Filomat 30:6 (2016), 1493–1496
DOI 10.2298/FIL1606493Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Arens Product and Approximate Identity in Locally Convex Algebras

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Abstract. Let $\mathcal{A}'$ and $\mathcal{A}''$ be the dual and bidual spaces of a locally convex algebra $\mathcal{A}$ with dual and weak* topology, respectively. In this paper, we show that $\mathcal{A}$ has a bounded right (left) approximate identity if and only if $\mathcal{A}''$ has a right (left) unit with respect to the first (second) Arens product.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. It is well known that on the second dual space $\mathcal{A}''$ of $\mathcal{A}$, there are two multiplications, called the first and second Arens products, which make $\mathcal{A}''$ into a Banach algebra [1]. In [3], Civin and Yood proved that the Banach algebra $\mathcal{A}$ has a weak right identity if and only if $\mathcal{A}''$ has a right unit with respect to the first Arens product. In the other word an element $E \in \mathcal{A}''$ is a right unit for $\mathcal{A}''$ if and only if it is a weak* cluster point of some bounded right approximate identity $(e_\alpha)_{\alpha\in I}$ in $\mathcal{A}$, [2]. In this paper, as a main theorem we extend this result for locally convex algebras and we obtain some related results.

2. Definitions and Notations

Throughout this paper we will assume that $\mathcal{A}$ is a locally convex algebra with hypo-continuous multiplication. We say that the product $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is left (right) hypo-continuous if for each neighborhood U of 0 and for each bounded set B of $\mathcal{A}$ there exists a second neighborhood V of 0 such that $VB \subset U$ $(BV \subset U)$. The multiplication in $\mathcal{A}$ is said to be hypo-continuous if it is both left and right hypo-continuous.

The dual $\mathcal{A}'$ of $\mathcal{A}$, is the space of all continuous, complex valued linear maps on $\mathcal{A}$. The dual topology (resp. weak* topology) on $\mathcal{A}'$ is the topology of uniform convergence on the bounded sets (resp. finite point sets) of $\mathcal{A}$. It is clear that if $\mathcal{A}$ is normable, then the dual topology on $\mathcal{A}'$ is the norm topology. In this paper we consider $\mathcal{A}'$ with dual topology, where $\mathcal{A}'$ with this topology is certainly a locally convex topological vector space. The bidual of $\mathcal{A}$ is the dual of $\mathcal{A}'$ which is denoted by $\mathcal{A}''$. The bidual topology on $\mathcal{A}''$ is the topology of uniform convergence on the bounded sets of $\mathcal{A}'$. The second topology on $\mathcal{A}''$ is the weak* topology (the uniform convergence topology on finite point sets of $\mathcal{A}'$).

Let π denotes the canonical embedding of $\mathcal{A}$ into $\mathcal{A}''$. Then for all $a \in \mathcal{A}$, $\pi(a)$ is linear and continuous for the weak* topology on $\mathcal{A}'$, and hence for the stronger dual topology on $\mathcal{A}'$. Therefore $\pi(a)$ is in $\mathcal{A}''$.

2010 *Mathematics Subject Classification*. Primary 46H05.
Keywords. Arens regular, bounded approximate identity, weakly quasi-complete, topological centre.
Received: 27 April 2014; Accepted: 08 August 2014
Communicated by Dragan S. Djordjević
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Also π is an algebra homomorphism and $\pi(\mathcal{A})$ is weak* dense in $\mathcal{A}''$ [5]. For each $a, b \in \mathcal{A}$, $f \in \mathcal{A}'$ and $\Phi \in \mathcal{A}''$, the elements $f \cdot a$, $a \cdot f$, $\Phi \cdot f$ and $f \cdot \Phi$ of $\mathcal{A}'$ are defined as follows:

$$(f \cdot a)b = f(ab), \quad (a \cdot f)b = f(ba).$$

$$(\Phi \cdot f)a = \Phi(f \cdot a), \quad (f \cdot \Phi)a = \Phi(a \cdot f).$$

The first and second Arens products of $\Phi, \Psi \in \mathcal{A}''$, which we denote by $\square$ and $\diamond$ respectively, are defined by the formula [5],

$$(\Phi \square \Psi)f = \Phi(\Psi \cdot f), \quad (\Phi \diamond \Psi)f = \Psi(f \cdot \Phi).$$

