Filomat 30:6 (2016), 1441–1456
DOI 10.2298/FIL1606441Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Improvement of Ostrowski Integral Type Inequalities with Application

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Abstract. The aim of this paper is to establish new inequalities which are more generalized than the inequalities of Dragomir, Wang and Cerone. The current article also obtains bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval. A variety of earlier results are recaptured as special cases of the inequalities obtained. Some new perturbed results and application for cumulative distribution function are also discussed.

1. Introduction

In the last few decades, the field of mathematical inequalities has proved to be an extensively applicable field. Integral inequalities play an important role in several branches of mathematics and statistics with reference to its applications. The elementary inequalities are proved to be helpful in the development of many other branches of mathematics. Ostrowski [9] proved his famous inequality in 1938 which, because of its applications in numerical analysis, attracted a lot of researchers in the past few years [3]-[7] . For recent results and generalizations concerning Ostrowski's inequality see [13]-[16].

The first generalization of Ostrowski's inequality was given by Milovanović and Pecarić in [8]. Further generalizations of Ostrowski's inequality were given by Qayyum and Hussain in [17] and Qayyum et.al in [18]. Cheng gave a sharp version of the inequality derived in [2]. Cerone [1], and Dragomir and Wang [3]-[7] generalized the Ostrowski's inequality for L_1, L_p and L_∞ norms. In this work, we define a new mapping which help refine the results of Cerone [1], and Dragomir and Wang [3]-[7] and also provide new results with wide ranging applications. We also derive some perturbed results by using Grüss and Čebyŝev inequalities.

The obtained inequalities are of supreme importance because they have immediate applications in numerical integration, probability theory, information theory and integral operator theory etc. In the last, we will apply our inequalities to cumulative distribution functions.

2010 *Mathematics Subject Classification*. Primary 26D15, 26D20; Secondary 65Xxx
Keywords. Ostrowski inequality, Numerical integration, Grüss and Čebyŝev inequalities
Received: 15 April 2014; Accepted: 09 September 2014
Communicated by Dragan S. Djordjević
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2. Preliminaries

Let the functional $S(f;a,b)$ represent the deviation of $f(x)$ from its integral mean over $[a,b]$ and be defined by

$$S(f;a,b) = f(x) - M(f;a,b) \tag{1}$$

and

$$M(f;a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2}$$

Ostrowski [9] proved the following interesting integral inequality:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\left\|f'\right\|_\infty = \sup_{t\in[a,b]}\left|f'(t)\right| < \infty$ then*

$$\left|S(f;a,b)\right| \le \left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\frac{M}{b-a}, \tag{3}$$

for all $x\in[a,b]$.

In this paper, we will use the usual L_p norms defined for a function k as follows:

$$\|k\|_\infty := ess\sup_{t\in[a,b]}|k(t)|$$

and

$$\|k\|_p := \left(\int_a^b |k(t)|^p\,dt\right)^{\frac{1}{p}}, 1\le p<\infty.$$

Dragomir and Wang [3]-[6] proved (3) and other variants for $f'\in L_p[a,b]$ for $p\ge 1$ and the Lebesgue norms making use of a peano kernel approach and Montgomery's identity [12].

Montgomery's identity states that for an absolutely continuous mappings $f:[a,b]\to\mathbb{R}$

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b P(x,t)f'(t)dt, \tag{4}$$

where the kernel p: $[a,b]^2\to\mathbb{R}$ is given by

$$P(x,t) = \begin{cases} t-a & \text{if } a\le t\le x\le b \\ t-b & \text{if } a\le x<t\le b. \end{cases}$$

If we assume that $f'\in L_\infty[a,b]$ and $\left\|f'\right\|_\infty = \underset{t\in[a,b]}{ess}\left|f'(t)\right|$ then M in (3) may be replaced by $\left\|f'\right\|_\infty$.

Dragomir and Wang [3]-[6] obtained the following inequality by using $P(x,t)$ and an integration by parts argument.

$$\left|S(f;a,b)\right| \tag{5}$$

$$\le \begin{cases} \frac{1}{b-a}\left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\left\|f'\right\|_\infty, f'\in L_\infty[a,b] \\ \frac{1}{b-a}\left[\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f'\right\|_p, f'\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1 \\ \frac{1}{b-a}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left\|f'\right\|_1, f'\in L_1[a,b], \end{cases}$$

where $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$.

Cerone [1], proved the following inequality:

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping. Define*

$$\tau(x; \alpha, \beta) := f(x) - \frac{1}{\alpha + \beta}\left[\alpha M(f; a, x) + \beta M(f; x, b)\right] \tag{6}$$

then

$$\left|\tau(x; \alpha, \beta)\right| \tag{7}$$
$$\leq \begin{cases} \frac{1}{2(\alpha+\beta)}\left[\alpha(x-a) + \beta(b-x)\right]\left\|f'\right\|_\infty, f' \in L_\infty[a, b] \\ \\ \frac{1}{(\alpha+\beta)(q+1)^{\frac{1}{q}}}\left[\alpha^q(x-a) + \beta^q(b-x)\right]^{\frac{1}{q}}\left\|f'\right\|_p \\ , f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \frac{1}{2}\left(1 + \frac{|\alpha-\beta|}{\alpha+\beta}\right)\left\|f'\right\|_1, f' \in L_1[a, b]. \end{cases}$$

Qayyum et. al [13] also proved Ostrowski's type integral inequalities.

