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CURVATURE THEORY OF VECTOR BUNDLES AND SUBBUNDLES

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Abstract. *The vector bundle is decomposed as summ of subbundle and complementary subbundle. Using the special coordinate transformations the relations between curvature tensors of the generalized connection in the vector bundle and subbundles are given and the Ricci equations are determined. When the generalized connection reduces to the Mirons d-connection the known formulae are obtained.*

1. Curvature tensors of a vector bundle

Let $\xi = (E, \pi, M)$ be a C^∞ vector bundle with $\dim M = n$, $\dim E = n + m$. In some local chart the point $u \in E$ has the coordinates

$$(x^1, \dots, x^n, y^1, \dots, y^m) = (x^i, y^a) = (x, y)$$

$$i, j, k, l, m = 1, \dots, n \qquad a, b, c, d, e, f = 1, \dots, m.$$

The adapted basis B of $T(E)$ with respect to the nonlinear connection N is given by

$$B = \{\delta_i, \partial_a\},$$

where

$$\delta_i = \partial_i - N_i^a(x, y)\partial_a. \tag{1.1}$$

Let X, Y, Z be three vector fields in $T(E)$ given by

$$\text{(a) } X = X^i\delta_i + X^a\partial_a \quad \text{(b) } Y = Y^j\delta_j + Y^b\partial_b \quad \text{(c) } Z = Z^h\delta_h + Z^c\partial_c. \tag{1.2}$$

The curvature vector $R(X, Y)Z$ as usually is determined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{1.3}$$

Theorem 1.1. *The curvature vector $R(X, Y)Z$ in the basis B has the form*

$$R(X, Y)Z = R^k\delta_k + R^d\partial_d, \tag{1.4}$$

where

$$\begin{aligned} R^x = {} & 2^{-1}(R^x_{h\,ji}Z^h + R^x_{c\,ji}Z^c)(X^iY^j - Y^iX^j) + \\ & (P^x_{h\,ja}Z^h + P^x_{c\,ja}Z^c)(X^aY^j - Y^aX^j) + \\ & 2^{-1}(S^x_{h\,ba}Z^h + S^x_{c\,ba}Z^c)(X^aY^b - Y^aX^b) \quad x \in \{k, d\}. \end{aligned} \tag{1.5}$$

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The components of the curvature tensors R, P and S are determined in the folloving form

$$R^{x}_{y\,ji} = K^{x}_{y\,ji} + C^{x}_{yb} K^{b}_{j\,i} \tag{1.6}$$

$$K^{x}_{y\,ji} = (\delta_i F^{x}_{y\,j} + F^{e}_{y\,j} F^{x}_{e\,i} + F^{l}_{y\,j} F^{x}_{l\,i}) - (i,j) \tag{1.7}$$

$$K^{b}_{j\,i} = \delta_i N^{b}_{j} - \delta_j N^{b}_{i} \qquad K^{b}_{a\,j} = \partial_a N^{b}_{j} - F^{b}_{a\,j} \tag{1.8}$$

$$\begin{aligned} P^{x}_{y\,j\,a} &= \partial_a F^{x}_{y\,j} - C^{x}_{y\,a|j} + K^{b}_{a\,j} C^{x}_{y\,b} \\ C^{x}_{y\,a|j} &= \delta_j C^{x}_{y\,a} + C^{e}_{y\,a} F^{x}_{e\,j} + C^{l}_{y\,a} F^{x}_{l\,j} - C^{x}_{e\,a} F^{e}_{y\,j} - C^{x}_{l\,a} F^{l}_{y\,j} - C^{x}_{y\,e} F^{e}_{a\,j} \end{aligned} \tag{1.9}$$

$$S^{x}_{y\,ab} = (\partial_b C^{x}_{y\,a} + C^{e}_{y\,a} C^{x}_{e\,b} + C^{l}_{y\,a} C^{x}_{l\,b}) - (a,b) \tag{1.10}$$

$$x \in \{k,d\} \qquad y \in \{h,c\}\,.$$

Let us consider the special transformation of coordinate system given by

$$\begin{aligned} &x^i = x^i(u^1,\dots,u^{\tilde n}) + \widehat{x}^i(\overline{u}^{\tilde n+1},\dots,\overline{u}^{n}) + C^i = x^i(u^\alpha) + \widehat{x}^i(\overline{u}^{\overline{\alpha}}) + C^i\,,\ C^i \in \mathbb{R} \\ &\alpha,\beta,\gamma,\delta,\epsilon,\dots = 1,\dots,\tilde n\,, \qquad \overline{\alpha},\overline{\beta},\overline{\gamma},\overline{\delta},\dots = \tilde n+1,\dots,n\,, \\ &y^a = B^a_A(u) v^A + B^a_{\overline{A}}(\overline{u}) \overline{v}^{\overline{A}}\,, \\ &A,B,C,D,E = 1,\dots,\tilde m\,, \qquad \overline{A},\overline{B},\overline{C},\overline{D},\overline{E} = \tilde m+1,\dots,m\,, \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} &rank \begin{bmatrix} B^i_\alpha \\ B^i_{\overline{\alpha}} \end{bmatrix} = n \qquad rank \begin{bmatrix} B^a_A \\ B^a_{\overline{A}} \end{bmatrix} = m \\ &B^i_\alpha = \partial_\alpha x^i \qquad B^i_{\overline{\alpha}} = \partial_{\overline{\alpha}} x^i \qquad \partial_\alpha = \partial/\partial u^\alpha, \quad \partial_{\overline{\alpha}} = \partial/\partial u^{\overline{\alpha}}, \\ &B^a_A = \partial_A y^a \qquad B^a_{\overline{A}} = \partial_{\overline{A}} y^a, \qquad \partial_A = \partial/\partial v^A, \quad \partial_{\overline{A}} = \partial/\partial v^{\overline{A}}\,. \end{aligned}$$

Under special conditions determined in [4, 3.] for $\overline{u}^{\overline{\alpha}} = C^{\overline{\alpha}}$, $\overline{v}^{\overline{A}} = C^{\overline{A}}$ ($C^{\overline{\alpha}}, C^{\overline{A}} \in \mathbb{R}$) (1.11) reduces to the subbundle $\tilde\xi = (\tilde E, \tilde\pi, \tilde M)$, $\dim \tilde E = \tilde n + \tilde m$, $\dim \tilde M = \tilde n$.

