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ON CONVERGENCE OF PADE'S APPROXIMATION

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1. Denote by $H(U_1)$ the set of all holomorphic function on the unit ball $U_1 = \{z \in C : |z| < 1\}$. Let $f \in H(U_1)$, let $\alpha = (\alpha_{n,1}; \alpha_{n,2}; \dots \alpha_{n,2n+1})_{n=1}^{\infty}$ he an interpolation table ($\alpha_{n,i} \in U_1, \forall i \leq 2n+1, n \in N$); denote by $\overline{\alpha}$ the following set $\overline{\alpha} = \{a \in \overline{U}_1 : \forall \epsilon > 0 \ \exists$ a point $\alpha_{n,k} \in \alpha$ such that $|a - \alpha_{n,k}| < \epsilon\} \equiv \{a : \exists$ a sequence $(\alpha_{n_i,k_i}) \subset \alpha$ such that $\alpha_{n_i,k_i} \to a, \ i \to \infty\}$.

Let p_n and q_n he polynomials with $deg(p_n) \leq n$, $deg(q_n) \leq n$, such that $q_n \neq 0$ and

$$(q_n f - p_n)(\alpha_{n,k}) = 0 \quad k = 0, 1, \dots, 2n+1. \tag{1}$$

The last relation is equivalent to the system of $2n+1$ equations with $2n+2$ variables and hence it always has a solution. Every solution of the system (1) defines a rational function $\pi_n(z) = (\frac{p}{q})(z)$; $\pi_n = \pi_n(f, \alpha)$ is called the multipoint Pade approximantion of f. From (1) is follows

$$(f - \pi_n)(\alpha_{n,k}) = 0 \tag{2}$$

for all $\alpha_{n,k}$ for which $q_n(\alpha_{n,k}) \neq 0$ holds; in other words, π_n interpolates f at all point of α for which the interpolation property is preserved when we pass from (1) to (2), i.e. $q_n(\alpha_{n,k}) \neq 0$ for all $k = 1, 2, \dots, 2n+1$, $k \in N$. The last condition is equivalent to the fact that (1) has a solution when $(p_n, q_n) = 1$ (for details see [1],[2]).

The classical Pade approximation of a power series is obtained from the previous constraction when $\alpha_{n,k} = 0$, $k = 1, 2, \dots, 2n+1$, $n \in N$. In this case the sequence $\{\pi_n\}_{n=1}^{\infty}$ coincides with the main diagonal $[n/n]_f$ of the Pade table of f.

Many papers (see the survey papers [5],[6],[7],[8]) are devoted to the problem of convergence of different sequences of the Pade table. In particular, the following question was considered: what one can say about the convergence of the sequence $[n/n]_f$, $n \in N$, if it is known that $[n/n]_f \in H(U_1)$, $n \in N$, (see [1],[2],[4]). This problem was solved by Gonchar in [4]. In this paper we prove that in the same case $[n/n]_f \in H(U_1)$ the sequence$[n/n]_f$, $n \in N$, uniformly converges to f on U_1.

One can consider another variant of the problem: what can say on convergence of the $[n/n]_f$, $n \in N$, sequence if we know that $[n/n]_f \in H(U_1)$, for $n \in \Lambda \subset N$. In

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this case, $[n/n]_f$, $n \in \Lambda$, need not converge to f on U_1, in general; the corresponding example was constructed by Rahmanov in [5]. This example shows that the radii r_0 of the biggest ball (with centre in zero) in which $[n/n]_f$, $n \in \Lambda$ converges does not exceed 0,8. In [4] Gonchar and Rahmanov have also found lower estimation for $r_0 : r_0 \geq r_1 = 0,629\ldots$.

In this paper we study the convergence of the sequence $\{\pi_n\}$, $n \in \Lambda$, $\Lambda \subset N$ of multipoint approximations of a function f that is holomorphic on U_1 if we know that $\pi_n \in H(U_1)$ for all $n \in \Lambda$. The main result of the paper is the following

Theorem. *Let $f \in H(U_1)$, $\overline{\alpha} \in \overline{U}_r$, $r < 1$ and let $\pi_n = \pi_n(f, \alpha)$, $n \in \Lambda$ be a sequence of the multipoint approximantions of f. If $\pi_n \in H(U_1)$ for all $n \in \Lambda$, then there exists an open set $\Omega = \Omega(f, \alpha) \subset U_1$ such that $\pi_n \to f$ uniformly on Ω $(n \to \infty,\ n \in \Lambda)$.*

2. In what follows "measure" means a positive Borel measure ν an the complex plane C such that $\nu(K)$ is finite for every compact set $K \subset C$. $supp(\nu)$ denotes the support of ν, $|\nu| = \nu(C)$, V^ν is the logarithmic potential of ν:

$$V^\nu(z) = \int \log \frac{1}{|t-z|} d\nu(t), \quad z \in C.$$

The potential V^ν is a superharmonic function on C. We shall write $\nu_n \to \nu$ if ν_n weakly converges to ν, i.e. if $\int \varphi d\nu_n \to \int \varphi d\nu$ for each continuous finite function φ. If $K \subset C$ is a compact in C and $supp(\nu_n) \subset K$, then

$$V^{\nu_n} \to V^\nu, \quad \text{uniformly on} \quad C \setminus K$$

and for any $z \in C$ the descent principle holds:

$$\liminf_{n\to\infty} V^{\nu_n}(z) \geq V^\nu(z); \tag{3}$$

if $K = \partial U_r$, then

$$\lim_{n\to\infty} \min_{z\in\partial U_r} V^{\nu_n}(z) = \min_{z\in\partial U_r} V^\nu(Z). \tag{4}$$

All notions and facts concerning the potential theory that we use can be found in [3].