Lemma 2.3. *Denote by $P(x, .) : [a, b] \to \mathbb{R}$ the kernel is given by*

$$P(x, t) := \begin{cases} \frac{\alpha}{2(\alpha+\beta)(x-a)}(t-a)^2, \ a \leq t \leq x \\ \\ \frac{\beta}{2(\alpha+\beta)(b-x)}(t-b)^2, \ x < t \leq b. \end{cases} \tag{8}$$

Then,

$$\left|\tau(x; \alpha, \beta)\right| \tag{9}$$
$$= \frac{1}{2(\alpha+\beta)}\left[\alpha(x-a) - \beta(b-x)\right]f'(x) - f(x)$$
$$+\frac{1}{\alpha+\beta}\left[\alpha M(f; a, x) + \beta M(f; x, b)\right]$$
$$\leq \begin{cases} \left[\alpha(x-a)^2 + \beta(b-x)^2\right]\frac{\|f''\|_\infty}{6(\alpha+\beta)}, f'' \in L_\infty[a, b] \\ \\ \frac{1}{(2q+1)^{\frac{1}{q}}}\left[\alpha^q(x-a)^{q+1} + \beta^q(b-x)^{q+1}\right]^{\frac{1}{q}}\frac{\|f''\|_p}{2(\alpha+\beta)}, \\ \\ f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \left(\alpha(x-a) + \beta(b-x) + \left|\alpha(x-a) - \beta(b-x)\right|\right)\frac{\|f''\|_1}{4(\alpha+\beta)} \\ , f'' \in L_1[a, b]. \end{cases}$$

Motivated by the result of Cerone [1] and Dragomir [3]-[6], we will present new inequalities which will be the extended and generalized form of Cerone [1] and Dragomir and Wang [3]-[6].

3. Main Results

Before stating the main result, we need to establish the following lemma.

Lemma 3.1. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous mapping. Let* $P(x,.) : [a,b] \to \mathbb{R}$, *the peano type kernel is given by*

$$P(x,t) = \begin{cases} \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\left[t-\left(a+h\frac{b-a}{2}\right)\right], \; a \le t \le x \\ \frac{\beta}{\alpha+\beta}\frac{1}{b-x}\left[t-\left(b-h\frac{b-a}{2}\right)\right], \; x < t \le b, \end{cases} \tag{10}$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ *and* $h \in [0,1]$, *where* $\alpha, \beta \in \mathbb{R}$ *are non negative and not both zero, then the identity:*

$$\begin{aligned}\int_a^b P(x,t)f'(t)dt &= \frac{1}{\alpha+\beta}\left[\begin{array}{c} \frac{\alpha}{x-a}\left\{x-\left(a+h\frac{b-a}{2}\right)\right\} \\ -\frac{\beta}{b-x}\left\{x-\left(b-h\frac{b-a}{2}\right)\right\} \end{array}\right] f(x) \\ &+\frac{h}{\alpha+\beta}\left(\frac{b-a}{2}\right)\left(\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right) \\ &-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)\,dt+\frac{\beta}{b-x}\int_x^b f(t)\,dt\right]\end{aligned} \tag{11}$$

holds.

Proof. From (10), we have

$$\begin{aligned}\int_a^b P(x,t)f'(t)dt &= \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\int_a^x \left(t-\left(a+h\frac{b-a}{2}\right)\right)f'(t)dt \\ &+\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\int_x^b \left(t-\left(b-h\frac{b-a}{2}\right)\right)f'(t)dt.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}&\int_a^b P(x,t)f'(t)dt \\ &= \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)f(x)+h\frac{b-a}{2}f(a)-\int_a^x f(t)\,dt\right] \\ &+\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\left[h\frac{b-a}{2}f(b)-\left(x-\left(b-h\frac{b-a}{2}\right)\right)f(x)-\int_x^b f(t)\,dt\right].\end{aligned}$$

Combining like terms and using algebraic manipulation, we get the required identity given in (11). □

We now state and prove our main result.

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous mapping. Using (11), we define*

$$\tau(x;\alpha,\beta) = \int_a^b P(x,t) f'(t)dt \tag{12}$$
$$:= \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\left\{x-\left(a+h\frac{b-a}{2}\right)\right\} - \frac{\beta}{b-x}\left\{x-\left(b-h\frac{b-a}{2}\right)\right\}\right] f(x)$$
$$+\frac{h}{\alpha+\beta}\left(\frac{b-a}{2}\right)\left(\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right)$$
$$-\frac{1}{\alpha+\beta}\left[\alpha M(f;a,x)+\beta M(f;x,b)\right],$$