For $u^\alpha = C^\alpha$, $v^A = C^A$ ($C^\alpha, C^A \in \mathbb{R}$) (1.11) reduces to the complementary subbundle $\tilde{\tilde\xi} = (\tilde{\tilde E}, \tilde{\tilde\pi}, \tilde{\tilde M})$, $\dim \tilde{\tilde E} = n + m - (\tilde n + \tilde m)$, $\dim \tilde{\tilde M} = n - \tilde n$.

Such nonlinear connections $N^A_\alpha(u,v)$ and $N^{\overline{A}}_{\overline{\alpha}}(\overline{u},\overline{v})$ in $T(\tilde E)$ and $T(\tilde{\tilde E})$ respectively are introducted, wich beside usual transformation law satisfy following relations

$$\begin{aligned} &(a) \quad B^a_A N^A_\alpha = B^i_\alpha N^a_i + (\partial_\alpha B^a_A) v^A \\ &(b) \quad B^a_{\overline{A}} N^{\overline{A}}_{\overline{\alpha}} = B^i_{\overline{\alpha}} N^a_i + (\partial_{\overline{\alpha}} B^a_{\overline{A}}) v^{\overline{A}}\,. \end{aligned} \tag{1.12}$$

Conditions (1.12) are not necessary, but if they are satisfied, then between the bases

$$B = \{\delta_i, \partial_a\} \quad \text{and} \quad \overline{B} = \{\delta_\alpha, \delta_{\overline{\alpha}}, \partial_A, \partial_{\overline{A}}\}\,,$$

where

$$\delta_\alpha = \partial_\alpha - N^A_\alpha(u,v)\partial_A \qquad \delta_{\overline{\alpha}} = \partial_{\overline{\alpha}} - N^{\overline{A}}_{\overline{\alpha}}(\overline{u},\overline{v})\partial_{\overline{A}} \tag{1.13}$$

the following relations are fulfilled [4, (3.15)]

$$\begin{aligned} \delta_\alpha &= B^i_\alpha \delta_i \qquad & \delta_{\overline{\alpha}} &= B^i_{\overline{\alpha}} \delta_i \\ \partial_A &= B^a_A \partial_a \qquad & \partial_{\overline{A}} &= B^a_{\overline{A}} \partial_a\,. \end{aligned} \tag{1.14}$$

Theorem 1.2. *The curvature vector $R(X,Y)Z$ in the basis $\overline{B}$ has the form:*

(1.15) $$R(X,Y)Z = R^{\delta}\delta_{\delta} + R^{\overline{\delta}}\delta_{\overline{\delta}} + R^{C}\partial_{C} + R^{\overline{C}}\partial_{\overline{C}},$$

where

(1.16)
$$\begin{aligned} R^{x} =& 2^{-1}(R^{x}_{(\beta,\alpha)} + R^{x}_{(\beta,\overline{\alpha})} + R^{x}_{(\overline{\beta},\alpha)} + R^{x}_{\overline{\beta},\overline{\alpha})}) + \\ &+(P^{x}_{(\beta,A)} + P^{x}_{(\beta,\overline{A})} + P^{x}_{(\overline{\beta},A)} + P^{x}_{(\overline{\beta},\overline{A})}) \\ &+2^{-1}(S^{x}_{(B,A)} + S^{x}_{(B,\overline{A})} + S^{x}_{(\overline{B},A)} + S^{x}_{(\overline{B},\overline{A})}), \quad x \in \{\delta, \overline{\delta}, C, \overline{C}\}. \end{aligned}$$

(1.17) $$R^{x}_{(\beta,\alpha)} = (R^{x}_{\chi\,\beta\alpha} Z^{\chi} + R^{x}_{\overline{\chi}\,\beta\alpha} Z^{\overline{\chi}} + R^{x}_{D\,\beta\alpha} Z^{D} + R^{x}_{\overline{D}\,\beta\alpha} Z^{\overline{D}})(X^{\alpha}Y^{\beta} - Y^{\alpha}X^{\beta})$$

(1.18) $$P^{x}_{(\beta,A)} = (P^{x}_{\chi\,\beta A} Z^{\chi} + P^{x}_{\overline{\chi}\,\beta A} Z^{\overline{\chi}} + P^{x}_{D\,\beta A} Z^{D} + P^{x}_{\overline{D}\,\beta A} Z^{\overline{D}})(X^{A}Y^{\beta} - Y^{A}X^{\beta})$$

(1.19) $$S^{x}_{(B,A)} = (S^{x}_{\chi\,BA} Z^{\chi} + S^{x}_{\overline{\chi}\,BA} Z^{\overline{\chi}} + S^{x}_{D\,BA} Z^{D} + P^{x}_{\overline{D}\,BA} Z^{\overline{D}})(X^{A}Y^{B} - Y^{A}X^{B})$$

If in (1.17) (β,α) is substituted by $(\beta,\overline{\alpha}), (\overline{\beta},\alpha)$ and $(\overline{\beta},\overline{\alpha})$, then $R^{x}_{(\beta\overline{\alpha})}$, $R^{x}_{(\overline{\beta},\alpha)}$ and $R^{x}_{\overline{\beta},\overline{\alpha})}$ are obtained respectively. If in (1.18) (β, A) is substituted by $(\beta,\overline{A})$, $(\overline{\beta}, A)$ and $(\overline{\beta},\overline{A})$, then $P^{x}_{(\beta\overline{A})}$, $P^{x}_{(\overline{\beta},A)}$ and $P^{x}_{\overline{\beta},\overline{A})}$ are obtained respectively. If in (1.19) (B, A) is substituted by $(B,\overline{A})$, $(\overline{B}, A)$ and $(\overline{B},\overline{A})$, then $S^{x}_{(B\overline{A})}$, $S^{x}_{(\overline{B},A)}$ and $S^{x}_{\overline{B},\overline{A})}$ are obtained respectively.

