Filomat 30:6 (2016), 1559–1565
DOI 10.2298/FIL1606559X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Distance Spectrum of Trees

Jie Xue[a], Ruifang Liu[a], Huicai Jia[b]

[a] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China*
[b] *College of Science, Henan Institute of Engineering, Zhengzhou, Henan 451191, China*

Abstract. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. $D(G) = (d_{ij})_{n\times n}$ is the distance matrix of G, where d_{ij} denotes the distance between v_i and v_j. Let $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$ be the distance spectrum of G. A graph G is said to be determined by its distance spectrum if any graph having the same distance spectrum as G is isomorphic to G. Trees can not be determined by its distance spectrum. Naturally, we prove that two kinds of special trees path P_n and double star $S(a, b)$ are determined by their distance spectra in this paper.

1. Introduction

All graphs in this paper are undirected, simple and connected. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $N_G(v)$ denote the neighbor set of v in G. The distance between vertices u and v of a graph G is denoted by d_{uv}. The diameter of G, denoted by d or $d(G)$, is the maximum distance between any pair of vertices of G. Let X be a subset of $V(G)$. The induced subgraph $G[X]$ is the subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. The complete product $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

The distance matrix $D(G) = (d_{ij})_{n\times n}$ of a connected graph G is the matrix indexed by the vertices of G, where d_{ij} denotes the distance between the vertices v_i and v_j. Let $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$ be the spectrum of $D(G)$, that is, the distance spectrum of G. The polynomial $P_D(\lambda) = det|\lambda I - D(G)|$ is defined as the distance characteristic polynomial of a graph G. A graph G is said to be determined by its distance spectrum if there is no other nonisomorphic graph with the same distance spectrum as G.

Spectral characterization problem was proposed by Dam and Haemers in [3]. In their paper, Dam and Haemers investigated the cospectrality of graphs up to order 11. They showed that the adjacency matrix appears to be the worst representation in terms of producing a large number of cospectral graphs. The Laplacian is superior in this regard and the signless Laplacian even better. Subsequently, Dam et al. [4, 5] wrote two excellent surveys on this topic.

So far, only a few families of graphs were shown to be determined by their spectra, and most of these results focused on adjacency, Laplacian or signless Laplacian spectra. Especially, there are much fewer results on which graphs are determined by their distance spectra. In [7], Lin et al. proved that the complete

2010 *Mathematics Subject Classification*. Primary 05C50
Keywords. Distance spectrum; Distance characteristic polynomial; Distance spectrum determined
Received: 06 May 2014; Accepted: 22 February 2015
Communicated by Francesco Belardo
Supported by NSFC (No. 11201432) and NSF-Henan (Nos. 15A110003 and 15IRTSTHN006).
Email addresses: 7000734@qq.com (Jie Xue), rfliu@zzu.edu.cn (Ruifang Liu), jhc607@yahoo.com.cn (Huicai Jia)

bipartite graph K_{n_1,n_2} and the complete split graph $K_a \vee K_b^c$ are determined by their distance spectra, and conjectured that the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is determined by its distance spectrum. Recently, Jin and Zhang [6] have confirmed the conjecture.

In fact, trees can not be determined by its distance spectrum. McKay [8] constructed the smallest distance cospectral trees on 17 vertices. Using *Nauty* (a computer program for generating graphs available at http://cs.anu.edu.au/~bdm/nauty/), Aouchiche and Hansen [1] constructed the distance cospectral mates with at most 20 vertices. Thus a question naturally arises: can some special trees be determined by their distance spectra? In this paper, we show that two kinds of special trees path and double star are determined by their distance spectra.

The double star $S(a, b)$ (see Fig. 1) is the graph consisting of the union of two stars $K_{1,a}$ and $K_{1,b}$ together with an edge joining their centers, where $a \geq 1, b \geq 1$ and $a + b = n - 2$.

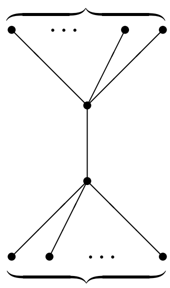

Fig. 1. The double star $S(a, b)$.

