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DECOMPOSING MAPPINGS AND TRANSFINITE DIMENSION

V.A. Chatyrko

Abstract. *Transfinite extensions zw_S and zw_H of Zarelua's function zw [9] are defined and some their properties are explored.*

A mapping $f : X \mapsto Y$ is called decomposing [9] if for every point $x \in X$ and every its open neighborhood Ox there exists an open neighborhood Ufx of fx such that $f^{-1}Ufx = V^1 \cap V^2$, where V^1 and V^2 are open disjoint subsets of X and $x \in V^1 Ox$. It is evident that every decomposing mapping is fiberwise inductively zero-dimensional (in the sense of ind), and every closed fiberwise inductively zero-dimensional mapping is decomposing.

By I we denote the unit segment [0,1] and by {*} we denote a point $\alpha = p(\alpha) + n(\alpha)$ in the natural decomposition of the ordinal number α into the sum of the limit ordinal number $p(\alpha)$ (0 is limit by definition) and the finite number $n(\alpha) \geq 0$. $X \subset Y$ means that X is embedded into Y as a closed subspace. The boundary of a set A in a space X is denoted by bdA.

In [9], A. Zarelua proved that if X is a fully paracompact space which admits a decomposing mapping into the Hilbert cube, then

$$indX = IndX = dimX = zwX. \tag{*}$$

Here $zwX = \min\{\tau :$ there exists a decomposing mapping$f : X \to I^\tau\}$. In particular, the equality (*) holds for separable metric spaces. It is well known that the classic dimension functions ind, Ind, dim possess natural transfi- nite extensins (about the transfinite extensions of the covering dimension dim see, for example, [2]). In this paper we shall define two transfinite extensions of the function zw and explore some their properties.

Smirnoff's compacta S [8] and Henderson's compacta H^α [4], $\alpha < \omega_1$, are transfinite analogs for finite-dimensional cubes $I^n, n = 1, 2, \ldots$, from the point of view of the dimension theory. So the following definition is natural.

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Definition 1. *Let X be a topological space. Then $zw_s X = \min\{\tau < \omega_1$: there exists a decomposing mapping $f : X \to S^\tau\}$. If there are no decomposing mappings from X into S^γ for any $\gamma < \omega_1$, then we put $zw_s X = \infty$.*

Analogously the function zw_H is defined. We should replace Smirnoff's compacta S^γ, $\gamma < \omega_1$, by Henderson's compacta H^1 in Definition 1.

It is clear that the functions zw and zw_H are transfinite extensions of zw, and $zw_H \leq zw_S$ (since $S^\gamma \subset H^\gamma, \gamma < \omega_1$).

1. Properties of zw_S and zw_H

Recall that Smirnoff's compactum S^α is:

(1) I^α, if $\alpha < \omega$;

(2) the one point Aleksandroff compactification $\{*\} \cup \oplus_{\beta<\alpha} S^\beta$ of the free topological sum $\oplus_{\beta<\alpha} S^\beta$, if α is a limit ordinal number;

(3) $S^{\alpha-1} \times I = S^{p(\alpha)} \times I^{n(\alpha)}$, if $n(\alpha) \geq 1$.

Henderson's compacta $H^\alpha, \alpha < \omega_1$, are defined as follows: $H^1 = I$ and $p_1 = \{0\} \in H^1$. Assume that for every $\beta < \alpha$ the compacta H^β and the points $p_\beta \in H^\beta$ have been defined. Let K_β denote the union of H^β and a semi-open arc A_β such that $A_\beta \cap H^\beta = \{p_\beta\}$ (the end of A_β). If $\alpha = \beta + 1$, then we put $H^{\beta+1} = H^\beta \times I = H^{p(\alpha)} \times I^{n(\alpha)}$ and $p_{\beta+1} = (p_\beta, p_1)$. If α is a limit ordinal number, then H^α is the one-point Aleksandroff compactification of the free sum $\oplus_{\beta<\alpha} K_\beta$ and p_α is the compactification point.

We shall use the following properties of decomposing mappings:

(1) Let $f : X \to Y$ be a decomposing mapping. Then

(a) [9] If $A \subseteq X$, then $f|A : A \to Y$ is decomposing;

(b) [9] If Y is a compactum, then there exists a compactification bX of X such that $wbX = \max\{wX, wY\}$ and f has a zero-dimensional extension on bX.