where $M(f;a,b)$ is the integral mean defined in (2), then

$$\left|\tau(x;\alpha,\beta)\right| \tag{13}$$
$$\leq \begin{cases} \left(\begin{array}{c} \frac{\alpha}{x-a}\left\{\frac{(x-a)^2}{4}+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\} \\ +\frac{\beta}{b-x}\left\{\frac{(b-x)^2}{4}+\left[\left(b-h\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2\right\}\end{array}\right)\frac{1}{(\alpha+\beta)}\left\|f'\right\|_\infty, \quad f'\in L_\infty[a,b] \\ \left[\begin{array}{c} \frac{\alpha^q}{(x-a)^q}\left\{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\} \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(b-\left(x+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\}\end{array}\right]^{\frac{1}{q}} \frac{1}{(q+1)^{\frac{1}{q}}(\alpha+\beta)}\left\|f'\right\|_p, \\ f'\in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1 \\ \left(\begin{array}{c} (\alpha+\beta)-h\frac{b-a}{2}\left[\frac{\alpha(b-x)+\beta(x-a)}{(x-a)(b-x)}\right] \\ +\left|(\alpha-\beta)+h\frac{b-a}{2}\left[\frac{\beta(x-a)-\alpha(b-x)}{(x-a)(b-x)}\right]\right|\end{array}\right)\frac{\left\|f'\right\|_1}{2(\alpha+\beta)}, \\ f'\in L_1[a,b]. \end{cases}$$

for all $x\in\left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ and $h\in[0,1]$.

Proof. Taking the modulus of (12) and using (2), we obtain

$$\left|\tau(x;\alpha,\beta)\right| = \left|\int_a^b P(x,t)f'(t)dt\right| \leq \int_a^b |P(x,t)|\left|f'(t)\right|dt. \tag{14}$$

By using the definition of L_∞ norm, we get

$$\left|\tau(x;\alpha,\beta)\right| \leq \left\|f'\right\|_\infty \int_a^b |P(x,t)|\,dt.$$

Now

$$\int_a^b |P(x,t)|\,dt$$
$$= \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\int_a^x \left|t-\left(a+h\frac{b-a}{2}\right)\right| dt+\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\int_x^b \left|t-\left(b-h\frac{b-a}{2}\right)\right| dt.$$

Again using integration by parts and some algebraic manipulation, we get

$$\int_a^b |P(x,t)|\,dt = \begin{array}{l} \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\left\{\frac{1}{4}(x-a)^2+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\} \\ +\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\left\{\frac{1}{4}(b-x)^2+\left[\left(b-h\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2\right\}. \end{array}$$

Hence the first inequality

$$\left|\tau(x;\alpha,\beta)\right| \le \left(\begin{array}{l} \frac{\alpha}{x-a}\left\{\frac{1}{4}(x-a)^2+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\} \\ +\frac{\beta}{b-x}\left\{\frac{1}{4}(b-x)^2+\left[\left(b-h\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2\right\} \end{array}\right)\frac{1}{\alpha+\beta}\left\|f'\right\|_\infty$$

is obtained.

Now using Hölder's integral inequality and definition of L_p norm, from (14) we get

$$\left|\tau(x;\alpha,\beta)\right| \le \left\|f'\right\|_p \left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}}.$$

Now

$$(\alpha+\beta)\left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}}$$
$$= \left[\frac{\alpha^q}{(x-a)^q}\int_a^x \left|t-\left(a+h\frac{b-a}{2}\right)\right|^q dt+\frac{\beta^q}{(b-x)^q}\int_x^b \left|\left(t-\left(b-h\frac{b-a}{2}\right)\right)\right|^q dt\right]^{\frac{1}{q}}.$$

Using integration by parts and some algebraic manipulation, we get

$$(\alpha+\beta)\left(\int_a^b |P(x,t)|^q\,dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{(q+1)^{\frac{1}{q}}}\left[\begin{array}{l} \frac{\alpha^q}{(x-a)^q}\left\{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\} \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(b-\left(x+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\} \end{array}\right]^{\frac{1}{q}}.$$

The second inequality

$$\begin{aligned}&\left|\tau\left(x;\alpha,\beta\right)\right|\\ &\leq\frac{1}{\left(q+1\right)^{\frac{1}{q}}\left(\alpha+\beta\right)}\left[\begin{array}{c}\frac{\alpha^{q}}{(x-a)^{q}}\left\{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\}\\+\frac{\beta^{q}}{(b-x)^{q}}\left\{\left(b-\left(x+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\}\end{array}\right]^{\frac{1}{q}}\left\|f'\right\|_{p}\end{aligned}$$

is obtained.

Finally, using definition of L_1 norm and $P(x,t)$, we have from (14)

$$\left|\tau\left(x;\alpha,\beta\right)\right|\leq\sup_{t\in[a,b]}|P(x,t)|\left\|f'\right\|_{1}$$

where

$$(\alpha+\beta)\sup_{t\in[a,b]}|P(x,t)|=\left(\begin{array}{c}(\alpha+\beta)-h\frac{b-a}{2}\left[\frac{\alpha(b-x)+\beta(x-a)}{(x-a)(b-x)}\right]\\+\left|(\alpha-\beta)+h\frac{b-a}{2}\left[\frac{\beta(x-a)-\alpha(b-x)}{(x-a)(b-x)}\right]\right|\end{array}\right)\frac{\left\|f'\right\|_{1}}{2}.$$

This completes the proof of the theorem. □

Remark 3.3. *If we put $h=0$, in (13), we get (7). If we put $\alpha=\beta$ and $h=0$, in (13), we get (5). This shows that the results of Cerone and Dragomir are our special cases.*