2. Preliminaries

For the proof of the main theorem, we first give some useful lemmas and results. The following lemma is well-known Cauchy Interlace Theorem.

Lemma 2.1. **([2])** *Let A be a Hermitian matrix of order n with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$, and B be a principal submatrix of A of order m with eigenvalues $\mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_m(B)$. Then $\lambda_{n-m+i}(A) \leq \mu_i(B) \leq \lambda_i(A)$ for $i = 1, 2, \ldots, m$.*

Applying Lemma 2.1 to the distance matrix of a graph, we have

Lemma 2.2. *Let G be a graph of order n with distance spectrum $\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \cdots \geq \lambda_n(D(G))$, and H be an induced subgraph of G on m vertices with the distance spectrum $\mu_1(D(H)) \geq \mu_2(D(H)) \geq \cdots \geq \mu_m(D(H))$. Moreover, if $D(H)$ is a principal submatrix of $D(G)$, $\lambda_{n-m+i}(D(G)) \leq \mu_i(D(H)) \leq \lambda_i(D(G))$ for $i = 1, 2, \ldots, m$.*

Lemma 2.3. *Let $G = S(a, b)$ be a double star. Then the distance characteristic polynomial of G is*

$$P_D(\lambda) = (\lambda + 2)^{n-4}[\lambda^4 - (2a + 2b - 4)\lambda^3 - (9a + 9b + 5ab - 3)\lambda^2 - (12a + 12b + 4ab + 4)\lambda - (4a + 4b + 4)].$$

Proof. Let J be the all-one matrix. Clearly, the distance matrix of G is

$$D(G) = \begin{pmatrix} 0 & 1 & J_{1\times a} & 2J_{1\times b} \\ 1 & 0 & 2J_{1\times a} & J_{1\times b} \\ J_{a\times 1} & 2J_{a\times 1} & 2J_{a\times a} - 2I & 3J_{a\times b} \\ 2J_{b\times 1} & J_{b\times 1} & 3J_{b\times a} & 2J_{b\times b} - 2I \end{pmatrix}_{n\times n}.$$

Then

$$det(\lambda I - D(G)) = \begin{vmatrix} \lambda & -1 & -J_{1\times a} & -2J_{1\times b} \\ -1 & \lambda & -2J_{1\times a} & -J_{1\times b} \\ -J_{a\times 1} & -2J_{a\times 1} & (\lambda+2)I-2J_{a\times a} & -3J_{a\times b} \\ -2J_{b\times 1} & -J_{b\times 1} & -3J_{b\times a} & (\lambda+2)I-2J_{b\times b} \end{vmatrix}$$

$$= (\lambda+2)^{a+b-2} \begin{vmatrix} \lambda & -1 & -a & -2b \\ -1 & \lambda & -2a & -b \\ -1 & -2 & \lambda+2-2a & -3b \\ -2 & -1 & -3a & \lambda+2-2b \end{vmatrix}$$

$$= (\lambda+2)^{n-4}[\lambda^4 - (2a+2b-4)\lambda^3 - (9a+9b+5ab-3)\lambda^2 - (12a+12b+4ab+4)\lambda - (4a+4b+4)].$$

□

Let $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$ be the distance spectrum of a graph G. Note that $\sum_{i=1}^{n} \lambda_i(D) = 0$. Consider $\sum_{i=1}^{n} {\lambda_i}^2(D)$, we have the following result.

