta_i : L^1(J, \mathbb{R}) \rightarrow \mathcal{X}_i$ by

$$\theta_i(v)(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v(s) ds.$$

By using Lemma 2.3, $\theta_i \circ S_{F_i}$ is a closed graph operator. Since $h_i^n \in \theta_i(S_{F_i, (u_1^n, \dots, u_k^n)})$ for all n , $1 \leq i \leq k$ and $(u_1^n, \dots, u_k^n) \rightarrow (u_1^0, \dots, u_k^0)$, there exists $v_i^0 \in S_{F_i, (u_1^0, \dots, u_k^0)}$ such that

$$h_i^0(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i^0(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i^0(s) ds.$$

Hence, $h_i^0 \in N_i(u_1^0, \dots, u_k^0)$ for all $1 \leq i \leq k$. This implies that N_i has a closed graph for all $1 \leq i \leq k$ and so N has a closed graph. Now, suppose that there exists $\lambda \in (0, 1)$ such that $(u_1, \dots, u_k) \in \lambda N(u_1, \dots, u_k)$. Then there exists

$$(v_1, \dots, v_k) \in S_{F_1, (u_1, \dots, u_k)} \times \dots \times S_{F_k, (u_1, \dots, u_k)}$$

such that

$$u_i(t) = \frac{\lambda}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i(s) ds + \int_0^1 \lambda \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i(s) ds$$

for all $t \in J$ and $1 \leq i \leq k$. Since $\frac{\|u_i\|}{(\Lambda_1^i + \Lambda_2^i + \Lambda_3^i) \|p_i\|_{\infty} \psi(\|(u_1, u_2, \dots, u_k)\|)} \leq 1$, $\|u_i\| < L_i$ for all $i = 1, \dots, k$. Now, put $U = \{(u_1, \dots, u_k) \in \mathcal{X} : \|(u_1, \dots, u_k)\| < \sum_{i=1}^k L_i + 1\}$. Thus, there are no $(u_1, \dots, u_k) \in \partial U$ and $\lambda \in (0, 1)$ such that $(u_1, \dots, u_k) \in \lambda N(u_1, \dots, u_k)$. Note that, the operator $N : \bar{U} \rightarrow P_{cp,cv}(\bar{U})$ is upper semi-continuous

because it is completely continuous and has closed graph. By considering definition of U , there is no $(u_1, \dots, u_k) \in \partial U$ such that $(u_1, \dots, u_k) \in \lambda N(u_1, \dots, u_k)$ for some $\lambda \in (0, 1)$. Now by using Lemma 2.4, N has a fixed point in \bar{U} . It is easy to check that each fixed point of N is a solution of the k -dimensional system of inclusions (1.2). This completes the proof. \square

Example 3.1. Consider the 2-dimensional system of fractional differential inclusions

$$\begin{cases} {}^c D^{\frac{5}{2}} u_1(t) \in F_1(t, u_1(t), u_2(t), {}^c D^{\frac{1}{2}} u_1(t), {}^c D^{\frac{3}{4}} u_2(t), {}^c D^{\frac{5}{4}} u_1(t), {}^c D^{\frac{3}{2}} u_2(t)), \\ {}^c D^{\frac{5}{3}} u_2(t) \in F_2(t, u_1(t), u_2(t), {}^c D^{\frac{1}{4}} u_1(t), {}^c D^{\frac{3}{4}} u_2(t), {}^c D^{\frac{5}{4}} u_1(t), {}^c D^{\frac{7}{4}} u_2(t)), \end{cases} \quad (3.1)$$