(2) Let $f_i : X_i \to Y_i, i = 1, 2$, be decomposing mappings. Then the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also decomposing.

(3) Let $f_i : X_i \to Y, i = 1, 2$, be decomposing mappings. Then the mapping $f : X_1 \oplus X_2 \to Y$ such that $f|X_1 = f_1, f|X_2 = f_2$ is also decomposing.

Theorem 1. *Let zw_* be one of the functions zw_S and zw_H. Then:*

(a) If X is a topological space and $A \subseteq X$, then $zw_* A \leq zw_* X$;

(b) If X is a topological space and $zw_* X = \alpha$, then there exists a com- pactification bX of X such that $wbX = wX$ and $zw_* bX = \alpha$;

(c) $zw_*(X_1 \oplus X_2) = \max\{zw_* X_1, zw_* X_2\}$;

(d) If $zw_* X_2 < \omega$, then $zw_*(X_1 \times X_2) \leq zw_* X_1 + zw_* X_2$ for any space X_1;

(e) Let $f : X \to Z$ be a continuous mapping from a compactum X onto a space Z such that for every zero-dimensional subcontinuum $A \subseteq X$, $f(A)$ is a zero- dimensional subcontinuum in Z. If $zw_* X = \alpha$, then there exist a compact space Y and continuous mappings $g : X \to Y$ and $h : Y \to Z$ such that $f = hg$, $zw_* Y \leq alpha$ and $wY \leq wZ$.

Proof. The statements (a) - (d) easily follow from the properties of decomposing mappings and the definitions of Smirnoff's and Henderson's compacta. For (e), let us assume that $zw_* = zw_S$. Let $r : X \to S^\alpha$ be a zerodimensional mapping (since $zw_S = \alpha$). Put $g = r\Delta f$, $Y = gX \subset S \times Z$, $h = pr_z|Y$, where $pr_z : S^\alpha \times Z \to Z$ is the projection. Obviously, $pr_{S^\alpha}|Y : Y \to S^\alpha$ is zero-dimensional (here $pr_{S^\alpha} : S \times Z \to S^\alpha$ is the projection). The theorem is proved. □

Let us note that H^α, $\alpha < \omega_1$, admit continuous fiwerwise zero-dimensional mappings onto I. It is easy to show (W. Olszewski) that a compactum $H^\omega \times H^\omega$ does not admit such a mapping. So, this space does not admit any zero-dimensional mapping into H^γ for every $\gamma < \omega_1$. Hence, $zw_H(H^\omega \times H^\omega) = \infty$, but $zw_H H = \omega$. In [3], it was proved that $S^\alpha \times S^\beta \subset S^{\alpha(+)\beta} \subset S^\alpha \times s^\beta$, where $\alpha(+)\beta$ is the sum of the ordinals $\alpha, \beta < \omega_1$. So we have the following

Theorem 2. *For topological spaces X and Y we have*

$$zw_S(X \times Y) \leq zw_S X(+) zw_S Y. \quad \square$$

Using the connectedness of H^γ, one can easily check that $zw_S H^\gamma = \infty$ and $H^\omega = S^\omega \cup X$, where X is the one-point compactification of a countable free sum of semi-open arcs; besides, $zw_S S^\omega = \omega$ and $zw_S X = 1$. For the function zw_H the finite sum theorem is also false.

2. Universal spaces

In this section we consider only separable metric spaces. Let us recall that a space X is called universal for a class $\mathcal{K}$ of topological spaces if X belongs to $\mathcal{K}$ and every space from $\mathcal{K}$ is embedded into X. Let φ be a function defined on a class of topological spaces with values which are ordinal numbers (for example, let φ be a dimension function) and let α be an ordinal number. Put $\mathcal{K}(\varphi, \alpha) = \{X : \varphi X \leq \alpha\}$,$\mathcal{K}(\varphi, \alpha) = \{X : X \text{is compact and } \varphi X \leq \alpha\}$. In this section we shall concern the following questions:

(A) Is there a universal space for a class $\mathcal{K}(\varphi, \alpha)$?

(B) Is there a universal space for a class $\mathcal{K}_c(\varphi, \alpha)$?