Lemma 2.4. *Let G be a connected graph on n vertices with distance matrix $D(G) = (d_{ij})_{n\times n}$. Then $\sum_{i=1}^{n} \lambda_i^2(D) = \sum_{i,j\in\{1,2,\ldots,n\};i\neq j} d_{ij}^2$.*

Proof. Obviously, $\lambda_1^2(D), \lambda_2^2(D), \ldots, \lambda_n^2(D)$ are the eigenvalues of $D^2(G)$. Let $D^2(G) = (d_{ij}^{\star})$. Then

$$d_{ii}^{\star} = d_{i1}^2 + d_{i2}^2 + \cdots + d_{in}^2 = \sum_{j=1}^{n} d_{ij}^2.$$

Since $\sum_{i=1}^{n} \lambda_i^2(D) = \sum_{i=1}^{n} d_{ii}^{\star}$, then

$$\sum_{i=1}^{n} \lambda_i^2(D) = \sum_{i=1}^{n} d_{ii}^{\star} = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 = \sum_{i,j\in\{1,2,\ldots,n\};i\neq j} d_{ij}^2.$$

This completes the proof. □

Corollary 2.5. *Let G be a graph with order n and $d(G) = 2$. If G' has the same distance spectrum as G, then*
•$|E(G)| = |E(G')|$ when $d(G') = 2$;
•$|E(G)| < |E(G')|$ when $d(G') \geq 3$.

Proof. Suppose that G and G' have the same distance spectra denoted by $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$. Let $D(G) = (d_{ij})_{n\times n}$ and $D(G') = (d'_{ij})_{n\times n}$ be the distance matrices of G and G', respectively. Let $|E(G)| = m$ and $|E(G')| = m'$. By Lemma 2.4, we have

$$\sum_{i=1}^{n} \lambda_i^2(D) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 = 2[m + (\frac{n(n-1)}{2} - m) \times 4] = 4n(n-1) - 6m$$

and

$$\sum_{i=1}^{n} \lambda_i^2(D) = \sum_{i=1}^{n}\sum_{j=1}^{n} {d'_{ij}}^2 \geq 2[m' + (\frac{n(n-1)}{2} - m') \times 4] = 4n(n-1) - 6m'.$$

If $d(G') = 2$, the latter formula is $\sum_{i=1}^{n} \lambda_i^2(D) = 4n(n-1) - 6m'$, and then $m = m'$. If $d(G') \geq 3$, the latter formula is $\sum_{i=1}^{n} \lambda_i^2(D) > 4n(n-1) - 6m'$, and then $4n(n-1) - 6m > 4n(n-1) - 6m'$, that is $m < m'$. □

3. P_n and $S(a, b)$ are Determined by Their Distance Spectra

First, We will prove that the path P_n is determined by its distance spectrum.

Lemma 3.1. *Let G be a connected graph with order $n \geq 3$ and $\phi(G) = \sum_{i=1}^{n} \lambda_i^2(D)$. Then $\phi(G) \leq \phi(P_n)$ and the equality holds if and only if $G \cong P_n$.*

Proof. If G is a tree. We adopt the induction on n. By Lemma 2.4, it is obviously true for $n = 3$. For $n \geq 4$, let u be a pendant vertex of G, and suppose that $\phi(G - u) \leq \phi(P_{n-1})$. Consider the case n,

$$\phi(G) = \phi(G - u) + 2 \sum_{v \in V(G-u)} d_{uv}^2 \leq \phi(G - u) + 2 \sum_{i=1}^{n-1} i^2 \leq \phi(P_{n-1}) + 2 \sum_{i=1}^{n-1} i^2 = \phi(P_n).$$

Then $\phi(G) \leq \phi(P_n)$. The equality holds if and only if $\sum_{v \in V(G-u)} d_{uv}^2 = \sum_{i=1}^{n-1} i^2$ and $\phi(G - u) = \phi(P_{n-1})$, that is $G \cong P_n$.

If G is not a tree. Then there exists an edge e such that $G - e$ is also connected, and it is easy to check that $\phi(G) < \phi(G - e)$. Repeating this step, we get a spanning tree T of G with $\phi(G) < \phi(T)$. According to the above case, we have $\phi(G) < \phi(T) \leq \phi(P_n)$. □

Lemma 3.1 implies that $\phi(P_n)$ is maximum, hence we obtain the following result directly.

Theorem 3.2. *P_n is determined by its distance spectrum.*

Next we will show that double star $S(a, b)$ is determined by its distance spectrum.