The components of the curvature tensors are given by the folloving formulae:

(1.20)
$$\begin{aligned} &(a) \quad R^{x}_{y\,\beta\alpha} = K^{x}_{y\,\beta\alpha} + C^{x}_{y\,B} K^{B}_{\beta\,\alpha} \\ &(b) \quad R^{x}_{y\,\overline{\beta\alpha}} = K^{x}_{y\,\overline{\beta\alpha}} + C^{x}_{y\,\overline{B}} K^{\overline{B}}_{\overline{\beta}\,\overline{\alpha}} \\ &(c) \quad R^{x}_{y\,\beta\overline{\alpha}} = K^{x}_{y\,\beta\overline{\alpha}} \qquad R^{x}_{y\,\overline{\beta}\alpha} = K^{x}_{y\,\overline{\beta}\alpha} \end{aligned}$$

(1.21) $$K^{x}_{y\,\beta\alpha} = (\delta_{\alpha} F^{x}_{y\,\beta} + F^{\epsilon}_{y\,\beta} F^{x}_{\epsilon\,\alpha} + F^{\overline{\epsilon}}_{y\,\beta} F^{x}_{\overline{\epsilon}\,\alpha} + F^{D}_{y\beta} F^{x}_{D_{\alpha}} + F^{\overline{D}}_{y\beta} F^{x}_{\overline{D}\,\alpha}) - (\alpha\beta)$$

(1.22) $$K^{B}_{\alpha\,\beta} = \delta_{\beta} N^{B}_{\alpha} - \delta_{\alpha} N^{B}_{\beta} \qquad K^{\overline{B}}_{\overline{\alpha\beta}} = \delta_{\overline{\beta}} N^{\overline{\beta}}_{\overline{\alpha}} - \delta_{\overline{\alpha}} N^{\overline{B}}_{\overline{\beta}}$$

$$x \in \{\delta, \overline{\delta}, C < \overline{C}\} \qquad y \in \{\chi, \overline{\chi}, D, \overline{D}\}.$$

If in (1.21) (β,α) is substituted by $(\beta,\overline{\alpha})$, $(\overline{\beta},\alpha)$ or $(\overline{\beta},\overline{\alpha})$, then $K^{x}_{y\,\beta\overline{\alpha}}$, $K^{x}_{y\,\overline{\beta}\alpha}$ and $K^{x}_{y\,\overline{\beta\alpha}}$ are obtained.

The components of the tensor P are given by

(1.23) $$P^{x}_{y\,\beta A} = \partial_{A} F^{x}_{y\,\beta} - C^{x}_{y\,A|\beta} - C^{x}_{y\,\overline{E}} F^{\overline{E}}_{A\,\beta} + C^{x}_{y\,E}(\partial_{A} N^{E}_{\beta} - F^{E}_{A\beta})$$

(1.24) $$P^{x}_{y\,\overline{\beta A}} = \partial_{\overline{A}} F^{x}_{y\,\overline{\beta}} - C^{x}_{y\,\overline{A}|\overline{\beta}} - C^{x}_{y\,E} F^{E}_{\overline{A}\,\overline{\beta}} + C^{x}_{y\,\overline{E}}(\partial_{\overline{A}} N^{\overline{E}}_{\overline{\beta}} - F^{\overline{E}}_{\overline{A}\,\overline{\beta}})$$

$$x \in \{\delta, \overline{\delta}, C, \overline{C}\} \qquad y \in \{\chi, \overline{\chi}, D, \overline{D}\}.$$

(1.25)
$$\begin{aligned} C^{x}_{y\,A|\beta} = & \delta_{\beta} C^{x}_{y\,A} + C^{\gamma}_{y\,A} F^{x}_{\gamma\,\beta} + C^{\gamma}_{y\,A} F^{x}_{\overline{\gamma}\beta} + C^{E}_{y\,A} F^{x}_{E\,\beta} + C^{\overline{E}}_{y\,A} F^{x}_{\overline{E}\,\beta} \\ & - C^{x}_{\gamma\,A} F^{\gamma}_{y\,\beta} - C^{x}_{\overline{\gamma}\,A} F^{\overline{\gamma}}_{y\,\beta} - C^{x}_{E\,A} F^{E}_{y\,\beta} - C^{x}_{\overline{E}\,A} F^{\overline{E}}_{y\,\beta} \\ & - C^{x}_{y\,E} F^{E}_{A\,\beta} - C^{x}_{y\,\overline{E}} F^{\overline{E}}_{A\,\beta}. \end{aligned}$$

If in (1.23) A is substituded by $\overline{A}$ $(\delta_{\overline{A}} N^{E}_{\beta} = 0)$we obtain $P^{x}_{y\,\beta\overline{A}}$ and if in (1.24) $\overline{A}$ is substituded by A $(\delta_{A} N^{\overline{E}}_{\overline{\beta}} = 0)$ we obtain $P^{x}_{y\,\overline{\beta}A}$. $C^{x}_{y\,A|\overline{\beta}}$, $C^{x}_{y\,\overline{A}|\beta}$, $C^{x}_{y\,\overline{A}|\overline{\beta}}$ can be obtained from (1.25) if $A|\beta$is substituded by $A|\overline{\beta}$, $\overline{A}|\beta$ and $\overline{A}|\overline{\beta}$ respectively. All terms in the tensors $P^x_{y\,\beta A}$, $P^x_{y\,\overline{\beta}\overline{A}}$, $P^x_{y\,\overline{\beta}A}$ and $P^x_{y\,\beta\overline{A}}$ are tensors, because all C's and $F^{E}_{\overline{A}\beta}$, $F^{\overline{E}}_{A\beta}$, $F^{\overline{E}}_{\overline{A}\beta}$, $F^{E}_{A\overline{\beta}}$, $F^{E}_{\overline{A}\overline{\beta}}$, $F^{\overline{E}}_{A\overline{\beta}}$ are tensors, but

$$\delta_A N^E_\beta - F^E_{A\beta} \qquad \text{and} \qquad \delta_{\overline{A}} N^{\overline{E}}_{\overline{\beta}} - F^{\overline{E}}_{\overline{A}\overline{\beta}}$$

are components of torsion tensors.

The components of the tensor S are given by

$$(1.26)\ S^x_{y\,BA} = (\partial_A C^x_{y\,B} + C^\gamma_{y\,A} C^x_{\gamma\,B} + C^{\overline{\gamma}}_{y\,A} C^x_{\overline{\gamma}\,B} + C^E_{y\,A} C^x_{E\,B} + C^{\overline{E}}_{y\,A} C^x_{\overline{E}\,B}) - (A,B).$$

If in (1.26) (B,A) is substituded by $(B,\overline{A})$, $(\overline{B},A)$ or $(\overline{B},\overline{A})$ we obtein $S^x_{y\,B\overline{A}}$, $S^x_{y\,\overline{B}A}$ and $S^x_{y\,\overline{B}\overline{A}}$ respectively.