Remark 3.4. *By substituting $h=1$ and $\alpha=\beta$ in (12) and (13), we get*

$$\begin{aligned}&\left|\left(f\left(a\right)+f\left(b\right)\right)-\frac{2}{\left(b-a\right)}\int_{a}^{b}f\left(t\right)dt\right|\\ &\leq\begin{cases}\left(\begin{array}{c}\frac{1}{x-a}\left\{\frac{(x-a)^{2}}{4}+\left[\frac{a+b}{2}-\frac{a+x}{2}\right]^{2}\right\}\\+\frac{1}{b-x}\left\{\frac{(b-x)^{2}}{4}+\left[\frac{a+b}{2}-\frac{x+b}{2}\right]^{2}\right\}\end{array}\right)\frac{1}{2}\left\|f'\right\|_{\infty},\\ \left[\begin{array}{c}\frac{1}{(x-a)^{q}}\left\{\left(x-\frac{a+b}{2}\right)^{q+1}-\left(\frac{a-b}{2}\right)^{q+1}\right\}\\+\frac{1}{(b-x)^{q}}\left\{\left(\frac{a+b}{2}-x\right)^{q+1}-\left(\frac{a-b}{2}\right)^{q+1}\right\}\end{array}\right]^{\frac{1}{q}}\frac{1}{2(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{p},\\ \left(2-\frac{1}{2}\left[\frac{(b-a)^{2}}{(x-a)(b-x)}\right]+\left|\frac{b-a}{2}\left[\frac{(x-a)-(b-x)}{(x-a)(b-x)}\right]\right|\right)\frac{\left\|f'\right\|_{1}}{4}.\end{cases}\end{aligned}$$

Remark 3.5. *If we substitute $h = \frac{1}{2}$, $\alpha = \beta$ and $x = \frac{a+b}{2}$ in (12) and (13), we get another result:*

$$\left|\frac{1}{2}f(x) + \frac{1}{4}(f(a) + f(b)) - \frac{1}{(b-a)}\int_a^b f(t)\,dt\right|$$

$$\leq \begin{cases} \left(\frac{b-a}{8}\right)\left\|f'\right\|_\infty \\ \left[\begin{array}{c} \left\{\left(\frac{b-a}{4}\right)^{q+1} - \left(\frac{a-b}{4}\right)^{q+1}\right\} \\ +\left\{\left(\frac{b-a}{4}\right)^{q+1} - \left(\frac{a-b}{4}\right)^{q+1}\right\} \end{array}\right]^{\frac{1}{q}} \frac{1}{(b-a)(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p \\ \frac{\left\|f'\right\|_1}{4}. \end{cases}$$

Similarly, for different values of h, we can obtain a variety of results.

Remark 3.6. *It should be noted that from (12) and (1)*

$$(\alpha + \beta)\,\tau(x;\alpha,\beta) = \alpha S(f;a,x) + \beta S(f;x,b). \tag{15}$$

From (13), we obtain

$$(\alpha + \beta)\left|\tau(x;\alpha,\beta)\right| \tag{16}$$

$$\leq \begin{cases} \frac{\alpha}{x-a}\left\{\frac{(x-a)^2}{4} + \left[\left(a + h\frac{b-a}{2}\right) - \frac{a+x}{2}\right]^2\right\}\left\|f'\right\|_{\infty,[a,x]} \\ +\frac{\beta}{b-x}\left\{\frac{(b-x)^2}{4} + \left[\left(b - h\frac{b-a}{2}\right) - \frac{x+b}{2}\right]^2\right\}\left\|f'\right\|_{\infty,[x,b]}, \\ \frac{\alpha}{(x-a)}\left\{\left(x - \left(a + h\frac{b-a}{2}\right)\right)^{q+1} - \left(h\frac{a-b}{2}\right)^{q+1}\right\}^{\frac{1}{q}}\frac{1}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{p,[a,x]} \\ +\frac{\beta}{(b-x)}\left\{\left(b - \left(x + h\frac{b-a}{2}\right)\right)^{q+1} - \left(h\frac{a-b}{2}\right)^{q+1}\right\}^{\frac{1}{q}}\frac{1}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_{p,[x,b]}, \\ \frac{\alpha}{x-a}\left[x - \left(a + h\frac{b-a}{2}\right)\right]\left\|f'\right\|_{1,[a,x]} + \frac{\beta}{b-x}\left[\left(b - h\frac{b-a}{2}\right) - x\right]\left\|f'\right\|_{1,[x,b]}. \end{cases}$$

That is,

$$(\alpha+\beta)\left|\tau(x;\alpha,\beta)\right| \tag{17}$$

$$\leq\begin{cases}\left[\begin{array}{l}\frac{\alpha}{x-a}\left\{\frac{(x-a)^2}{4}+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\}\\+\frac{\beta}{b-x}\left\{\frac{(b-x)^2}{4}+\left[\left(b-h\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2\right\}\end{array}\right]\left\|f'\right\|_\infty,\\ \left[\begin{array}{l}\frac{\alpha}{(x-a)}\left\{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\}^{\frac{1}{q}}\\+\frac{\beta}{(b-x)}\left\{\left(b-\left(x+h\frac{b-a}{2}\right)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\}^{\frac{1}{q}}\end{array}\right]\frac{1}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p,\\ \left(\frac{\alpha}{x-a}\left[x-\left(a+h\frac{b-a}{2}\right)\right]+\frac{\beta}{b-x}\left[\left(b-h\frac{b-a}{2}\right)-x\right]\right)\left\|f'\right\|_1.\end{cases}$$