Let $S(a, b)$ be a double star where $a \geq 1$ and $b \geq 1$. If $a = b = 1$, then $S(a, b) = P_4$. Clearly, by Theorem 3.2, it is determined by its distance spectrum.

Next let $c = \max\{a, b\}$ and $c \geq 2$. Obviously, $D(S(1, 2))$ is a principal submatrix of $D(S(a, b))$. Using Lemma 2.2, one can obtain the distance spectrum distribution of $S(a, b)$. By a simple calculation, the distance spectrum of $S(1, 2)$ is as follows:

$\lambda_1(D)$	$\lambda_2(D)$	$\lambda_3(D)$	$\lambda_4(D)$	$\lambda_5(D)$
7.4593	-0.5120	-1.0846	-2.0000	-3.8627

.

Then we have

$$\begin{cases} \lambda_1(D(S(a, b))) \geq \lambda_1(D(S(1, 2))) = 7.4593, \\ \lambda_2(D(S(a, b))) \geq \lambda_2(D(S(1, 2))) = -0.5120, \\ \lambda_3(D(S(a, b))) \geq \lambda_3(D(S(1, 2))) = -1.0846, \\ \lambda_4(D(S(a, b))) \geq \lambda_4(D(S(1, 2))) = -2, \\ \lambda_n(D(S(a, b))) \leq \lambda_5(D(S(1, 2))) = -3.8627. \end{cases}$$

Similarly, $D(S(a, b))$ is a principal submatrix of $D(S(c, c))$. Suppose that $S(c, c)$ has n' vertices. By Lemma 2.3, the distance characteristic polynomial of $S(c, c)$ is as follows:

$$\begin{aligned} P_{D(S(c,c))}(\lambda) &= (\lambda + 2)^{n'-4}[\lambda^4 - (4c - 4)\lambda^3 - (18c + 5c^2 - 3)\lambda^2 - (24c + 4c^2 + 4)\lambda - (8c + 4)] \\ &= (\lambda + 2)^{n'-4}[\lambda^2 + (c + 3)\lambda + 2][\lambda^2 - (5c - 1)\lambda - (4c + 2)]. \end{aligned}$$

Considering the equation

$$f(\lambda) = [\lambda^2 + (c + 3)\lambda + 2][\lambda^2 - (5c - 1)\lambda - (4c + 2)]$$

and solving it, we have

• $\lambda_1(D) = \frac{5c-1+\sqrt{25c^2+6c+9}}{2} > 0$;
• $\lambda_2(D) = \frac{-c-3+\sqrt{c^2+6c+1}}{2} < 0$, it is easy to check that $\lambda_2(D)$ is an increasing function on c, $\lambda_2(D)|_{c=2} = -0.4384$, and $\lim_{c\to+\infty} \frac{-c-3+\sqrt{c^2+6c+1}}{2} = 0^-$;
• $\lambda_3(D) = \frac{5c-1-\sqrt{25c^2+6c+9}}{2}$, it is also an increasing function on c, $\lambda_3(D)|_{c=2} = -1$, and $\lim_{c\to+\infty} \frac{5c-1-\sqrt{25c^2+6c+9}}{2} = -0.8^-$;
• $\lambda_4(D) = -2$.
By Lemma 2.2,

$$\begin{cases} \lambda_2(D(S(a,b))) \le \lambda_2(D(S(c,c))) < 0, \\ \lambda_3(D(S(a,b))) \le \lambda_3(D(S(c,c))) < -0.8, \\ \lambda_4(D(S(a,b))) \le \lambda_4(D(S(c,c))) = -2. \end{cases}$$

Thus the distance spectrum of $S(a,b)$ ($\max\{a,b\} \ge 2$) is as follows:

$\lambda_1(D)$	$\lambda_2(D)$	$\lambda_3(D)$	$\lambda_4(D)$	$\cdots$	$\lambda_{n-1}(D)$	$\lambda_n(D)$
[7,4593, $+\infty$)	[-0.5120, 0)	[-1.0846, -0.8)	-2	$\cdots$	-2	($-\infty$, -3.8627]

Lemma 3.3. *Let $D'_{m\times m}$ be a principal submatrix of $D(S(a,b))$, then*
• $\lambda_2(D') < 0$ *and* $\lambda_3(D') < -0.8$;
• $\lambda_4(D') = -2$ *when* $m = 5$.