with the boundary value conditions $t^{\frac{1}{2}} {}^c D^{\frac{1}{2}} u_1(t)|_{t \rightarrow 0} = -t^{\frac{1}{2}} {}^c D^{\frac{1}{2}} u_1(t)|_{t=1}$, $u_1(0) = -u_1(1)$, $u_2(0) = -u_2(1)$, $t^{\frac{2}{3}} {}^c D^{\frac{1}{3}} u_2(t)|_{t \rightarrow 0} = -t^{\frac{2}{3}} {}^c D^{\frac{1}{3}} u_2(t)|_{t=1}$, $t^{\frac{3}{4}} {}^c D^{\frac{5}{4}} u_1(t)|_{t \rightarrow 0} = -t^{\frac{3}{4}} {}^c D^{\frac{5}{4}} u_1(t)|_{t=1}$ and $t^{\frac{1}{2}} {}^c D^{\frac{3}{2}} u_2(t)|_{t \rightarrow 0} = -t^{\frac{1}{2}} {}^c D^{\frac{3}{2}} u_2(t)|_{t=1}$. In fact, $k = 2$, $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{8}{3}$, $\gamma_{11}^1 = \frac{1}{2}$, $\gamma_{12}^1 = \frac{1}{3}$, $\gamma_{21}^1 = \frac{5}{4}$, $\gamma_{22}^1 = \frac{3}{2}$, $\gamma_{11}^2 = \frac{1}{4}$, $\gamma_{12}^2 = \frac{3}{4}$, $\gamma_{21}^2 = \frac{5}{4}$, $\gamma_{22}^2 = \frac{7}{4}$, $\beta_{11} = \frac{1}{2}$, $\beta_{12} = \frac{1}{3}$, $\beta_{21} = \frac{5}{4}$, $\beta_{22} = \frac{3}{2}$ and the multifunction $F_1, F_2 : J \times \mathbb{R}^6 \rightarrow 2^{\mathbb{R}}$ are given by

$$F_1(t, x_1, \dots, x_6) = \left[\sin t + \frac{|x_1|}{1 + |x_1|} + \cos x_2, 4 + 3t^2 + \frac{1}{1 + e^{|x_3|}} + \sin x_4 + \frac{|x_6|^3}{|x_6|^3 + 2} \right]$$

and $F_2(t, x_1, \dots, x_6) = \left[0, \frac{e^t}{\pi} \left(\frac{|x_1| + |x_2| + |x_3| + |x_4| + |x_6|}{1 + |x_1| + |x_2| + |x_3| + |x_4| + |x_6|} \right) + \sin x_5 \right]$. Thus,

$$\|F_1(t, x_1, \dots, x_6)\| = \sup\{|y| : y \in F_1(t, x_1, \dots, x_6)\} \leq 10$$

and $\|F_2(t, x_1, \dots, x_6)\| = \sup\{|y| : y \in F_2(t, x_1, \dots, x_6)\} \leq 2$. We show that F_1 and F_2 are Caratheodory multifunctions. In this way, consider the continuous maps

$$f_1(t) = \sin t + \frac{|x_1|}{1 + |x_1|} + \cos x_2,$$

$$g_1(t) = 4 + 3t^2 + \frac{1}{1 + e^{|x_3|}} + \sin x_4 + \frac{|x_6|^3}{|x_6|^3 + 2}, f_2(t) = 0 \text{ and}$$

$$g_2(t) = \frac{e^t}{\pi} \left(\frac{|x_1| + |x_2| + |x_3| + |x_4| + |x_6|}{1 + |x_1| + |x_2| + |x_3| + |x_4| + |x_6|} \right) + \sin x_5$$

for all t . Since the maps f_1, g_1, f_2 and g_2 are continuous, it is easy to check that the maps $t \mapsto d(y, F_1(t, x_1, \dots, x_6))$ and $t \mapsto d(y, F_2(t, x_1, \dots, x_6))$ are continuous and so measurable ones for all $y \in \mathbb{R}$. Also, one can check that $t \mapsto F_1(t, x_1, \dots, x_6)$ and $t \mapsto F_2(t, x_1, \dots, x_6)$ are completely continuous multifunctions via compact values and closed graph for all $t \in J$. Now by using Lemma 2.5, we get $t \mapsto F_1(t, x_1, \dots, x_6)$ and $t \mapsto F_2(t, x_1, \dots, x_6)$ are upper semi-continuous multifunctions for all $x_1, \dots, x_6 \in \mathbb{R}$. Thus, F_1 and F_2 are Caratheodory multifunctions. Now, put $p_1(t) = 5$, $p_2(t) = 1$ and $\psi(t) = 2$. If $L_1 > 55.37$ and $L_2 > 10.046$, then we get $\frac{L_1}{(\Lambda_1^1 + \Lambda_2^1 + \Lambda_3^1) \|p_1\|_{\infty} \psi(\|(u_1, u_2)\|)} > 1$ and $\frac{L_2}{(\Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2) \|p_2\|_{\infty} \psi(\|(u_1, u_2)\|)} > 1$. Thus, the assumptions of Theorem 3.1 hold and so the 2-dimensional system of fractional differential inclusions (3.1) has at least one solution.