The locally convex algebra $\mathcal{A}$ is said to be Arens regular if the products $\square$ and $\diamond$ coincide on $\mathcal{A}''$. The bilinear mapping $(\Phi, \Psi) \longrightarrow \Phi \square \Psi$ is a separately continuous with bidual topology on $\mathcal{A}''$, therefore $(\mathcal{A}'', \square)$ with bidual topology is an associative locally convex topological algebra [5].
Also for any fixed $\Phi \in \mathcal{A}''$, the map $\Psi \longmapsto \Psi \square \Phi$ is weak*-weak* continuous on $\mathcal{A}''$, but in general, the map $\Psi \longmapsto \Phi \square \Psi$ is not weak*-weak* continuous on $\mathcal{A}''$. We define the first topological centre $Z_t^1(\mathcal{A}'')$ of $\mathcal{A}''$ by

$$Z_t^1(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \longmapsto \Phi \square \Psi \text{ is } w^* - w^* \text{ continuous on } \mathcal{A}''\}.$$

It is easy to check that

$$Z_t^1(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Phi \square \Psi = \Phi \diamond \Psi \ (\Psi \in \mathcal{A}'')\}.$$

The algebra $\mathcal{A}$ is called left strongly Arens irregular if $Z_t^1(\mathcal{A}'') = \mathcal{A}$, [4]. For more information about the Arens product and topological centres, we refer the reader to Memoire [4]. An element E of $\mathcal{A}''$ is said to be a mixed unit if E is a right unit for the first Arens product and a left unit for the second Arens product, i.e, for each Φ in $\mathcal{A}''$, $\Phi \square E = E \diamond \Phi = \Phi$. A bounded net $(e_\alpha)_{\alpha \in I}$ in $\mathcal{A}$ is a bounded left approximate identity (BLAI for short) if, for each $a \in \mathcal{A}$, $e_\alpha a \longrightarrow a$. Bounded right approximate identity (BRAI) and bounded approximate identity (BAI) can be defined similarly.
The quasi-product of elements a and b in $\mathcal{A}$ is the element aob of $\mathcal{A}$ defined by $aob = a + b - ab$.

The proof of the following result contained in [7].

Theorem 2.1. *Let G be an infinite locally compact group. Then $L^1(G)$ is not Arens regular.*

Throughout the paper we identify an element of $\mathcal{A}$ with its canonical image in $\mathcal{A}''$.

3. First Arens Product and Right Approximate Identity

For the proof of the main theorem we need the following result which generalized proposition 2, § 11 of [2].

Proposition 3.1. *Let B be a bounded subset of $\mathcal{A}$ such that for each $a \in \mathcal{A}$ and for every neighborhood U of 0 there exists $b \in B$ such that $a - ab \in U$. Then $\mathcal{A}$ has a BAI.*

Proof. We first show that for every neighborhood U of 0 and for each finite subset F of $\mathcal{A}$ there exist $w \in BoB = \{bob' : b, b' \in B\}$ such that

$$x - xw \in U \qquad (x \in F).$$

Let U be a neighborhood of 0. Choose the balanced neighborhoods V and W of 0 such that

$$V \subseteq W, \ V^2 \subseteq W \quad \text{and} \qquad W + W \subseteq U.$$

Since B is bounded, so we can choose $\lambda > 1$ such that $B \subset \lambda V$. Given $F = \{x_1, x_2\}$, then there exist $b, b' \in B$ such that

$$(x_1 - x_1 b) \in \lambda^{-1} V \quad \text{and} \qquad (x_2 - x_2 b) - (x_2 - x_2 b)b' \in U.$$

Put $w = bob'$, then we have $x_i - x_i w \in U \quad (i = 1, 2)$. Assume that the result has been established for sets of n elements. Let $F = \{x_1, ..., x_{n+1}\}$, and U be a neighborhood of 0. For $\lambda > 1$ suppose that $\{x_1, ...x_n\}B \subseteq \lambda V$. By assumption there exists $y \in BoB$ such that $(x_i - x_i y) \in \lambda^{-1}V$ for $i = 1, ..., n$. Hence for $H = \{y, x_{n+1}\}$, there exists $w \in BoB$ such that

$$(y - yw) \in \lambda^{-1}V \quad and \quad (x_{n+1} - x_{n+1}w) \in \lambda^{-1}V.$$