For $\varphi = ind$, Ind, dim, zw and every finite α, the answers to the questions (A) and (B) are affirmative and belong to the classical results of dimension theory [1]. For $\varphi = ind$ and every infinite α, the answer to the question (A) is positive [7] and the answer to the question (B) for $\alpha = \omega$ is negative [5]. For $\varphi = Ind$, dim, dim and $\alpha = \omega$, the answers are negative, too. (Here dim_W and dim_c are Borst's transfinite extensions of the covering dimension dim; see [2]). We are going to examine the questions (A) and (B) for the functions zw_S and zw_H.

In [6], B. Pasynkov proved the following statement:

For every compactum X and every cardinal $\tau \geq wX$ there are a compactum $P(X)$ with $wP(X) = \tau g$ which is universal for all compacta Y with $wY \leq \tau$ and a zero-dimensional mapping $f : Y \mapsto X$.

From this statement and Theorem 1.(b) we get

Corollary 1. *There exist universal spaces for the classes $\mathcal{K}(zw_S, \alpha)$, $\mathcal{K}(zw_H, \alpha)$, $\mathcal{K}_c(zw_S, \alpha)$ and $\mathcal{K}_c(zw_H, \alpha)$, $\alpha < \omega$.* □

Let us note that the universal spaces from Corollary 1 are constructed as partial products [6]. Now we shall get other, more simple, universal spaces for the function zw_S.

Theorem 3. *The space $S^{p(\alpha)} \times M_{n(\alpha)}$ is universal for the classes $\mathcal{K}(zw_S, \alpha)$ and $\mathcal{K}_c(zw_S, \alpha)$, $\alpha < \omega_1$ (where M_k, $k = 0, 1, \dots$ are universal k-dimensional compacta [1]).*

PROOF. It is necessary to apply Theorem 1.(b) and Lemmas 2 and 3 below. □

Lemma 1. (A corollary to the Freudenthal theorem [1]) *Let X be a compactum, F a closed subset of X, $dimF = n$ and $F = \cap_{i=1}^{\infty} C_i$, where C_i are clopen in X and $C_i \supset C_{i+1}$ for every i. Then there are an inverse sequence $\mathcal{S} = \{P_i; ⋒p_1^{i+1}\}$ of at most n-dimensional polyhedra and a continuous mapping $g : X \to \lim\mathcal{S}$, such that $g|F : F \to gF$ is a homeomorphism.* □

Lemma 2. *For every $\alpha < \omega_1$, $zw_S(S^{p(\alpha)} \times M_{n(\alpha)}) \leq \alpha$.* □

Lemma 3. *Let X be a compactum with $zw_S X \leq \alpha$. Then $X \subset S^{p(\alpha)} \times M_{n(\alpha)}) \leq \alpha$.*

PROOF OF LEMMA 3. We shall use induction. For $\alpha < \omega$ see (*). Let $\alpha \geq \omega$, let $f : X \to S^\alpha$ be a decomposing mapping and let $g : S^\alpha \to S^\alpha/(* \times I^{n(\alpha)})$ be a factorization (here * is the compactification point of $S^{p(\alpha)}$. Clearly, there exists an embedding $i : S^\alpha/(* \times I^{n(\alpha)}) \to S^{p(\alpha)}$. The mapping $h = igf : X \to S^{p(\alpha)}$ is decomposing on the set $X_1 = \cup_{\beta < p(\alpha)} h^{-1} S^\beta = X \setminus g^{-1}(\{*\} \times I^{n(\alpha)})$. By induction there is a mapping $q : X \to S^{p(\alpha)}$ (recall that $M \subset I^{2k+1}$ [1]) such that q is injective on the subspace X^1 of X, $q(X \setminus X^1) = \{*\}$ and $qX^1 \cap q(X \setminus X^1) = 0$. From the remark to Lemma 1 there is a mapping $l : X \to M_{n(\alpha)}$ which is an embedding on the set $f^{-1}(\{*\} \times I^{n(\alpha)}) = q^{-1}(\{*\})$. Obviously, the diagonal product $q\Delta l : X \to S^{p(\alpha)} \times M_{n(\alpha)}$ is a continuous injective mapping, so an embedding. The lemma is proved. □

Corollary 2. (Transfinite extension of the Nobeling-Pontrjagin theorem [1]) *Every space X with $zw_S X \leq \alpha$ is topologically embedded in $S^{p(\alpha+2n(\alpha)+1}$, $\alpha < \omega_1$.* □