3.2. The Lipschitz case

Now, we provide our next result about the existence of solution for the k -dimensional system of inclusions (1.2) by deleting the assumption of convex values for the multifunctions.

Theorem 3.2. Let $m_1, \dots, m_k \in C(J, \mathbb{R}^+)$ be such that $L = \sum_{i=1}^k \|m_i\|_{\infty} (\Lambda_1^i + \Lambda_2^i + \Lambda_3^i) < 1$, where $\Lambda_1^i = \frac{3}{\Gamma(\alpha_i + 1)} + \frac{3\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i} + 1)} + \frac{5\Gamma(3 - \beta_{2i})}{4\Gamma(\alpha_i - \beta_{2i} + 1)}$,

$$\Lambda_2^i = \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^1 + 1)} + \frac{\Gamma(2 - \beta_{1i})}{2\Gamma(\alpha_i - \beta_{1i} + 1)\Gamma(2 - \gamma_{ii}^1)} + \frac{\Gamma(3 - \beta_{2i})}{4\Gamma(\alpha_i - \beta_{2i} + 1)\Gamma(2 - \gamma_{ii}^1)(2 - \beta_{1i})}$$

$$+ \frac{\Gamma(3 - \beta_{2i})}{2\Gamma(\alpha_i - \beta_{2i} + 1)\Gamma(3 - \gamma_{ii}^1)}$$

and $\Lambda_3^i = \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^2 + 1)} + \frac{\Gamma(3 - \beta_{2i})}{2\Gamma(\alpha_i - \beta_{2i} + 1)\Gamma(3 - \gamma_{ii}^2)}$ for $i = 1, \dots, k$. Suppose that $F_i : J \times \mathbb{R}^{3k} \rightarrow P_{cp}(\mathbb{R})$ is a multifunction such that the map $t \mapsto F_i(t, x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k)$ is integrable bounded, measurable and

$$\begin{aligned} &H_d(F_i(t, u_1, \dots, u_k, y_1, \dots, y_k, z_1, \dots, z_k), F_i(t, x'_1, \dots, x'_k, y'_1, \dots, y'_k, z'_1, \dots, z'_k)) \\ &\leq m_i(t) \left(\sum_{i=1}^k |x_i - x'_i| + |y_i - y'_i| + |z_i - z'_i| \right) \end{aligned}$$

for almost all $t \in J, x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k, x'_1, \dots, x'_k, y'_1, \dots, y'_k, z'_1, \dots, z'_k \in \mathbb{R}$ and $i = 1, \dots, k$. Then the k -dimensional system of inclusions with the boundary value conditions (1.2) has a solution.

Proof. Note that, the multifunction

$$t \rightarrow F_i(t, u_1(t), \dots, u_k(t), {}^c D^{\gamma_{ii}^1} u_1(t), \dots, {}^c D^{\gamma_{ik}^1} u_k(t), {}^c D^{\beta_{1i}^2} u_1(t), \dots, {}^c D^{\beta_{1i}^2} u_k(t))$$

is measurable and closed valued for all $(u_1, \dots, u_k) \in \mathcal{X}$ and $i = 1, \dots, k$. Hence, it has measurable selection and so the set $S_{F_i, (u_1, \dots, u_k)}$ is nonempty for all $i = 1, \dots, k$. Again, consider the operator $N : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ defined

by $N(u_1, \dots, u_k) = \left(\begin{array}{c} N_1(u_1, \dots, u_k) \\ N_2(u_1, \dots, u_k) \\ \vdots \\ N_k(u_1, \dots, u_k) \end{array} \right)$, where

$$N_i(u_1, \dots, u_k) = \left\{ h \in \mathcal{X}_i : \text{there exists } v \in S_{F_i, (u_1, \dots, u_k)} \text{ such that} \right.$$

$$\begin{aligned} h(t) = &\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} \right. \\ &\left. + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v(s) ds, \end{aligned}$$