Proof. Theorem 1.2 can be proved by direct calculation using the linearity of ∇ and the fact that vector fields X, Y, Z of $T(E)$ in the basis $\overline{B}$ have the form

$$(1.27)\quad \begin{aligned} &(a) \quad X = X^\alpha \delta_\alpha + X^{\overline{\alpha}} \delta_{\overline{\alpha}} + X^A \partial_A + X^{\overline{A}} \partial_{\overline{A}} \\ &(b) \quad Y = Y^\beta \delta_\beta + Y^{\overline{\beta}} \delta_{\overline{\beta}} + Y^B \partial_B + Y^{\overline{B}} \partial_{\overline{B}} \\ &(c) \quad Z = Z^\gamma \delta_\gamma + Z^{\overline{\gamma}} \delta_{\overline{\gamma}} + Z^C \partial_C + Z^{\overline{C}} \partial_{\overline{C}}. \end{aligned}$$

From (1.4) and (1.5) it can be seen that the curvature tensor in the basis B has 4 types of tensor R, 4 types P and 4 types S, i.e. in the basis B it exist $4^2 - 4$ different types of curvature tensors.

From (1.15)-(1.25) it can be seen that in the basis $\overline{B}$ there exist 64 types ot tensor R, 64 types of tensor P and 64 types of tensors S i.e in the basis $\overline{B}$ there are $4^4 - 4^3$ different types of the curvature tensors.

Theorem 1.3. *In the basis $\overline{B}$ $R(X,Y)Z$ can be decomposed in the following way:*

$$(1.28)\qquad R(X,Y)Z = R_\chi Z^\chi + R_{\overline{\chi}} Z^{\overline{\chi}} + R_D Z^D + R^{\overline{D}} Z_{\overline{D}},$$

where

$$(1.29)\quad \begin{aligned} R_\chi = \Big\{ 2^{-1} \Big[&(X^\alpha Y^\beta - Y^\alpha X^\beta) R(\delta_\alpha, \delta_\beta) + (X^\alpha Y^{\overline{\beta}} - Y^\alpha X^{\overline{\beta}}) R(\delta_\alpha, \delta_{\overline{\beta}}) + \\ &(X^{\overline{\alpha}} Y^\beta - Y^{\overline{\alpha}} X^\beta) R(\delta_{\overline{\alpha}}, \delta_\beta) + (X^{\overline{\alpha}} Y^{\overline{\beta}} - Y^{\overline{\alpha}} X^{\overline{\beta}}) R(\delta_{\overline{\alpha}}, \delta_{\overline{\beta}}) \Big] + \\ \Big[&(X^A Y^\beta - Y^A X^\beta) R(\partial_A, \delta_\beta) + (X^A Y^{\overline{\beta}} - Y^A X^{\overline{\beta}}) R(\partial_A, \delta_{\overline{\beta}}) + \\ &(X^{\overline{A}} Y^\beta - Y^{\overline{A}} X^\beta) R(\delta_{\overline{A}} \delta_\beta) + (X^{\overline{A}} Y^{\overline{\beta}} - Y^{\overline{A}} X^{\overline{\beta}}) R(\delta_{\overline{A}}, \delta_{\overline{\beta}}) \Big] + \Big\} \\ 2^{-1} \Big[&(X^A Y^B - Y^A X^B) R(\partial_A, \partial_B) + (X^A Y^{\overline{B}} - Y^A X^{\overline{B}}) R(\partial_A, \partial_{\overline{B}}) + \\ &(X^{\overline{A}} Y^B - Y^{\overline{A}} X^B) R(\partial_{\overline{A}}, \partial_B) + (X^{\overline{A}} Y^{\overline{B}} - Y^{\overline{A}} X^{\overline{B}}) R(\partial_{\overline{A}}, \partial_{\overline{B}}) \Big] \Big\} \delta_\chi \end{aligned}$$

$R_{\overline{\chi}}$, R_D, $R_{\overline{D}}$ can be obtained from (1.29) if in (1.29) the last term δ_χ is substituted by $\delta_{\overline{\chi}}$, ∂_D and $\partial_{\overline{D}}$ respectively.

We have

(1.30) $$R(\delta_\alpha,\delta_\beta)\delta_\chi = R_\chi{}^\delta{}_{\beta\alpha}\delta_\delta + R_\chi{}^{\overline{\delta}}{}_{\beta\alpha}\delta_{\overline{\delta}} + R_\chi{}^C{}_{\beta\alpha}\partial_C + R_\chi{}^{\overline{C}}{}_{\beta\alpha}\partial_{\overline{C}}\,.$$

$R(\delta_\alpha,\delta_{\overline{\beta}})\delta_\chi$, $R(\delta_{\overline{\alpha}},\delta_\beta)\delta_\chi$ and $R(\delta_{\overline{\alpha}},\delta_{\overline{\beta}})\delta_\chi$ can be obtained from (1.30) if in (1.30) (α,β) is substituted by $(\alpha,\overline{\beta})$, $(\overline{\alpha},\beta)$ and $(\overline{\alpha},\overline{\beta})$ respectively. The curvature tensors R are determined by (1.20)-(1.22).

Further we have

(1.31) $$R(\partial_A,\delta_\beta)\delta_\chi = P_\chi{}^\delta{}_{\beta A}\delta_\delta + P_\chi{}^{\overline{\delta}}{}_{\beta A}\delta_{\overline{\delta}} + P_\chi{}^C{}_{\beta A}\partial_C + P_\chi{}^{\overline{C}}{}_{\beta A}\partial_{\overline{C}}\,.$$

$R(\partial_A,\delta_{\overline{\beta}})\delta_\chi$, $R(\partial_{\overline{A}},\delta_\beta)\delta_\chi$, $R(\partial_{\overline{A}},\delta_{\overline{\beta}})\delta_\chi$ can be obtained from (1.31) if (A,β) is substituted by $(A,\overline{\beta})$, $(\overline{A},\beta)$ and $(\overline{A},\overline{\beta})$ respectively. The curvature tensors P are determined by (1.23)–(1.25). At last

(1.32) $$R(\partial_A,\partial_B)\delta_\chi = S_\chi{}^\delta{}_{BA}\delta_\delta + S_\chi{}^{\overline{\delta}}{}_{BA}\delta_{\overline{\delta}} + S_\chi{}^C{}_{BA}\partial_C + S_\chi{}^{\overline{C}}{}_{BA}\partial_{\overline{C}}\,.$$

$R(\partial_A,\partial_{\overline{B}})\delta_\chi$, $R(\partial_{\overline{A}},\partial_B)\delta_\chi$, $R(\partial_{\overline{A}},\partial_{\overline{B}})\delta_\chi$ can be obtained from (1.32) if (A,B) is substituted by $(A,\overline{B})$, $(\overline{A},B)$ and $(\overline{A},\overline{B})$ respectively. The curvature tensors S are determined by (1.26).