Remark 3.7. *We may write*

$$\begin{aligned}&\alpha M(f;a,x)+\beta M(f;x,b)\\=\;&\alpha M(f;a,x)+\frac{\beta}{b-x}\left(\int_a^b f(u)\,du-\int_a^x f(u)\,du\right)\\=\;&\alpha M(f;a,x)-\frac{\beta}{b-x}\int_a^x f(u)\,du+\frac{\beta}{b-x}\int_a^b f(u)\,du\\=\;&(\alpha+\beta-\beta\sigma(x))M(f;a,x)+\beta\sigma(x)M(f;a,b),\end{aligned}$$

where

$$\frac{b-a}{b-x}=\sigma(x). \tag{18}$$

Thus, from (12), we get

$$\begin{aligned}&\tau(x;\alpha,\beta) \\=\;&\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\left\{x-\left(a+h\frac{b-a}{2}\right)\right\}-\frac{\beta}{b-x}\left\{x-\left(b-h\frac{b-a}{2}\right)\right\}\right]f(x)\\&+\frac{h}{\alpha+\beta}\left(\frac{b-a}{2}\right)\left(\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right)\\&-\left[\left(1-\frac{\beta}{\alpha+\beta}\sigma(x)\right)M(f;a,x)+\frac{\beta}{\alpha+\beta}\sigma(x)M(f;a,b)\right]\end{aligned} \tag{19}$$

so that for fixed $[a,b]$, $M(f;a,b)$ *is also fixed.*

Corollary 3.8. *If we take $\alpha=\beta$ and $x=\frac{a+b}{2}$ in (12) and (13), we get*

$$\left|(1-h)f\left(\frac{a+b}{2}\right)+\frac{h}{2}(f(a)+f(b))-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{20}$$

$$\leq\begin{cases}\left(\begin{array}{c}\frac{1}{x-a}\left\{\frac{(b-a)^2}{16}+\left[\left(a+h\frac{b-a}{2}\right)-\frac{3a+b}{4}\right]^2\right\}\\+\frac{1}{b-x}\left\{\frac{(b-a)^2}{16}+\left[\left(b-h\frac{b-a}{2}\right)-\frac{a+3b}{4}\right]^2\right\}\end{array}\right)\frac{1}{2}\left\|f'\right\|_\infty\\ \left[\begin{array}{c}\frac{2^q}{(b-a)^q}\left\{\left(\frac{b-a}{2}(1-h)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\}\\+\frac{2^q}{(b-a)^q}\left\{\left(\frac{b-a}{2}(1-h)\right)^{q+1}-\left(h\frac{a-b}{2}\right)^{q+1}\right\}\end{array}\right]^{\frac{1}{q}}\frac{1}{2(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p\\ \frac{(1-h)}{2}\left\|f'\right\|_1.\end{cases}$$

4. Some Perturbed Results

In 1882, Čebyŝev [10] gave the following inequality:

$$\left|T(f,g)\right|\leq\frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty, \tag{21}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous bounded functions

$$\begin{aligned}&T(f,g) \\ =\;& \frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\\ =\;& M(f,g;a,b)-M(f;a,b)M(g;a,b).\end{aligned} \tag{22}$$

In 1935, Grüss [11] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx\right|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{23}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the condition

$$\varphi\leq f(x)\leq\Phi \text{ and } \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b]. \tag{24}$$

The constant $\frac{1}{4}$ is the best possible. We will obtain the perturbed version of the results of Theorem 3.2, by using Grüss type results involving the Čebyŝev functional.

$$T(f,g)=M(f,g;a,b)-M(f;a,b)M(g;a,b), \tag{25}$$

where M is the integral mean defined in (2).

Theorem 4.1. *Let $f\colon [a,b]\to\mathbb{R}$ be an absolutely continuous mapping and $\alpha\geq 0$, $\beta\geq 0$, $(\alpha+\beta)\neq 0$, then*

$$\begin{aligned}\left|\tau(x;\alpha,\beta)-\frac{R}{2(\alpha+\beta)}S\right| &\leq (b-a)N(x)\left[\frac{1}{b-a}\left\|f''\right\|_2^2-S^2\right]^{\frac{1}{2}} \\ &\leq (b-a)\frac{(\Gamma-\gamma)}{2}\lambda\end{aligned} \tag{26}$$

where, $\tau(x;\alpha,\beta)$ is as given by (11), $\lambda=\Phi-\varphi$ and

$$\begin{aligned} R &= \frac{\alpha}{x-a}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2-\left(h\frac{b-a}{2}\right)^2\right] \\ &\quad+\frac{\beta}{b-x}\left[\left(h\frac{b-a}{2}\right)^2-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right], \\ S &= \frac{f(b)-f(a)}{b-a},\end{aligned} \tag{27}$$

$$\begin{aligned} N^2(x) &= \frac{1}{3}\left(\frac{1}{\alpha+\beta}\right)^2\left(\begin{array}{c}\left(\frac{\alpha}{x-a}\right)^2\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^3+\left(h\frac{b-a}{2}\right)^3\right] \\ +\left(\frac{\beta}{b-x}\right)^2\left[\left(h\frac{b-a}{2}\right)^3-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^3\right]\end{array}\right) \\ &\quad-\left(\frac{R}{2(\alpha+\beta)(b-a)}\right)^2.\end{aligned} \tag{28}$$