First, we show that $N(u_1, \dots, u_k)$ is a closed subset of \mathcal{X} for all $(u_1, \dots, u_k) \in \mathcal{X}$. Let $\{(u_1^n, \dots, u_k^n)\}_{n \geq 1}$ be a sequence in $N(u_1, \dots, u_k)$ such that $(u_1^n, \dots, u_k^n) \rightarrow (u_1^0, \dots, u_k^0)$. Choose $(v_1^n, \dots, v_k^n) \in S_{F_1, (u_1, \dots, u_k)} \times S_{F_2, (u_1, \dots, u_k)} \times \dots \times S_{F_k, (u_1, \dots, u_k)}$ such that

$$\begin{aligned} u_i^n(t) = &\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i^n(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} \right. \\ &\left. + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \right. \\ &\left. \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i^n(s) ds \end{aligned}$$

for all $t \in J$ and $i = 1, \dots, k$. Since F_i is compact valued for all $i, \{v_i^n\}_{n \geq 1}$ has a subsequence which converges to some $v_i^0 \in L^1(J, \mathbb{R})$. Denote the subsequence again by $\{v_i^n\}_{n \geq 1}$. It is easy to check that $v_i^0 \in S_{F_i, (u_1, \dots, u_k)}$ and

$$\begin{aligned} u_i^n(t) \rightarrow u_i^0(t) = &\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i^0(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} \right. \\ &\left. + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i^0(s) ds \end{aligned}$$

for all $t \in J$. This implies that $u_i^0 \in N_i(u_1, \dots, u_k)$ for all $i = 1, \dots, k$. This implies that $(u_1^0, \dots, u_k^0) \in N(u_1, \dots, u_k)$. Now, we show that N is a contractive multifunction with the constant $L < 1$, where $L = \sum_{i=1}^k \|m_i\|_\infty (\Lambda_1^i + \Lambda_2^i + \Lambda_3^i) < 1$. Let $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathcal{X}$ and $(h_1, \dots, h_k) \in N(y_1, \dots, y_k)$ be given. Then, we can choose

$$(v_1, \dots, v_k) \in S_{F_1, (y_1, \dots, y_k)} \times S_{F_2, (y_1, \dots, y_k)} \times \dots \times S_{F_k, (y_1, \dots, y_k)}$$

such that

$$h_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v_i(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v_i(s) ds$$

for all $t \in J$ and $i = 1, \dots, k$. Since

$$\begin{aligned} &H_d \left(F_i(t, x_1(t), \dots, x_k(t), {}^c D^{\gamma_{i1}} x_1(t), \dots, {}^c D^{\gamma_{ik}} x_k(t), {}^c D^{\gamma_{i1}} x_1(t), \dots, {}^c D^{\gamma_{ik}} x_k(t)), \right. \\ & \left. F_i(t, y_1(t), \dots, y_k(t), {}^c D^{\gamma_{i1}} y_1(t), \dots, {}^c D^{\gamma_{ik}} y_k(t), {}^c D^{\gamma_{i1}} y_1(t), \dots, {}^c D^{\gamma_{ik}} y_k(t)) \right) \\ &\leq m_i(t) \sum_{i=1}^k (|x_i(t) - y_i(t)| + |{}^c D^{\gamma_{i1}} x_i(t) - {}^c D^{\gamma_{i1}} y_i(t)| + |{}^c D^{\gamma_{i2}} x_i(t) - {}^c D^{\gamma_{i2}} y_i(t)|) \end{aligned}$$

for almost all $t \in J$ and $i = 1, \dots, k$, there exists

$$w_i \in F_i(t, x_1(t), \dots, x_k(t), {}^c D^{\gamma_{i1}} x_1(t), \dots, {}^c D^{\gamma_{ik}} x_k(t), {}^c D^{\gamma_{i1}} x_1(t), \dots, {}^c D^{\gamma_{ik}} x_k(t))$$

such that

$$|v_i(t) - w_i| \leq m_i(t) \sum_{i=1}^k (|x_i(t) - y_i(t)| + |{}^c D^{\gamma_{i1}} x_i(t) - {}^c D^{\gamma_{i1}} y_i(t)| + |{}^c D^{\gamma_{i2}} x_i(t) - {}^c D^{\gamma_{i2}} y_i(t)|)$$

for almost all $t \in J$ and $i = 1, \dots, k$. Consider the multifunction $U_i : J \rightarrow 2^{\mathbb{R}}$ by