Then for $i = 1, ..., n$ we have

$$\begin{aligned} x_i - x_i w &= (x_i - x_i y) + (x_i y - x_i yw) + (x_i yw - x_i w) \\ &= (x_i - x_i y) + x_i(y - yw) - (x_i y - x_i)w \\ &\in (\lambda^{-1}V) + (\lambda V)(\lambda^{-1}V) + (\lambda^{-1}V)(\lambda V) \\ &\subseteq W + W + W \subseteq U. \end{aligned}$$

Now let I denotes the set of all pairs (U, F), where $U \subset \mathcal{A}$ is a neighborhood of 0 and $F \subset \mathcal{A}$ is a finite set. Define an order on I by

$$(U_1, F_1) \leq (U_2, F_2) \iff U_1 \supset U_2 \quad and \quad F_1 \subset F_2.$$

Then I is a directed set. For each $\alpha = (U, F) \in I$ there exists $e_\alpha \in B$ such that $a - ae_\alpha \in U$ for all $a \in F$. Therefore $(e_\alpha)_{\alpha\in I}$ is a BRAI for $\mathcal{A}$. Similarly, $\mathcal{A}$ has a BLAI and so has a BAI, as required.

Now we can prove the main result.

Theorem 3.2. *$\mathcal{A}$ has a BRAI if and only if the topological algebra $(\mathcal{A}'', \Box)$ has a right unit.*

Proof. Assume that $\mathcal{A}$ have a BRAI $(e_\alpha)_{\alpha\in I}$ and let

$$\Gamma = \{\pi(e_\alpha) : \alpha \in I\}.$$

Then Γ is a equicontinuous family on $\mathcal{A}'$, so we may suppose, by passing to a subnet, that $\pi(e_\alpha)$ is weak* convergent to $E \in \mathcal{A}''$. Then for all $a \in \mathcal{A}$, $f \in \mathcal{A}'$ we have

$$\pi(e_\alpha)(f \cdot a) = (f \cdot a)e_\alpha = f(ae_\alpha) \longrightarrow f(a),$$

and so $(E \cdot f)a = E(f \cdot a) = f(a)$. Hence for each $\Phi \in \mathcal{A}''$ and $f \in \mathcal{A}'$,

$$(\Phi\Box E)f = \Phi(E \cdot f) = \Phi(f).$$

Therefore $\Phi\Box E = \Phi$ and so E is a right unit for $(\mathcal{A}'', \Box)$.

Conversely assume that $\mathcal{A}''$ has a right unit, namely E. Since $\pi(\mathcal{A})$ is weak* dense in $\mathcal{A}''$, so there exists net $(x_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ such that $\pi(x_\alpha) \longrightarrow E$ in weak* topology of $\mathcal{A}''$. Hence $\{\pi(x_\alpha) : \alpha \in I\}$ is bounded subset in $\mathcal{A}''$. It follows that $(x_\alpha)_{\alpha\in I}$ is weakly bounded and therefore is a bounded net in $\mathcal{A}$ by Theorem 3.18 of [6]. Suppose that B is the convex hull of (x_α), then B is a bounded subset in $\mathcal{A}$ and for all $a \in \mathcal{A}$, the weak closure and original closure of aB is equal by Theorem 3.12 of [6]. Let $a \in \mathcal{A}$ and $f \in \mathcal{A}'$, then

$$\begin{aligned} f(ax_\alpha) &= (f \cdot a)x_\alpha = \pi(x_\alpha)(f \cdot a) \longrightarrow E(f \cdot a) \\ &= \pi(a)(E \cdot f) = (\pi(a)\Box E)f = \pi(a)(f) = f(a). \end{aligned}$$

Hence $ax_\alpha \longrightarrow a$ in the weak topology, and so

$$a \in \overline{\{(ax_\alpha) : \alpha \in I\}}^w \subseteq \overline{(aB)}^w = \overline{(aB)}.$$

Thus for every neighborhood U of 0 there exist $b \in B$ such that $a - ab \in U$. Now the result follows from above proposition.

One can verify that the left case of theorem 3.2 is also valid, i.e., $\mathcal{A}$ has a BLAI if and only if $(\mathcal{A}'', \diamond)$ has a left unit.

As an consequence of this theorem we have the following results.