Proof. Associating $f(t)$ with $P(x,t)$ and $g(t)$ with $f'(t)$, and using (25), we obtain the following

$$T(P(x,.),f'(.);a,b)=M(P(x,.),f'(.);a,b)-M(P(x,.);a,b)M(f'(.);a,b).$$

Now using identity (11), we obtain

$$(b-a)T(P(x,.),f'(.);a,b)=\tau(x;\alpha,\beta)-(b-a)M(P(x,.);a,b)S. \tag{29}$$

Now from (11) and (22), we get

$$\begin{aligned}(b-a)M(P(x,.);a,b) &= \int_a^b P(x,t)dt \\ &= \frac{\alpha}{\alpha+\beta}\frac{1}{x-a}\int_a^x\left(t-\left(a+h\frac{b-a}{2}\right)\right)dt \\ &\quad+\frac{\beta}{\alpha+\beta}\frac{1}{b-x}\int_x^b\left(t-\left(b-h\frac{b-a}{2}\right)\right)dt \\ &= \frac{\alpha}{2(\alpha+\beta)}\frac{1}{x-a}\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^2-\left(h\frac{b-a}{2}\right)^2\right] \\ &\quad+\frac{\beta}{2(\alpha+\beta)}\frac{1}{b-x}\left[\left(h\frac{b-a}{2}\right)^2-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^2\right] \\ &= R\frac{1}{2(\alpha+\beta)}.\end{aligned} \tag{30}$$

By combining (30) with (28), the left hand side of (26) can easily be obtained.

Let $f\colon [a,b] \to \mathbb{R}$ and $fg\colon [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, then [1]

$$\begin{aligned} \left|T(f,g)\right| &\le T^{\frac{1}{2}}(f,f)\,T^{\frac{1}{2}}(g,g) \qquad (f,g \in L_2[a,b]) \\ &\le \frac{(\Gamma-\gamma)}{2} T^{\frac{1}{2}}(f,f) \qquad (\gamma \le g(x) \le \Gamma,\ t \in [a,b]) \\ &\le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \qquad (\varphi \le f(x) \le \Phi,\ t \in [a,b]). \end{aligned} \tag{31}$$

Note that

$$\begin{aligned} 0 &\le T^{\frac{1}{2}}(f'(.);a,b,f'(.);a,b) = \left[M\left(f'(.)^2;a,b\right) - M^2(f'(.);a,b)\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{b-a}\int_a^b \left|f'(t)\right|^2 dt - \left(\frac{1}{b-a}\int_a^b f'(t)\,dt\right)^2\right] \\ &= \frac{1}{b-a}\left\|f'\right\|_2^2 - S^2 \\ &\le \frac{(\Gamma-\gamma)}{2}, \text{ where } \gamma \le f'(t) \le \Gamma,\ t \in [a,b]. \end{aligned} \tag{32}$$

Now, for the bounds on (29), we have to determine

$$T^{\frac{1}{2}}((P(x,.);a,b),(P(x,.);a,b)) \text{ and } \varphi \le P(x,.) \le \Phi.$$

Using the definition of $P(x,t)$ and from (10), we have

$$T((P(x,.);a,b),(P(x,.);a,b)) = M\left(P^2(x,.);a,b\right) - M^2(P(x,.);a,b). \tag{33}$$

From (30), we obtain

$$M(P(x,.);a,b) = \frac{R}{2(\alpha+\beta)(b-a)},$$

and

$$\begin{aligned} M\left(P^2(x,.);a,b\right) &= \left(\frac{\alpha}{\alpha+\beta}\right)^2\left(\frac{1}{x-a}\right)^2\int_a^x \left(t-\left(a+h\frac{b-a}{2}\right)\right)^2 dt \\ &\quad+\left(\frac{\beta}{\alpha+\beta}\right)^2\left(\frac{1}{b-x}\right)^2\int_x^b \left(t-\left(b-h\frac{b-a}{2}\right)\right)^2 dt \\ &= \frac{1}{3}\left(\frac{1}{\alpha+\beta}\right)^2\left(\begin{array}{c} \left(\frac{\alpha}{x-a}\right)^2\left[\left(x-\left(a+h\frac{b-a}{2}\right)\right)^3+\left(h\frac{b-a}{2}\right)^3\right] \\ +\left(\frac{\beta}{b-x}\right)^2\left[\left(h\frac{b-a}{2}\right)^3-\left(x-\left(b-h\frac{b-a}{2}\right)\right)^3\right] \end{array}\right). \end{aligned}$$