If in (1.30), (1.31) and (1.32) on the left hand side δ_χ is substituted by $\delta_{\overline{\chi}}$, δ_D, $\delta_{\overline{D}}$ then on the right hand side χ should be substituted by $\overline{\chi}$, D and $\overline{D}$ respectively.

The proof of Theorem 1.3 can be obtained by direct calculation using (1.27).

2. Relations between the curvature tensors of vector bundles and subbundles

The vector fields X, Y, Z in the bases B and $\overline{B}$ are given by (1.2) and (1.27) respectively. Under conditions (1.12) the relations between the coordinates of the vector fields in the bases B and $\overline{B}$ are given by ((3.20) in [4]).

(2.1)
$$\begin{aligned}
&(a)\quad X^i = B^i_\alpha X^\alpha + B^i_{\overline{\alpha}}X^{\overline{\alpha}} && X^a = B^a_A X^A + B^a_{\overline{A}}X^{\overline{A}}.\\
&(b)\quad Y^j = B^j_\beta Y^\beta + B^j_{\overline{\beta}}Y^{\overline{\beta}} && Y^b = B^b_B Y^B + B^b_{\overline{B}}Y^{\overline{B}}.\\
&(c)\quad Z^k = B^k_\gamma Y^\gamma + B^k_{\overline{\gamma}}X^{\overline{\gamma}} && Z^c = B^c_C Z^C + B^c_{\overline{C}}Z^{\overline{C}}.
\end{aligned}$$

From (1.28) it can be seen that R_χ is the factor beside Z^χ in the expression $R(X,Y)Z$. From (1.4), (1.5) and (1.27c) we have

(2.2)
$$\begin{aligned}
R_\chi = B^h_\chi[2^{-1}(R_h{}^k{}_{ji}\delta_k + R_h{}^c{}_{ji}\partial_c)(X^iY^j - Y^iX^j)+\\
(P_h{}^k{}_{ja}\delta_k + P_h{}^c{}_{ja}\partial_c)(X^aY^j - Y^aX^j)+\\
2^{-1}(S_h{}^k{}_{ba}\delta_k + S_h{}^c{}_{ba}\partial_c)(X^aY^b - Y^aX^b)]\,.
\end{aligned}$$

If (2.1) is substituted into (2.2) this becomes

(2.3) $$R_\chi = \tilde{R}^k\delta_k + \tilde{R}^c\partial_c\,,$$

where

(2.4)
$$\begin{aligned}
\tilde{R}^k = B^h_\chi\{2^{-1}[R_h{}^k{}_{(\beta,\alpha)} + R_h{}^k{}_{(\beta,\overline{\alpha})} + R_h{}^k{}_{(\overline{\beta},\alpha)} + R_h{}^k{}_{(\overline{\beta},\overline{\alpha})}]+\\
[P_h{}^k{}_{(\beta,A)} + P_h{}^k{}_{(\beta,\overline{A})} + P_h{}^k{}_{(\overline{\beta},A)} + P_h{}^k{}_{(\overline{\beta},\overline{A})}]+\\
2^{-1}[S_h{}^k{}_{(B,A)} + S_h{}^k{}_{(B,\overline{A})} + S_h{}^k{}_{(\overline{B},A)} + S_h{}^k{}_{(\overline{B},\overline{A})}]\}\,.
\end{aligned}$$

$\tilde{R}^c$ can be obtained from (2.4) if in both side of (2.4) k is substituted by c. We have

$$(2.5)\quad \begin{aligned} &(a)\quad R_h{}^k{}_{(\beta,\alpha)} = R_h{}^k{}_{ji} B^j_\beta B^i_\alpha (X^\alpha Y^\beta - Y^\alpha X^\beta) \\ &(b)\quad P_h{}^k{}_{(\beta,A)} = P_h{}^k{}_{ja} B^j_\beta B^a_A (X^A Y^\beta - Y^A X^\beta) \\ &(c)\quad S_h{}^k{}_{(B,A)} = S_h{}^k{}_{ba} B^b_B B^a_A (X^A Y^B - Y^A X^B) \end{aligned}$$

Remark 1. $R_h{}^k{}_{(\alpha,\overline{\beta})}$, $R_h{}^k{}_{(\overline{\alpha},\beta)}$, $R_h{}^k{}_{(\overline{\alpha},\overline{\beta})}$ can be obtained from (2.5a) if in both side of this equation (β,α) is substituted by $(\beta,\overline{\alpha})$, $(\overline{\beta},\alpha)$ and $(\overline{\beta},\overline{\alpha})$ respectively. $P_h{}^k{}_{(\beta,\overline{A})}$, $P_h{}^k{}_{(\overline{\beta},A)}$, $P_h{}^k{}_{(\overline{\beta},\overline{A})}$ can be obtained from (2.5b) if in foth side of this equation (β,A) is substituted by $(\beta,\overline{A})$, $(\overline{\beta},A)$ and $(\overline{\beta},\overline{A})$ respectively. $S_h{}^k{}_{(B,\overline{A})}$, $S_h{}^k{}_{(\overline{B},A)}$, $S_h{}^k{}_{(\overline{B},\overline{A})}$ can be obtained from (2.5c) if (B,A) is substituted by $(B,\overline{A})$, $(\overline{B},A)$ and $(\overline{B},\overline{A})$ respectively.

Comparing (1.29) with the equation obtained from (2.2) after substitution of (2.3), (2.4), (2.5) and (2.1) into (2.2) (and make the similar procedure for $R_{\overline{X}}$, R_D and $R_{\overline{D}}$ we obtain