$$U_i(t) = \{w \in \mathbb{R} : |v_i(t) - w| \leq m_i(t)g(t) \text{ for almost all } t \in J\},$$

where $g(t) = \sum_{i=1}^k (|x_i(t) - y_i(t)| + |{}^c D^{\gamma_{i1}} x_i(t) - {}^c D^{\gamma_{i1}} y_i(t)| + |{}^c D^{\gamma_{i2}} x_i(t) - {}^c D^{\gamma_{i2}} y_i(t)|)$. Since v_i and $\varphi_i = m_i \sum_{i=1}^k (|x_i - y_i| + |{}^c D^{\gamma_{i1}} x_i - {}^c D^{\gamma_{i1}} y_i| + |{}^c D^{\gamma_{i2}} x_i - {}^c D^{\gamma_{i2}} y_i|)$ are measurable for all i , $U_i(\cdot) \cap F_i(t, x_1(\cdot), \dots, x_k(\cdot), {}^c D^{\gamma_{i1}} x_1(\cdot), \dots, {}^c D^{\gamma_{ik}} x_k(\cdot), {}^c D^{\gamma_{i1}} x_1(\cdot), \dots, {}^c D^{\gamma_{ik}} x_k(\cdot))$ is a measurable multifunction. Thus, we can choose

$$v'_i(t) \in F_i(t, x_1(t), \dots, x_k(t), {}^c D^{\gamma_{i1}} x_1(t), \dots, {}^c D^{\gamma_{ik}} x_k(t), {}^c D^{\gamma_{i1}} x_1(t), \dots, {}^c D^{\gamma_{ik}} x_k(t))$$

such that

$$h'_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} v'_i(s) ds + \int_0^1 \left[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right] v'_i(s) ds$$

for all $t \in J$ and $i = 1, \dots, k$. Since

$$|h_i(t) - h'_i(t)| \leq \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} |v_i(s) - v'_i(s)| ds$$

$$\begin{aligned}
 & + \int_0^1 \left\| \frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i-1} \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})} (1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \right. \\
 & \left. \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})} (1-s)^{\alpha_i-\beta_{2i}-1} \right\| |v_i(s) - v'_i(s)| ds \leq \Lambda_1^i \|m_i\|_\infty \| (x_1 - y_1, \dots, x_k - y_k) \|, \\
 & |{}^c D^{\gamma_{ii}^1} h_i(t) - {}^c D^{\gamma_{ii}^1} h'_i(t)| \leq \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^1)} \int_0^t (t-s)^{\alpha_i - \gamma_{ii}^1 - 1} |v_i(s) - v'_i(s)| ds \\
 & \quad + \frac{t^{1-\gamma_{ii}^1} \Gamma(2-\beta_{1i})}{2\Gamma(2-\gamma_{ii}^1) \Gamma(\alpha_i - \beta_{1i})} \int_0^1 (1-s)^{\alpha_i - \beta_{1i} - 1} |v_i(s) - v'_i(s)| ds \\
 & \quad + \frac{t^{1-\gamma_{ii}^1} \Gamma(3-\beta_{2i})}{4\Gamma(2-\gamma_{ii}^1) \Gamma(\alpha_i - \beta_{1i})(2-\beta_{1i})} \int_0^1 (1-s)^{\alpha_i - \beta_{2i} - 1} |v_i(s) - v'_i(s)| ds \\
 & \quad + \frac{t^{2-\gamma_{ii}^1} \Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^1) \Gamma(\alpha_i - \beta_{2i})} \int_0^1 (1-s)^{\alpha_i - \beta_{2i} - 1} |v_i(s) - v'_i(s)| ds \\
 & \leq \|m_i\|_\infty \| (x_1 - y_1, \dots, x_k - y_k) \| \Lambda_2^i
 \end{aligned}$$

and

$$\begin{aligned}
 & |{}^c D^{\gamma_{ii}^2} h_i(t) - {}^c D^{\gamma_{ii}^2} h'_i(t)| \leq \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^2)} \int_0^t (t-s)^{\alpha_i - \gamma_{ii}^2 - 1} |v_i(s) - v'_i(s)| ds \\
 & \quad + \frac{t^{2-\gamma_{ii}^2} \Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^2) \Gamma(\alpha_i - \beta_{2i})} \int_0^1 (1-s)^{\alpha_i - \beta_{2i} - 1} |v_i(s) - v'_i(s)| ds \\
 & \leq \|m_i\|_\infty \| (x_1 - y_1, \dots, x_k - y_k) \| \Lambda_3^i,
 \end{aligned}$$