Proof. By Lemma 2.2 and the distance spectrum distribution of $S(a,b)$, then $\lambda_2(D') \le \lambda_2(D(S(a,b))) < 0$ and $\lambda_3(D') \le \lambda_3(D(S(a,b))) < -0.8$. If $m = 5$, then $-2 = \lambda_{n-1}(D(S(a,b))) \le \lambda_4(D') \le \lambda_4(D(S(a,b)) = -2$, hence $\lambda_4(D') = -2$. □

We call H a forbidden subgraph of a graph G if G contains no H as an induced subgraph.

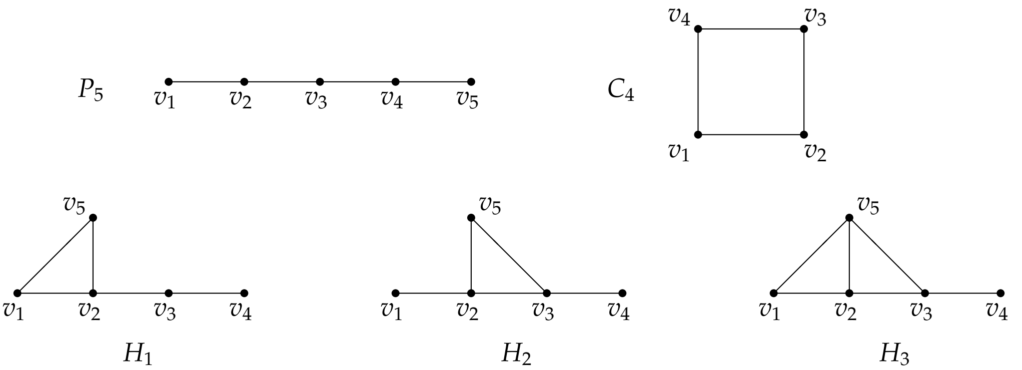

Fig. 2. Graphs P_5, C_4, H_1, H_2 and H_3.

Lemma 3.4. *If G and $S(a,b)$ have the same distance spectrum, then C_4, P_5, H_1, H_2 and H_3 are forbidden subgraphs of G.*

Proof. We prove this by contradiction. For $S \subseteq V(G)$, we denote by $D_G(S)$ the principal submatrix of $D(G)$ induced by S.

Consider C_4. Suppose that C_4 is an induced subgraph of G, then $D_G[\{v_1v_2v_3v_4\}] = D(C_4)$. By Lemma 2.2, we have $\lambda_2(D(G)) \ge \lambda_2(D(C_4)) = 0$, this contradicts $\lambda_2(D(G)) < 0$. Hence C_4 is a forbidden subgraph of G.

Consider P_5. Suppose that P_5 is an induced subgraph of G, then

$$D_G[\{v_1v_2v_3v_4v_5\}] = \begin{pmatrix} 0 & 1 & 2 & a & b \\ 1 & 0 & 1 & 2 & c \\ 2 & 1 & 0 & 1 & 2 \\ a & 2 & 1 & 0 & 1 \\ b & c & 2 & 1 & 0 \end{pmatrix}.$$

Note that $a, c \in \{2, 3\}$ and $b \in \{2, 3, 4\}$. Moreover, if $b = 4$, then $a = c = 3$. By a simple calculation, we have

(a,b,c)	(3,4,3)	(3,3,3)	(2,3,3)	(3,2,3)	(3,3,2)	(3,2,2)	(2,3,2)	(2,2,3)	(2,2,2)
$\lambda_4(D)$	-1.7304	-2.1467	-1.8864	-2.6300	-1.8864	-2.2442	-1.7557	-2.2442	-2.1388

By Lemma 3.3 ($m = 5$), we get a contradiction. So P_5 is a forbidden subgraph of G.