Theorem 2.1. *The curvature tensors of the vector bundle and vector subbundles are connected by*

$$(2.6)\quad \begin{aligned} &(a)\quad R_h{}^k{}_{ji} B^h_X B^j_\beta B^i_\alpha = R_X{}^\gamma{}_{\beta\alpha} B^k_\gamma + R_X{}^{\overline{\gamma}}{}_{\beta\alpha} B^k_{\overline{\gamma}} \\ &(b)\quad R_h{}^c{}_{ji} B^h_X B^j_\beta B^i_\alpha = R_X{}^C{}_{\beta\alpha} B^c_C + R_X{}^{\overline{C}}{}_{\beta\alpha} B^c_{\overline{C}} \end{aligned}$$

$$(2.7)\quad \begin{aligned} &(a)\quad P_h{}^k{}_{ja} B^h_X B^j_\beta B^a_A = P_X{}^\gamma{}_{\beta A} B^k_\gamma + P_X{}^{\overline{\gamma}}{}_{\beta A} B^k_{\overline{\gamma}} \\ &(b)\quad P_h{}^c{}_{ja} B^h_X B^j_\beta B^a_A = P_X{}^C{}_{\beta A} B^c_C + P_X{}^{\overline{C}}{}_{\beta A} B^c_{\overline{C}} \end{aligned}$$

$$(2.8)\quad \begin{aligned} &(a)\quad S_h{}^k{}_{ba} B^h_X B^b_B B^a_A = S_X{}^\gamma{}_{BA} B^k_\gamma + S_X{}^{\overline{\gamma}}{}_{BA} B^k_{\overline{\gamma}} \\ &(b)\quad S_h{}^c{}_{ba} B^h_X B^b_B B^a_A = S_X{}^C{}_{BA} B^c_C + S_X{}^{\overline{C}}{}_{BA} B^c_{\overline{C}} \end{aligned}$$

$$(2.9)\quad \begin{aligned} &(a)\quad R_d{}^k{}_{ji} B^d_D B^j_\beta B^i_\alpha = R_D{}^\gamma{}_{\beta\alpha} B^k_\gamma + R_D{}^{\overline{\gamma}}{}_{\beta\alpha} B^k_{\overline{\gamma}} \\ &(b)\quad R_d{}^c{}_{ji} B^d_D B^j_\beta B^i_\alpha = R_D{}^C{}_{\beta\alpha} B^c_C + R_D{}^{\overline{C}}{}_{\beta\alpha} B^c_{\overline{C}} \end{aligned}$$

$$(2.10)\quad \begin{aligned} &(a)\quad P_d{}^k{}_{ja} B^d_D B^j_\beta B^a_A = P_D{}^\gamma{}_{\beta A} B^k_\gamma + P_D{}^{\overline{\gamma}}{}_{\beta A} B^k_{\overline{\gamma}} \\ &(b)\quad P_d{}^c{}_{ja} B^d_D B^j_\beta B^a_A = P_D{}^C{}_{\beta A} B^c_C + P_D{}^{\overline{C}}{}_{\beta A} B^c_{\overline{C}} \end{aligned}$$

$$(2.11)\quad \begin{aligned} &(a)\quad S_d{}^k{}_{ba} B^d_D B^b_B B^a_A = S_D{}^\gamma{}_{BA} B^k_\gamma + S_D{}^{\overline{\gamma}}{}_{BA} B^k_{\overline{\gamma}} \\ &(b)\quad S_d{}^c{}_{ba} B^d_D B^b_B B^a_A = S_D{}^C{}_{BA} B^c_C + S_D{}^{\overline{C}}{}_{BA} B^c_{\overline{C}} \end{aligned}$$

Remark 2. If in (2.6), (2.9), (β,α) is substituted by $(\beta,\overline{\alpha})$, $(\overline{\beta},\alpha)$ or $(\overline{\beta},\overline{\alpha})$, the obtained formulae are valid.

If in (2.7), (2.10), (β, A) is substituted by $(\beta, \overline{A})$, $(\overline{\beta}, A)$ or $(\overline{\beta}, \overline{A})$, the obtained formulae are valid.

If in (2.8), (2.11), (B, A) is substituted by $(B, \overline{A})$, $(\overline{B}, A)$ or $(\overline{B}, \overline{A})$, the obtained formulae are valid.

If in (2.6), (2.7) and (2.8) χ is substituted by $\overline{\chi}$ the obtained formulae are valid.

If in (2.9), (2.10) and (2.11) D is substituted by $\overline{D}$ the obtained formulae are valid.

The subbundle $\tilde{\xi}$ of the vector bundle ξ is given by the equations

$$\text{(2.12)} \quad \begin{array}{lll} (a) & x^k = x^k(u^1, \dots, u^{\tilde{n}}) & y^c = B^c_C(u) v^C \\ (b) & B^k_\gamma(u) = \partial x^k / \partial u^\gamma & B^c_C(u) = \partial y^c / \partial v^C \\ (c) & B^k_{\overline{\gamma}} = \partial x^k / \partial \overline{u}^{\overline{\gamma}} = 0 & B^c_{\overline{C}} = \partial y^c / \partial \overline{v}^{\overline{C}} = 0 \, . \end{array}$$

Theorem 2.2. *The curvature tensors of the vector bundle ξ and subbundle $\tilde{\xi}$ are related by*

$$\text{(2.6)}' \qquad (a) \quad R_h{}^k{}_{ji} B^h_\chi B^j_\beta B^i_\alpha = R_\chi{}^\gamma{}_{\alpha\beta} B^k_\gamma$$

$$\vdots$$

$$\text{(2.11)}' \qquad (b) \quad S_d{}^c{}_{ba} B^d_D B^b_B B^a_A = S_D{}^C{}_{BA} B^c_C \, .$$

Proof. The above formulae are obtained from (2.6)–(2.11) if in them (2.12c) are substituted. Then in formulae (2.6)-(2.11) the last terms are equal zero. Remark 1. and Remark 2. are valid for (2.6)′–(2.11)′.

$\tilde{\tilde{\xi}}$, the complementary subbundle to $\tilde{\xi}$ in the vector bundle ξ is determined by

$$\text{(2.13)} \quad \begin{array}{lll} (a) & x^k = x^k(\overline{u}^{n+1}, \dots, \overline{u}^n) & y^c = B^c_{\overline{C}}(\overline{u}) \overline{v}^{\overline{C}} \\ (b) & B^k_{\overline{\gamma}}(\overline{u}) = \partial x^k / \partial \overline{u}^{\overline{\gamma}} & B^c_{\overline{C}}(\overline{u}) = \partial y^c / \partial \overline{v}^{\overline{C}} \\ (c) & B^k_\gamma = \partial x^k / \partial u^\gamma = 0 & B^c_C = \partial y^c / \partial v^C = 0 \, . \end{array}$$

Theorem 2.3. *The curvature tensors of the vector bundle ξ and subbundle $\tilde{\tilde{\xi}}$ are connected in the following way*

$$\text{(2.6)}'' \qquad (a) \quad R_h{}^k{}_{ji} B^h_\chi B^j_\beta B^i_\alpha = R_\chi{}^{\overline{\gamma}}{}_{\beta\alpha} B^k_{\overline{\gamma}}$$

$$\vdots$$

$$\text{(2.11)}'' \qquad (b) \quad S_d{}^c{}_{ba} B^d_D B^b_B B^a_A = S_D{}^{\overline{C}}{}_{BA} B^c_{\overline{C}} \, .$$

Proof. The above formulae are obtained from (2.6)–(2.11) if in them (2.13c) are substituted. Then in all formulae (2.6)–(2.11) the first terms on the right hand side are equal to zero. Remark 1. and Remark 2. are valid for (2.6)″–(2.11)″.