Corollary 3. *For every $\alpha < \omega_1$, $zw_S S^\alpha = \alpha$.*

PROOF. Let us only notice that $IndS^\alpha = \alpha$ [8] and $Ind(S^{p(\alpha)} \times M_{n(\alpha)} = p(\alpha) + n$, $n = 1, 2, \dots$ (see for example [3]). □

To prove another statement we must consider the following family of compacta $S^{\alpha,k}, 0 \leq \alpha < \omega_1, k < \omega$. Put:

(i) $S^{\alpha,k} = S^{\alpha+k}$, if $\alpha < w$;

(ii) $S^{\alpha,k} = (\{*\} \cup \oplus_{\beta<\alpha} S^{\beta,k}) \times I$ (where $\{*\} \cup \oplus_{\beta<\alpha} S^{\beta,k}$ is the one-point compactification of the free sum $\oplus_{\beta<\alpha} S^{\beta,k}$), if α is a limit odrinal number;

(iii) $S^{\alpha,k} = S^{p(\alpha),k} \times I^{n(\alpha)}$, if $n(\alpha) \geq 1$.

One can easily check (by induction) that the following holds:

$$S^{\alpha+k} \subset S^{\alpha,k} \subset S^{\alpha+k}, \quad \alpha < \omega_1, \quad k < \omega.$$

Statement 1. *For every $\alpha < \omega_1$ we have $zw_H S^\alpha = zw_H^\alpha = \alpha$.*

PROOF. Clearly, $zw_H S^\alpha \leq \alpha$. Besides, $zw_H S^\alpha \leq zw_H^\alpha$, since $S^\alpha \subset H^\alpha$. We shall prove that $zw_H S^\alpha \geq \alpha$. We use induction. It is evident that $zw_H S^\alpha \geq \alpha$ for

$\alpha < \omega$ (see (*)). The case when α is a limit ordinal is evident too. Let $n(\alpha) \geq 1$ and suppose $zw_H S^\alpha < \alpha$. Then there is a zero-dimensional mapping $f : S^\alpha \mapsto H^{\alpha-1}$. By the above ramark $S^\alpha \supset S^{p(\alpha),n(\alpha)}$. Let us consider the restriction $f_1 = f|_{S^{p(\alpha),n(\alpha)}}: S^{p(\alpha),n(\alpha)} \mapsto H^{\alpha-1}$ which is also zero-dimensional. All the connected components in the compactum $S^{p(\alpha),n(\alpha)}$ except the only limit one are cubes having dimension $> n(\alpha)$. It is not difficult to see that $f_1(S^{p(\alpha),n(\alpha)} \subset H^\beta \subset H^{\alpha-1}$ for some $\beta < p(\alpha)$. But $S^{p(\alpha),n(\alpha)} \supset S^\alpha \supset S^{\beta+1}$, so that by the assumtion there is no any zero-dimensional mapping from $S^{\beta+1}$ into H^β. It is a contradiction. Hence, $zw_H S^\alpha \geq \alpha$ for $n(\alpha) \geq 1$. The statement is proved. □

3. Relations between zw_S, zw_H and other transfinite dimensions

Recall the following definition from [3]. $IdX = -1$ iff $X = \emptyset$. Put $IdX \leq \alpha$, where is an ordinal number, if there are systems $\sigma_{-1}, \sigma_0, \ldots, \sigma_\delta$, $\delta \leq \alpha$, of closed subsets of X such that:

(a) $\sigma_{-1} = \{\emptyset\}$, $X \in \sigma_\delta$, and $\sigma_\beta \subseteq \sigma_\gamma$ for every β, γ, with $-1 \leq \beta \leq \gamma \leq \delta$;

(b) for every $0 \leq \gamma \leq \delta$, every $F \in \sigma_\gamma$ and for every pair of disjoint closed subsets A and B in X there exist $\beta < \gamma$ and $\Psi \in \sigma_\beta$, $\Psi \subseteq F$ such that Ψ is a partition between $A \cap F$ and $B \cap F$ in F.

(c) for every F_1 and F_2 from σ_γe and every closed subset $F_3 \subset F_1$ we have $F_1 \cup F_2 \in \sigma_\gamma$ and $F_3 \in \sigma_\gamma$.

If in the above definition the set A is replaced by a point, we get the definition of the dimension function id. Note [3] that $indX \leq idX$, $IndX \leq IdX$ and if X is compact, then $idX = IdX$.