we get $\|h_i - h'_i\|_i \leq (\Lambda_1^i + \Lambda_2^i + \Lambda_3^i) \|m_i\|_\infty \| (x_1 - y_1, \dots, x_k - y_k) \|$ for all $i = 1, \dots, k$. Hence,

$$\begin{aligned}
 & \| (h_1, \dots, h_k) - (h'_1, \dots, h'_k) \| = \sum_{i=1}^k \|h_i - h'_i\|_i \\
 & \leq \sum_{i=1}^k (\Lambda_1^i + \Lambda_2^i + \Lambda_3^i) \|m_i\|_\infty \| (x_1 - y_1, \dots, x_k - y_k) \| \\
 & \leq L \| (x_1, \dots, x_k) - (y_1, \dots, y_k) \|.
 \end{aligned}$$

This implies that $H(N(x_1, \dots, x_k), N(y_1, \dots, y_k)) \leq L \| (x_1, \dots, x_k) - (y_1, \dots, y_k) \|$ and so N is a closed valued contractive multifunction. By using Lemma 2.6, N has a fixed point. One can check that each fixed point of N is a solution for the k -dimensional system of inclusions with the boundary value conditions (1.2). \square

Example 3.2. Consider the 2-dimensional system of fractional differential inclusions

$$\begin{cases}
 {}^c D^{\frac{7}{3}} u_1(t) \in F_1(t, u_1(t), u_2(t), {}^c D^{\frac{1}{3}} u_1(t), {}^c D^{\frac{1}{2}} u_2(t), {}^c D^{\frac{5}{3}} u_1(t), {}^c D^{\frac{5}{4}} u_2(t)), \\
 {}^c D^{\frac{13}{6}} u_2(t) \in F_2(t, u_1(t), u_2(t), {}^c D^{\frac{1}{3}} u_1(t), {}^c D^{\frac{6}{7}} u_2(t), {}^c D^{\frac{7}{6}} u_1(t), {}^c D^{\frac{7}{4}} u_2(t)),
 \end{cases} \tag{3.2}$$

with the boundary conditions $t^{\frac{1}{2}} {}^c D^{\frac{1}{2}} u_1(t)|_{t \rightarrow 0} = -t^{\frac{1}{2}} {}^c D^{\frac{1}{2}} u_1(t)|_{t=1}$, $u_1(0) = -u_1(1)$, $u_2(0) = -u_2(1)$, $t^{\frac{3}{4}} {}^c D^{\frac{1}{4}} u_2(t)|_{t \rightarrow 0} = -t^{\frac{3}{4}} {}^c D^{\frac{1}{4}} u_2(t)|_{t=1}$, $t^{\frac{1}{2}} {}^c D^{\frac{3}{2}} u_1(t)|_{t \rightarrow 0} = -t^{\frac{1}{2}} {}^c D^{\frac{3}{2}} u_1(t)|_{t=1}$ and

$$t^{\frac{3}{5}} {}^c D^{\frac{7}{5}} u_2(t)|_{t \rightarrow 0} = -t^{\frac{3}{5}} {}^c D^{\frac{7}{5}} u_2(t)|_{t=1}.$$

In fact, $k = 2$, $\alpha_1 = \frac{7}{3}$, $\alpha_2 = \frac{13}{6}$, $\gamma_{11}^1 = \frac{1}{3}$, $\gamma_{12}^1 = \frac{1}{2}$, $\gamma_{11}^2 = \frac{5}{3}$, $\gamma_{12}^2 = \frac{5}{4}$, $\gamma_{21}^1 = \frac{1}{3}$, $\gamma_{22}^1 = \frac{6}{7}$, $\gamma_{21}^2 = \frac{7}{6}$, $\gamma_{22}^2 = \frac{7}{4}$, $\beta_{11} = \frac{1}{2}$, $\beta_{12} = \frac{1}{4}$, $\beta_{21} = \frac{3}{2}$, $\beta_{22} = \frac{7}{5}$ and the multifunctions $F_1, F_2 : J \times \mathbb{R}^6 \rightarrow 2^{\mathbb{R}}$ are given by $F_1(t, x_1, \dots, x_6) = \left[0, \frac{t \sin x_1}{9\pi(2+t^2)} + \frac{t|x_2|}{75(1+|x_2|)} + \frac{|x_3|}{80(1+|x_3|)} + \frac{t \cos x_4}{10e^2} + \frac{t|x_5|}{70(1+|x_5|)} + \frac{t^2|x_6|}{75(1+|x_6|)}\right]$ and