Corollary 3.3. *Let $\mathcal{A}$ be an Arens regular. Then $\mathcal{A}''$ has a unit element if and only if $\mathcal{A}$ has a BAI.*

Corollary 3.4. *Let $(\mathcal{A}'', \Box)$ has a unit element and $\pi(\mathcal{A})$ is an ideal in $\mathcal{A}''$. Then $\mathcal{A}$ is Arens regular.*

Proof. Assume that $\Phi, \Psi, \Lambda \in \mathcal{A}''$. Then there exist net $(x_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ such that $\pi(x_\alpha) \longrightarrow \Lambda$ in the weak* topology of $\mathcal{A}''$. Since $\pi(\mathcal{A})$ is an ideal of $\mathcal{A}''$, we have

$$\begin{aligned}\pi(x_\alpha)\Box(\Phi\Box\Psi) &= (\pi(x_\alpha)\Box\Phi)\Box\Psi \\ &= (\pi(x_\alpha)\Box\Phi)\Diamond\Psi \\ &= (\pi(x_\alpha)\Diamond\Phi)\Diamond\Psi \\ &= \pi(x_\alpha)\Box(\Phi\Diamond\Psi).\end{aligned}$$

Therefore for all Λ in $\mathcal{A}''$, $\Lambda\Box(\Phi\Box\Psi) = \Lambda\Box(\Phi\Diamond\Psi)$ by the right weak* continuity of the first Arens product. Take $\Lambda = E$, where E is the unit element of $(\mathcal{A}'', \Box)$, thus we have $\Phi\Box\Psi = \Phi\Diamond\Psi$, as desired.

Example 3.5. *Let G be a non-discrete compact group and $\mathcal{A} = L^1(G)$ as a group algebra. Then $(\mathcal{A}'', \Box)$ does not have a unit element, in otherwise, since $\pi(\mathcal{A})$ is an ideal in $\mathcal{A}''$, so by the preceding corollary $\mathcal{A}$ is Arens regular, which is contradiction by theorem* 2.1.

We recall that the locally convex algebra $\mathcal{A}$ is called weakly quasi-complete, if every weakly cauchy net in $\mathcal{A}$ is weakly convergent.

Proposition 3.6. *Let $\mathcal{A}$ be weakly quasi-complete with a BRAI $(e_\alpha)_{\alpha\in I}$ and let $\Phi \in Z_t^1(\mathcal{A}'')$ satisfy $\Phi\mathcal{A} \subseteq \mathcal{A}$. Then $\mathcal{A}$ is left strongly Arens irregular.*

Proof. Let $(e_\alpha)_{\alpha\in I}$ be a BRAI for $\mathcal{A}$. By theorem 3.2 each weak* cluster point E of $\pi(e_\alpha)$ is a right identity for $\mathcal{A}''$. Since $\pi(e_\alpha) \longrightarrow E$ in the weak* topology of $\mathcal{A}''$, $\Phi\Box\pi(e_\alpha) \longrightarrow \Phi\Box E = \Phi$ for all $\Phi \in Z_t^1(\mathcal{A}'')$. Hence $\Phi\Box\pi(e_\alpha)$ is weakly cauchy and so convergent in $\mathcal{A}$. It follows that $\Phi \in \mathcal{A}$.

Corollary 3.7. *Let $\mathcal{A}$ be weakly quasi-complete with a BRAI $(e_\alpha)_{\alpha\in I}$, and let for each f in $\mathcal{A}'$, the net $(f \cdot e_\alpha)_{\alpha\in I}$ converges weakly to f. Then $(\mathcal{A}'', \Box)$ is unital and the unit element of $(\mathcal{A}'', \Box)$ is in $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ has a BRAI $(e_\alpha)_{\alpha\in I}$. Then $(\mathcal{A}'', \Box)$ has a right unit, say E. Therefore for each $f \in \mathcal{A}'$, we have

$$\begin{aligned}(E\Box\Phi)(f \cdot e_\alpha) &= \pi(e_\alpha)(E\Box\Phi)f \\ &= (\pi(e_\alpha)\Box\Phi)f = (\Phi \cdot f)e_\alpha = \Phi(f \cdot e_\alpha) \longrightarrow \Phi(f).\end{aligned}$$

Since $(f \cdot e_\alpha)_{\alpha\in I}$ tend to f in the weak topology of $\mathcal{A}'$, so we have

$$(E\Box\Phi)f = \Phi(f), \quad (f \in \mathcal{A}').$$

Therefore E is a unit element of $(\mathcal{A}'', \Box)$. The rest of result follows from proposition 3.6.

Acknowledgments

The author would like to thank the referee for giving useful suggestion to improve this work.

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