Lemma 4. *Let $f : X \mapsto Y$ be a decomposing mapping. Then $idX \leq idY$.*

PROOF. Let $idY \leq \alpha$. Then there exist systems σ_{-1}, $\sigma_0, \ldots, \sigma_\delta$, $\delta \leq \alpha$, of closed subsets of Y satisfying the conditions (a), (b) and (c) in the previous definition. Put $\tau_{-1} = \{\emptyset\}$, $\tau_\gamma = \{C : C$ is closed in X and $C \subset f_{-1}F$, where $F \in \delta_\gamma\}$, $0 \leq \gamma \leq \delta$. Evidently, the systems $\tau_{-1}, \tau_0, \ldots, \tau_\delta$, $\delta \leq \alpha$, satisfy (a) and (c) of the definition of id. We shall prove that these systems satisfy also (b) from that definition. Let $K = f^{-1}F$, $F \in \sigma_\gamma$, $x \in K$ and $x \notin A \subset K$. The mapping $f_1 = f|K : K \mapsto F$ is decomposing. Hence, there is a closed set $B \subseteq Y$ such that $f_1 x \notin B \subset F$ and $f^{-1}(F \setminus B) = U \cup V$, where $U \cap V = \emptyset$, U and V are open in K and $x \in U \subset K \setminus A$. According to the definition of id there are a $\beta < \gamma$ and a partition $G \in \delta_\beta$ between $f_1 x$ and B in F. We may suppose that $G = bd\,O$, where O is open in F, $f_1 x \in O$ and $O \cap B = \emptyset$. It is clear that the set $f^{-1}O \cap U$ is open in K and contains x. Besides, $bd(f^{-1}O \cap U) \subset f^{-1}bd\,O = f^{-1}G$. Consequently, $bd(f^{-1}O \cap U)$ is a partition between x and A in K and belongs to τ_γ. The lemma is proved. □

Statement 2. *Let X be a topological space. Then $ind\,X \leq id\,X \leq zw_H X \leq zw_S X$. In particular, if X is compact, then $ind\,X \leq Ind\,X \leq id\,X = Id\,X \leq zw_H X \leq zw_S X$.*

PROOF. Let us note that $id\,H^\alpha = \alpha$ [3]. Using now Lemma 4 and Statement 1 we finish the proof. □

The main result of this section is the following.

Theorem 4. *For every $\alpha < \omega_1$ there exists a compactum L_α such that $zw_S L_\alpha = \alpha$ and $ind\, L_\alpha = Ind\, L_\alpha = dim_w L_\alpha = id\, L_\alpha = Id\, L_\alpha = \omega$.*

PROOF. Let $(I^n, 0)$ be the n-dimensional cube with fixed the zero point $\{\emptyset\}$. Put: (1) $L_\alpha = I^\alpha$, if $\alpha < \omega$; (2) $L_\alpha = \{*\} \cup \bigoplus_{\beta \leq \alpha} L_\beta$ – the one-point compactification of the free sum $\bigoplus_{\beta \leq \alpha} L_\beta$, if α is a limit ordinal number; (3) $L_\alpha =$ the subset $(L_{p(\alpha)} \times \{\emptyset\}) \cup (\{\emptyset\} \times I^{n(\alpha)})$ of the product $L_{p(\alpha)} \times I^{n(\alpha)}$, if $n(\alpha) \geq 1$. One easily checks that $L_\alpha \subset S^\alpha$. Consequently, $zw_S L_\alpha \leq \alpha$. It is also clear that $ind\, L_\alpha = \omega$. So we have $Ind\, L_\alpha = dim_w L_\alpha = id\, L_\alpha = Id\, L_\alpha = \omega$. We are going to show the inequality $zw_w L_\alpha \geq \alpha$. If α is a limit ordinal, then it is true according to the monotonity of zw_S. If $n(\alpha) \geq 1$, then we get this from the position of the point $\{*\} \times \{\emptyset\}$ under an embedding of L_α into $S^{p(\beta)} \times M_{n(\beta)}$, $\beta < \alpha$ (see Theorem 3) and connectedness of the cubes I^k, $k < \omega$. The theorem is proved. □

Note that $zw_H L_{\omega+1} = \omega < \omega + 1 = zw_S L_{\omega+1}$ and $zw_H L_{\omega+2} = \omega + 1$.

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NIISI RAN, Avtozavodskaya st. 23, 109280 Moscow, Russia