By a similar analysis,

$$D_G[V(H_1)] = \begin{pmatrix} 0 & 1 & 2 & a & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ a & 2 & 1 & 0 & b \\ 1 & 1 & 2 & b & 0 \end{pmatrix}$$

(a,b)	(3,3)	(2,3)	(3,2)	(2,2)
$\lambda_4(D)$	-1.1774	-1.5650	-1.5650	-1.6007

$$D_G[V(H_2)] = \begin{pmatrix} 0 & 1 & 2 & a & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ a & 2 & 1 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

a	2	3
$\lambda_4(D)$	-2.2212	-1.8010

$$D_G[V(H_3)] = \begin{pmatrix} 0 & 1 & 2 & a & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ a & 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

a	2	3
$\lambda_4(D)$	-1.6403	-1.1971

Assume that H_1, H_2 and H_3 are induced subgraphs of G, respectively. By Lemma 3.3, we also get a contradiction. Thus H_1, H_2 and H_3 are also forbidden subgraphs of G. This completes the proof. □

Theorem 3.5. *Double star $S(a, b)$ is determined by its distance spectrum.*

Proof. Let G be a connected graph with the same distance spectrum as $S(a, b)$. According to Lemma 3.4, P_5 is a forbidden subgraph of G, then $d(G) \le 3$. Clearly, $d(G) \neq 1$. Suppose that $d(G) = 2$. By Corollary 2.5, then $|E(G)| < |E(S(a, b))|$, this contradicts the connectivity of G. Hence $d(G) = 3$.

Obviously, there exist two vertices $u, v \in V(G)$ such that $d_{uv} = 3$. Suppose that $P = uu'v'v$ is the path with length 3 in G. Let $X = \{u, u', v', v\}$, then $G[X] = P_4$. Denote by V_i ($i = 0, 1, 2, 3, 4$) the vertex subset of $V \backslash X$, whose each vertex is adjacent to i vertices of X. Clearly $V \backslash X = \cup_{i=0}^{4} V_i$.

Claim 1. $V_0 = \emptyset$.

Suppose not, then there exists a vertex $w \in V_0$ such that the distance between w and the vertices in X is 2 or 3. Thus

$$D_G[\{uu'v'vw\}] = \begin{pmatrix} 0 & 1 & 2 & 3 & a \\ 1 & 0 & 1 & 2 & b \\ 2 & 1 & 0 & 1 & c \\ 3 & 2 & 1 & 0 & d \\ a & b & c & d & 0 \end{pmatrix}.$$

Note that $D_G[\{uu'v'vw\}]$ is a principal submatrix of $D(G)$ and $a, b, c, d \in \{2, 3\}$. By a calculation, we have

(a,b,c,d)	(2,2,2,2)	(3,2,2,2)	(2,3,2,2)	(2,2,3,2)	(2,2,2,3)	(3,3,2,2)	(3,2,3,2)	(3,2,2,3)
$\lambda_4(D)$	-2.3956	-2.3810	-3.0586	-3.0586	-2.3810	-2.6028	-3.1163	-3.1436
(a,b,c,d)	(2,3,3,2)	(2,3,2,3)	(2,2,3,3)	(2,3,3,3)	(3,2,3,3)	(3,3,2,3)	(3,3,3,2)	(3,3,3,3)
$\lambda_4(D)$	-3.4142	-3.1163	-2.6028	-3.1014	-3.2798	-3.2798	-3.1014	-3.4142

By Lemma 3.3, we get a contradiction. Therefore Claim 1 holds.

Claim 2. Each vertex in V_1 is adjacent to u' or v'. Moreover $G[V_1]$ is empty.

Certainly, $w \in V_1$ is adjacent to one of u, u', v' and v. Since P_5 is a forbidden subgraph of G, w is only adjacent to u' or v'. In fact, if there exist two vertices $w, w' \in V_1$ such that $ww' \in E(G)$. Then $G[u'v'ww'] = C_4$, $G[ww'u'v'v] = H_1$ or $G[uu'v'ww'] = H_1$, we also get contradictions. Thus Claim 2 holds.