Formulae (2.6)′–(2.11)′ are the relations between the curvature tensors of the vector bundle ξ and the family of subbundles $\tilde{\xi}$, which can be obtained from (1.11) by $\overline{u}^{\overline{\alpha}} = C^{\overline{\alpha}}$ and $\overline{v}^{\overline{A}} = C^{\overline{A}}$. Similarly for $\tilde{\tilde{\xi}}$.

3. The curvature tensors and the covariant derivatives of the vector fields

The Ricci identities for the vector bundle ξ (using the basic B of $T(E)$) can be obtained if in calculation of $R(X,Y)Z$ (1.20) from [4] is used. For $W = \nabla_Y Z$ we have

$$W = (Y^j Z^k_{|j} + Y^b Z^k\big|_b)\delta_k + (Y^j Z^c_{|j} + Y^b Z^c\big|_b)\partial_c$$

and

$$\nabla_X \nabla_Y Z = \nabla_X W = (X^i W^k_{|i} + X^a W^k\big|_a)\delta_k + (X^i W^c_{|i} + X^a W^c\big|_a)\partial_c. \tag{3.1}$$

We may obtain $\nabla_{[X,Y]}Z$ from

$$\nabla_{[X,Y]}Z = ([X,Y]^j Z^k_{|j} + [X,Y]^b Z^k\big|_b)\delta_k + ([X,Y]^j Z^c_{|j} + [X,Y]^b Z^c\big|_b)\partial_c, \tag{3.2}$$

where

$$\begin{aligned} [X,Y] =& [(X^i\delta_i Y^j - Y^i\delta_i X^j) + (X^a\partial_a Y^j - Y^b\partial_b X^j)]\delta_j + \\ & [(X^i\delta_i Y^b - Y^i\delta_i X^b) + (X^a\partial_a Y^b - Y^a\partial_a X^b) + \\ & 2^{-1}(X^iY^j - Y^iX^j)K_i{}^b{}_j + (X^iY^a - Y^iX^a)\partial_a N^b_i]\partial_b. \end{aligned} \tag{3.3}$$

Substituting (3.1) - (3.3) into (1.3) we obtain

$$\begin{aligned} & R(X,Y)Z = \\ & (X^iY^j - Y^iX^j)\{[Z^k_{|[j|i]} + F_{[j}{}^l{}_{i]}Z^k_{|l} + (F_{[j}{}^b{}_{i]} + 2^{-1}K_j{}^b{}_i)Z^k\big|_b]\delta_k + \\ & \qquad [Z^c_{|[j|i]} + F_{[j}{}^l{}_{i]}Z^c_{|l} + (F_{[j}{}^b{}_{i]} + 2^{-1}K_j{}^b{}_i)Z^c\big|_b]\partial_c\} + \\ & (X^aY^j - Y^aX^j)\{[Z^k_{|j}\big|_a - Z^k\big|_{a|j} + (C_j{}^l{}_a - F_a{}^l{}_j)Z^k_{|l} + (C_j{}^b{}_a + K_a{}^b{}_j)Z^k\big|_b]\delta_k \\ & \qquad + [Z^c_{|j}\big|_a - Z^c\big|_{a|j} + (C_j{}^l{}_a - F_a{}^l{}_j)Z^c_{|l} + (C_j{}^b{}_a + K_a{}^b{}_j)Z^c\big|_b]\partial_c\} \\ & +(X^aY^b - Y^aX^b)\{[Z^k\big|_{[b}\big|_{a]} + C_{[b}{}^d{}_{a]}Z^k\big|_d + C_{[b}{}^j{}_{a]}Z^k_{|j}]\delta_k + \\ & \qquad [Z^c\big|_{[b}\big|_{a]} + C_{[b}{}^d{}_{a]}Z^c\big|_d + C_{[b}{}^j{}_{a]}Z^c_{|j}]\partial_c\}. \end{aligned} \tag{3.4}$$

Comparation of (3.4) with (1.5) gives

Theorem 3.1. *The Ricci identities for the vector bundle ξ with respect to the basis B of $T(E)$ are*

$$\begin{aligned} &(a)\quad Z^x_{|[j|i]} + F_{[j}{}^l{}_{i]}Z^x_{|l} + (F_{[j}{}^b{}_{i]} + 2^{-1}K_j{}^b{}_i)Z^x\big|_b = 2^{-1}(R_h{}^x{}_{ji}Z^h + R_d{}^x{}_{ji}Z^d) \\ &(b)\quad Z^x_{|j}\big|_a - Z^x\big|_{a|j} + (C_j{}^l{}_a - F_a{}^l{}_j)Z^x_{|l} + (C_j{}^b{}_a - K_a{}^b{}_j)Z^x\big|_b = \\ &\qquad P_h{}^x{}_{ja}Z^h + P_d{}^x{}_{ja}Z^d \\ &(c)\quad Z^x\big|_{[b}\big|_{a]} + C_{[b}{}^d{}_{a]}Z^x\big|_d + C_{[b}{}^j{}_{a]}Z^x_{|j} = 2^{-1}(S_h{}^x{}_{ba}Z^h + S_d{}^x{}_{ba}Z^d) \\ &\qquad x \in \{k,c\}. \end{aligned} \tag{3.5}$$

Theorem 3.2. *The Ricci identities for the vector bundle ξ for the torsion free connection ∇ in the basis B of $T(E)$ are*

$$(3.6)\quad \begin{aligned} &(a)\quad Z^{x}_{|[j|i]} = 2^{-1}(R_{h}{}^{x}{}_{ji}Z^{h} + R_{d}{}^{x}{}_{ji}Z^{d}) \\ &(b)\quad Z^{x}_{|j}\big|_{a} - Z^{x}\big|_{a|j} = P_{h}{}^{x}{}_{ja}Z^{h} + P_{d}{}^{x}{}_{ja}Z^{d} \\ &(c)\quad Z^{x}\big|_{[b}\big|_{a]} = 2^{-1}(S_{h}{}^{x}{}_{ba}Z^{h} + S_{d}{}^{x}{}_{ba}Z^{d}) \\ &\qquad x\in\{k,c\}. \end{aligned}$$

Proof. Using (1.27) [4] and equating the coordinates of the torsion vector with zero in (3.5) we obtain (3.6). In (3.6) the curvature tensors are determined by (1.6) - (1.10) but the connection coefficients are torsion free.