Thus, substituting the above results into (33), we obtain

$$0 \le N(x) = T^{\frac{1}{2}}((P(x,.);a,b)(P(x,.);a,b)), \tag{34}$$

where $N(x)$ is given explicitly by (28). Combining (29), (32) and (33) gives the first inequality in (31), and the first inequality in (26). Now utilizing inequality (32) produces the second result in (26). Further it can be seen from the definition of $P(x,t)$ in (10), that for $\alpha,\beta \ge 0$

$$\Phi = \sup_{t\in[a,b]} P(x,t) \text{ and } \varphi = \inf_{t\in[a,b]} P(x,t),$$

where

$$\Phi = \sup_{t\in[a,b]} \frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}\left[x-\left(a+h\frac{b-a}{2}\right)\right], \frac{\beta}{b-x}h\frac{b-a}{2}\right)$$

and

$$\varphi = \inf_{t\in[a,b]} \frac{1}{\alpha+\beta}\left(\frac{\alpha}{x-a}h\frac{b-a}{2}, \frac{\beta}{b-x}\left[x-\left(b-h\frac{b-a}{2}\right)\right]\right).$$

□

5. An Application to the Cumulative Distribution Function

Let X be a random variable taking values in the finite interval $[a,b]$ with Cumulative Distributive Function

$$F(x) = P_r(X \le x) = \int_a^x f(u)\,du,$$

where f is the probability density function. In particular,

$$\int_a^b f(u)\,du = 1.$$

The following theorem holds.

Theorem 5.1. *Let X and F be as above, then*

$$\begin{aligned} &|\left[\alpha(b-x)\left\{x-\left(a+h\tfrac{b-a}{2}\right)\right\}-\beta(x-a)\left\{x-\left(b-h\tfrac{b-a}{2}\right)\right\}\right]f(x) \\ &+h\left(\tfrac{b-a}{2}\right)\{\alpha(b-x)f(a)+\beta(x-a)f(b)\} \\ &-[\alpha(b-x)-\beta(x-a)]F(x)-\beta(x-a)| \end{aligned} \tag{35}$$

$$\le \begin{cases} (b-x)(x-a)\begin{pmatrix} \frac{\alpha}{x-a}\left\{\frac{(x-a)^2}{4}+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\} \\ +\frac{\beta}{b-x}\left\{\frac{(b-x)^2}{4}+\left[\left(b-h\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2\right\}\end{pmatrix}\|f'\|_\infty, \\ (b-x)(x-a)\begin{bmatrix} \frac{\alpha^q}{(x-a)^q}\left\{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{q+1}+\left(h\frac{b-a}{2}\right)^{q+1}\right\} \\ +\frac{\beta^q}{(b-x)^q}\left\{\left(h\frac{b-a}{2}\right)^{q+1}+\left(b-\left(x-h\frac{b-a}{2}\right)\right)^{q+1}\right\}\end{bmatrix}^{\frac{1}{q}} \frac{1}{(q+1)^{\frac{1}{q}}}\|f'\|_p, \\ (b-x)(x-a)\begin{pmatrix} (\alpha+\beta)+h\frac{b-a}{2}\left[\frac{\beta(x-a)-\alpha(b-x)}{(x-a)(b-x)}\right] \\ +\left|(\alpha-\beta)-h\frac{b-a}{2}\left[\frac{\beta(x-a)+\alpha(b-x)}{(x-a)(b-x)}\right]\right|\end{pmatrix}\frac{\|f'\|_1}{2}. \end{cases}$$

Proof. From (12), we have

$$\begin{aligned}\tau(x;\alpha,\beta) \quad : \quad &= \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\left\{x-\left(a+h\frac{b-a}{2}\right)\right\}-\frac{\beta}{b-x}\left\{x-\left(b-h\frac{b-a}{2}\right)\right\}\right]f(x)\\ &+\frac{h}{\alpha+\beta}\left(\frac{b-a}{2}\right)\left(\frac{\beta}{b-x}f(b)-\frac{\alpha}{x-a}f(a)\right)\\ &-\frac{1}{\alpha+\beta}\left[\alpha M(f;a,x)+\beta M(f;x,b)\right]\\ &= \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\left\{x-\left(a+h\frac{b-a}{2}\right)\right\}-\frac{\beta}{b-x}\left\{x-\left(b-h\frac{b-a}{2}\right)\right\}\right]f(x)\\ &+\frac{h}{\alpha+\beta}\left(\frac{b-a}{2}\right)\left(\frac{\beta}{b-x}f(b)-\frac{\alpha}{x-a}f(a)\right)-I\end{aligned}$$

where

$$\begin{aligned} I &= \frac{1}{\alpha+\beta}\left[\alpha M(f;a,x)+\beta M(f;x,b)\right]\\ &= \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)\,dt+\frac{\beta}{b-x}\int_x^b f(t)\,dt\right]\\ &= \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)\,dt+\frac{\beta}{b-x}\left\{\int_a^x f(t)\,dt-\int_a^x f(t)\,dt+\int_x^b f(t)\,dt\right\}\right]\\ &= \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(t)\,dt+\frac{\beta}{b-x}\left\{\int_a^b f(t)\,dt-\int_a^x f(t)\,dt\right\}\right].\end{aligned}$$

By using the definition of Cumulative Distributive Function and Probability Density Function, we get

$$I=\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}F(x)-\frac{\beta}{b-x}F(x)+\frac{\beta}{b-x}\right].$$