Claim 3. $V_2 = \emptyset$.

Suppose not, then there exists a vertex $w \in V_2$ is adjacent to two vertices in X. If $wu, wv \in E(G)$, a contradiction, since $d_{uv} = 3$. If w is adjacent to u and v' (or u' and v), $G[wuu'v'] = C_4$ (or $G[wu'v'v] = C_4$), a contradiction. If w is adjacent to u and u' (or v' and v), $G[wuu'v'v] = H_1$, a contradiction. If w is adjacent to u' and v', $G[wuu'v'v] = H_2$, we also get a contradiction. Thus Claim 3 holds.

Claim 4. $V_3 = \emptyset$.

Suppose that $w \in V_3$. Since $d_{uv} = 3$, w is adjacent to u, u' and v' (or u', v' and v), then $G[wuu'v'v] = H_3$. This contradicts that H_3 is a forbidden subgraph of G. Therefore $V_3 = \emptyset$.

Claim 5. $V_4 = \emptyset$.

Suppose not. Let $w \in V_4$, then $wu, wv \in E(G)$, this contradicts $d_{uv} = 3$. Thus Claim 5 holds.

By Claims 1-5, G is a double star. Without loss of generality, we may assume that $G = S(a', b')$. Since $G = S(a', b')$ and $S(a, b)$ have the same distance spectrum, $|V(S(a', b'))| = |V(S(a, b))|$, that is $a' + b' = a + b = n - 2$. By Lemma 2.3,

$$P_{D(S(a,b))}(\lambda) = (\lambda + 2)^{n-4}[\lambda^4 - (2n - 8)\lambda^3 - (9n - 21 + 5ab)\lambda^2 - (12n - 20 + 4ab)\lambda - (4n - 4)],$$

and

$$P_{D(S(a',b'))}(\lambda) = (\lambda + 2)^{n-4}[\lambda^4 - (2n - 8)\lambda^3 - (9n - 21 + 5a'b')\lambda^2 - (12n - 20 + 4a'b')\lambda - (4n - 4)].$$

Note that they have the same distance characteristic polynomial, then

$$\begin{cases} a' + b' = a + b, \\ a'b' = ab. \end{cases}$$

Solving these two equations, we get that $a' = a, b' = b$ or $a' = b, b' = a$. Therefore $G = S(a', b') \cong S(a, b)$. This completes the proof. □

Acknowledgment

The authors would like to thank the anonymous referees very much for valuable suggestions and corrections which improve the original manuscript.

References

[1] M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, Linear Algebra Appl. 439 (2013), 21–33.
[2] D. M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, third edition, Johann Abrosius Barth Verlag, 1995.
[3] E. R. van Dam and W. H. Haemers, Which graphs are determined by their spectra, Linear Algebra Appl. 373 (2003), 241–272.
[4] E. R. van Dam, W. H. Haemers and J. H. Koolen, Cospectral graphs and the generalized adjacency matrix, Linear Algebra Appl. 423 (2007), 33–41.
[5] E. R. van Dam and W. H. Haemers, Developments on spectral characterizations of graphs, Discrete Math. 309 (2009), 576–586.
[6] Y.-L. Jin and X.-D. Zhang, Complete multipartite graphs are determined by their distance spectra, Linear Algebra Appl. 448 (2014), 285–291.
[7] H. Q. Lin, Y. Hong, J. F. Wang and J. L. Shu, On the distance spectrum of graphs, Linear Algebra Appl. 439 (2013), 1662–1669.
[8] B. D. McKay, On the spectral characterisation of trees, Ars Combin. 3 (1977), 219–232.
[9] R. P. Stanley. Enumerative combinatorics, Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.
[10] X. L. Zhang, H. P. Zhang, Some graphs determined by their spectra, Linear Algebra Appl. 431 (2009), 1443–1454.