For every polynomial $p(z) = z^n + \cdots = \prod_{i=1}^n (z - z_i)$ we define the measure $\nu_n = \nu_n(p)$ associated with p as follows

$$\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\eta_i}.$$

where δ_{η_i} is the Dirac measure af the point η. We have

$$V^{\nu_n(p)}(z) = \frac{1}{n} \ln \frac{1}{|p(z)|},$$

(5) $$|p(z)|^{\frac{1}{n}} = e^{-V^{\nu_n(p)}}(z).$$

Proof of the theorem. The proof uses the Hermit formula for difference $(f - \pi_n)(z)$; let us fix ρ, $r < \rho < 1$; then

(6) $$(f - \pi_n)(z) = \frac{1}{2\pi i}(\frac{\omega_n}{Qq_n})(z) \int_{|t|=\rho} (\frac{fQq_n}{\omega_n})(t)\frac{dt}{t-z}$$

where $\omega_n(z) = \prod_{j=1}^{2n+1}(z - \alpha_{n,j})$, $Q(z) = z^n + \dots$ is an arbitrary polynomial, $z \in K \subset U$. From (6) it follows this estimation:

(7) $$|(f - \pi_n)(z)| \le \frac{M}{\delta(z, \partial U_r)}|(\frac{\omega_n}{Qq_n})(z)|||\frac{Qq_n}{\omega_n}||_{\partial U_\rho}$$

where $M = ||f||_{\partial U_\rho} = \max_{t\in\partial U_\rho} |f(t)|$ and $\delta(z, \partial U_\rho)$ is the distance from z to ∂U_ρ. Let ν_n, λ_n, μ_n be measures associated to polynomials Q, q_n, ω_n, respectively; Then from (5) it follows

(8) $$\frac{1}{n}\ln|(f - \pi_n)(z)| < (V^{\nu_n} + V^{\lambda_n} - V^{\mu_n})(z) \\ - \min_{t\in\partial U_\rho}(V^{\nu_n} + V^{\lambda_n} - V^{\mu_n})(t) + \frac{1}{n}\ln\frac{M}{\delta(z, \partial_\rho)}, \quad z \in K.$$

Choose a measure ν_n on ∂U_1 so that $supp(\nu_n) \subset \partial U_\rho$; by condition $supp(\lambda_n) \subset C \setminus U$, let λ_n^* be " vymitanie" of the measure λ_n on ∂U_1 i.e. $(V^{\lambda_n^*} - V_n^\lambda = const$, for every $z \in U_1$. Then (8) can be written in the form

(9) $$\frac{1}{n}\ln|(f - \pi_n)(z)| \le (V^{\nu_n} + V^{\lambda_n} - V^{\mu_n})(z) - \min_{t\in\partial U_\rho}(V^{\nu_n} + V^{\lambda_n} - V^{\mu_n})(t) \\ + o(1),\ z \in K,\ o(1) \to 0,\ n \to \infty.$$

Without loss of generality we can suppose that

$$\limsup_{n\in\Lambda} \frac{1}{n}\ln|(f - \pi_n)(z)| = \lim_{n\in\Lambda} \frac{1}{n}\ln|(f - \pi_n)(z)|,$$

(otherwise we pass to subsequences and denote them again by Λ). The supports of measures in (9) belong to $\overline{U}_1$ and $supp(\mu_n) \subset \partial U_r, supp(\nu_n) \subset \partial U_\rho$, $supp(\lambda_n^*) \subset \partial U_1$. There exists a subsequence $\Lambda_1 \subseteq \Lambda$ such that $\mu_n \to 2\mu$, $\nu_n \to \nu$, $\lambda_n^* \to \lambda^*$, where μ, ν, λ^* are unit measures on $\overline{U}_r$, ∂U_ρ, U_1 respectively. The descent principle implies

$$\lim_{n\in\Lambda_1} \frac{1}{n}\ln|(f - \pi_n)(z)| \le (V^\nu + V^{\lambda^*} - V^\mu)(z) - \min_{t\in\partial U_\rho}(V^\nu + V^{\lambda^*} - 2V^\mu)(t),\ z \in K \subset U_\rho.$$

Let $\varphi(z) = (V^\nu + V^{\lambda^*} - 2V^\mu)(z)$ and $\omega = \min_{t\in\partial U_\rho}\varphi(t)$. Denote by D the connected component of $C \setminus supp(\mu)$ containing ∞. Let $\Omega = \Omega(f, \alpha) = \{z \in$

$U_\rho : \varphi(z) - \omega < 0\}$. Let us show that Ω is a non-empty open set. The function φ is superharmonic on D. Let $\omega_1 = \min_{z\in D} \varphi(z)$. The minimum principle for superharmonic functions implies that $\omega_1 < \omega$, so that there exists a point $z_0 \in U_\rho$ such that $\varphi(z_0) - \omega < 0$. Therefore Ω in non-empty.

On the other hand, φ is subharmonic on U_r, hence it is a upper semicontinuous function, i.e. the set $G_a = \{z \in U : \varphi(z) < a\}$ in open. This implies that Ω is also open.

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