Thus, $\tau(x;\alpha,\beta)$ becomes

$$\begin{aligned}&\left|\tau(x;\alpha,\beta)\right| :=\\ &\left|\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\left\{x-\left(a+h\frac{b-a}{2}\right)\right\}-\frac{\beta}{b-x}\left\{x-\left(b-h\frac{b-a}{2}\right)\right\}\right]f(x)\right.\\ &+\frac{h}{\alpha+\beta}\left(\frac{b-a}{2}\right)\left(\frac{\alpha}{x-a}f(a)+\frac{\beta}{b-x}f(b)\right)\\ &\left.-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}F(x)-\frac{\beta}{b-x}F(x)+\frac{\beta}{b-x}\right]\right|.\end{aligned}$$

Using (13) and value of $\tau(x;\alpha,\beta)$ from the above equation, we get required result (35). □

Putting $\alpha=\beta=\frac{1}{2}$ in Theorem 5.1 gives the following result.

Corollary 5.2. *Let X be a random variable, F (x) Cumulative Distributive Function and f is a Probability Density Function, then*

$$
\begin{aligned}
&\left| \left[(b-x)\left\{x-\left(a+h\tfrac{b-a}{2}\right)\right\} - (x-a)\left\{x-\left(b-h\tfrac{b-a}{2}\right)\right\}\right] \tfrac{1}{2} f(x) \right.\\
&\quad + \tfrac{h}{2}\left(\tfrac{b-a}{2}\right)\{(x-a)f(b)+(b-x)f(a)\} \qquad (36)\\
&\quad \left. - \left(\tfrac{a+b}{2}-x\right)F(x) - \tfrac{1}{2}(x-a) \right|
\end{aligned}
$$

$$
\leq \begin{cases}
(b-x)(x-a)\begin{pmatrix} \frac{1}{x-a}\left\{\frac{(x-a)^2}{4}+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\} \\ +\frac{1}{b-x}\left\{\frac{(b-x)^2}{4}+\left[\left(b-h\frac{b-a}{2}\right)-\frac{x+b}{2}\right]^2\right\}\end{pmatrix}\frac{\|f'\|_\infty}{2}, \\
(b-x)(x-a)\begin{bmatrix} \frac{1}{(x-a)^q}\left\{\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{q+1}+\left(h\frac{b-a}{2}\right)^{q+1}\right\} \\ +\frac{1}{(b-x)^q}\left\{\left(h\frac{b-a}{2}\right)^{q+1}+\left(b-\left(x-h\frac{b-a}{2}\right)\right)^{q+1}\right\}\end{bmatrix}^{\frac{1}{q}} \frac{1}{2(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p, \\
(b-x)(x-a)\left(1+h\frac{b-a}{4}\left[\frac{2x-a-b}{(x-a)(b-x)}\right]+\frac{h}{4}\left|\frac{(b-a)^2}{(x-a)(b-x)}\right|\right)\frac{\|f'\|_1}{2}.
\end{cases}
$$

Remark 5.3. *The above result allow the approximation of F (x) in terms of f (x). The approximation of*

$$R(x) = 1 - F(x)$$

could also be obtained by a simple substitution. R (x) is of importance in reliability theory where f (x) is the Probability Density Function of failure.

We put $\beta = 0$ in (35), assuming that $\alpha \neq 0$ to obtain

$$
\begin{aligned}
&\left| \left[\alpha(b-x)\left\{x-\left(a+h\tfrac{b-a}{2}\right)\right\}\right] f(x) \right. \qquad (37)\\
&\left. + h\left(\tfrac{b-a}{2}\right)\{\alpha(b-x)f(a)\} - \alpha(b-x)F(x) \right|
\end{aligned}
$$

$$
\leq \begin{cases}
(b-x)(x-a)\left(\frac{\alpha}{x-a}\left\{\frac{(x-a)^2}{4}+\left[\left(a+h\frac{b-a}{2}\right)-\frac{a+x}{2}\right]^2\right\}\right)\left\|f'\right\|_\infty, \\
(b-x)(x-a)\left(\frac{\alpha^q}{(x-a)^q}\left\{\left[x-\left(a+h\frac{b-a}{2}\right)\right]^{q+1}-\left(h\frac{b-a}{2}\right)^{q+1}\right\}\right)^{\frac{1}{q}}\frac{1}{(q+1)^{\frac{1}{q}}}\left\|f'\right\|_p, \\
(b-x)(x-a)\begin{pmatrix} \alpha+h\frac{b-a}{2}\left[\frac{(x-a)-\alpha(b-x)}{(x-a)(b-x)}\right] \\ +\left|\alpha-h\frac{b-a}{2}\left[\frac{(x-a)+\alpha(b-x)}{(x-a)(b-x)}\right]\right|\end{pmatrix}\frac{\|f'\|_1}{2}.
\end{cases}
$$

6. Conclusion

Cerone [1], obtained bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval and applied these results to approximate the cumulative distribution function in terms of the probability density function. On similar lines, we establish new inequalities, which are more generalized as compared to the inequalities developed in [1], [3]-[6]. The approach that we used not only generalized the results of [1] and [3]-[6] but also gave some other interesting inequalities as special cases. Approximation of the cumulative distribution function is also provided.

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