$$F_2(t, x_1, \dots, x_6) = \left[0, \frac{\cos t(|x_1| + |x_2| + |x_3| + |x_4|)}{25\sqrt{\pi}(1 + |x_1| + |x_2| + |x_3| + |x_4|)} + \frac{|x_5|}{45(1 + |x_5|)} + \frac{t^2 \sin x_6}{30(t^2 + 1)}\right].$$

Since $\|F_1(t, x_1, \dots, x_6)\| = \sup\{|y| : y \in F_1(t, x_1, \dots, x_6)\} \leq \frac{t}{9\pi(2+t^2)} + \frac{t}{75} + \frac{1}{80} + \frac{t}{10e^2} + \frac{t}{70} + \frac{t^2}{75}$ and $\|F_2(t, x_1, \dots, x_6)\| = \sup\{|y| : y \in F_2(t, x_1, \dots, x_6)\} \leq \frac{\cos t}{25\sqrt{\pi}} + \frac{1}{45} + \frac{t^2}{30(t^2+1)}$ for all $x_1, \dots, x_6 \in \mathbb{R}$ and $t \in J$, we get $t \mapsto F_1(t, x_1, \dots, x_6)$ and $t \mapsto F_2(t, x_1, \dots, x_6)$ are integrable bounded multifunctions for all $x_1, \dots, x_5 \in \mathbb{R}$. By using a similar argument in Example 3.1, we can show that $t \mapsto F_1(t, x_1, \dots, x_6)$ and $t \mapsto F_2(t, x_1, \dots, x_6)$ are measurable. On the other hand, it is easy to see that

$$H(F_1(t, x_1, \dots, x_6), F_1(t, y_1, \dots, y_6)) \leq \left(\frac{t}{9\pi(2+t^2)} + \frac{t}{75} + \frac{1}{80} + \frac{t}{10e^2} + \frac{t}{70} + \frac{t^2}{75}\right) \sum_{i=1}^6 |x_i - y_i|$$

and $H(F_2(t, x_1, \dots, x_6), F_2(t, y_1, \dots, y_6)) \leq \left(\frac{\cos t}{25\sqrt{\pi}} + \frac{1}{45} + \frac{t^2}{30(t^2+1)}\right) \sum_{i=1}^6 |x_i - y_i|$ for all $t \in J$ and $x_1, \dots, x_6, y_1, \dots, y_6 \in \mathbb{R}$.

Now, put

$$m_1(t) = \frac{t}{9\pi(2+t^2)} + \frac{t}{75} + \frac{1}{80} + \frac{t}{10e^2} + \frac{t}{70} + \frac{t^2}{75}$$

and $m_2(t) = \frac{\cos t}{25\sqrt{\pi}} + \frac{1}{45} + \frac{t^2}{30(t^2+1)}$ for all $t \in J$. Then, we have

$$H(F_1(t, x_1, \dots, x_6), F_1(t, y_1, \dots, y_6)) \leq m_1(t) \sum_{i=1}^6 |x_i - y_i|$$

and $H(F_2(t, x_1, \dots, x_6), F_2(t, y_1, \dots, y_6)) \leq m_2(t) \sum_{i=1}^6 |x_i - y_i|$ for all $x_1, \dots, x_6, y_1, \dots, y_6 \in \mathbb{R}$ and $t \in J$. On the other hand, we have

$$L = \sum_{i=1}^2 \|m_i\|_{\infty} (\Lambda_1^i + \Lambda_2^i + \Lambda_3^i)$$

$$\leq (0.09 \times (2.654 + 1.74 + 1.63)) + (0.06 \times (1.962 + 1.15 + 1.46)) \approx 0.8178 < 1.$$

Thus, the assumptions of Theorem 3.2 hold and so the 2-dimensional system (3.2) has at least one solution.

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