4. The Ricci identities of the subbundle of vector bundle

Let X, Y, Z be vector fields in $T(E)$. In the basis $\overline{B}$ they are expressed by (1.27). Using Theorem 5.1. in [4] we can write

$$\nabla_Y Z = W = W^{\gamma}\delta_{\gamma} + W^{\overline{\gamma}}\delta_{\overline{\gamma}} + W^{C}\partial_{C} + W^{\overline{C}}\partial_{\overline{C}},$$

where

$$\begin{aligned} W^{x} =& Z^{x}_{|\beta}Y^{\beta} + Z^{x}_{|\overline{\beta}}Y^{\overline{\beta}} + Z^{x}\big|_{B}Y^{B} + Z^{x}\big|_{\overline{B}}Y^{\overline{B}} \\ & x\in\{\gamma,\overline{\gamma},C,\overline{C}\}. \end{aligned}$$

On the similar way we obtain

$$(4.1)\quad \nabla_X\nabla_Y Z = \nabla_X W = U = U^{\delta}\delta_{\delta} + U^{\overline{\delta}}\delta_{\overline{\delta}} + U^{C}\delta_{C} + U^{\overline{C}}\delta_{\overline{C}},$$

where

$$(4.2)\quad U^{x} = W^{x}_{|\alpha}X^{\alpha} + W^{x}_{|\overline{\alpha}}X^{\overline{\alpha}} + W^{x}\big|_{A}X^{A} + W^{x}\big|_{\overline{A}}X^{\overline{A}}$$

$$(4.3)\quad \begin{aligned} W^{x}_{|\alpha} =& Z^{x}_{|\beta|\alpha}Y^{\beta} + Z^{x}_{|\beta}Y^{\beta}_{|\alpha} + Z^{x}_{|\overline{\beta}|\alpha}Y^{\overline{\beta}} + Z^{x}_{|\overline{\beta}}Y^{\overline{\beta}}_{|\alpha} + \\ & Z^{x}\big|_{B|\alpha}Y^{B} + Z^{x}\big|_{B}Y^{B}_{|\alpha} + Z^{x}\big|_{\overline{B}|\alpha}Y^{\overline{B}} + Z^{x}\big|_{\overline{B}}Y^{\overline{B}}_{|\alpha}, \\ & x\in\{\delta,\overline{\delta},C,\overline{C}\}. \end{aligned}$$

If in (4.3) $|\alpha$ is substituted by $|\overline{\alpha}$, $|A$, $|\overline{A}$ we obtain $W^{x}_{|\overline{\alpha}}$, $W^{x}_{|A}$, $W^{x}_{|\overline{A}}$ respectively. On the other side, under condition (1.12) we have

$$(4.4)\quad \begin{aligned} [X,Y] =& ((XY^{\beta})\delta_{\beta} + (XY^{\overline{\beta}})\delta_{\overline{\beta}} + (XY^{B})\partial_{B} + (XY^{\overline{B}})\partial_{\overline{B}}) \\ & -((YX^{\alpha})\delta_{\alpha} + (YX^{\overline{\alpha}})\delta_{\overline{\alpha}} + (YX^{A})\partial_{A} + (YX^{\overline{A}})\partial_{\overline{A}}) + B, \end{aligned}$$

where

$$\begin{aligned}B =& X^\alpha Y^\beta K_{\alpha}{}^{B}{}_{\beta}\partial_B + X^{\overline{\alpha}} Y^{\overline{\beta}} K_{\overline{\alpha}}{}^{\overline{B}}{}_{\overline{\beta}}\partial_{\overline{B}}\\ &-X^A Y^\beta(\partial_A N^B_\beta)\partial_B - X^{\overline{A}} Y^{\overline{\beta}}(\partial_{\overline{A}} N^{\overline{B}}_{\overline{\beta}})\partial_{\overline{B}}\\ &+X^\alpha Y^B(\partial_B N^A_\alpha)\partial_A + X^{\overline{\alpha}} Y^{\overline{B}}(\partial_{\overline{B}} N^{\overline{A}}_{\overline{\alpha}})\partial_{\overline{A}}.\end{aligned}$$

After very long calculation we obtain

$$R(X,Y)Z = R^\delta \delta_\delta + R^{\overline{\delta}}\delta_{\overline{\delta}} + R^C\delta_C + R^{\overline{C}}\delta_{\overline{C}}, \tag{4.5}$$

where

$$\begin{aligned}R^x =& 2^{-1}\big(R^x(\beta,\alpha) + R^x(\beta,\overline{\alpha}) + R^x(\overline{\beta},\alpha) + R^x(\overline{\beta},\overline{\alpha})\big)+\\ &\big(R^x(\beta,A) + R^x(\beta,\overline{A}) + R^x(\overline{\beta},A) + R^x(\overline{\beta},\overline{A})\big)+\\ &2^{-1}\big(R^x(B,A) + R^x(B,\overline{A}) + R^x(\overline{B},A) + R^x(\overline{B},\overline{A})\big),\\ &x \in \{\delta,\overline{\delta},C,\overline{C}\}\end{aligned} \tag{4.6}$$

$$\begin{aligned}R^x(\beta,\alpha) =& \big(Z^x_{|\beta|\alpha} - Z^x_{|\alpha|\beta} + Z^x_{|\gamma} T_{\beta}{}^{\gamma}{}_{\alpha} + Z^x_{|\overline{\gamma}} T_{\beta}{}^{\overline{\gamma}}{}_{\alpha}+\\ &Z^x\big|_D T_{\beta}{}^{D}{}_{\alpha} + Z^x\big|_{\overline{D}} T_{\beta}{}^{\overline{D}}{}_{\alpha}\big)(X^\alpha Y^\beta - Y^\alpha X^\beta),\end{aligned} \tag{4.7}$$

($R^x(\beta,\overline{\alpha})$, $R^x(\overline{\beta},\alpha)$, $R^x(\overline{\beta},\overline{\alpha})$ are obtained if in (4.7) (β,α) is substituted by $(\beta,\overline{\alpha})$, $(\overline{\beta},\alpha)$ and $(\overline{\beta},\overline{\alpha})$ respectively),

$$\begin{aligned}R^x(\beta,A) =& \big(Z^x_{|\beta}\big|_A - Z^x\big|_{A|\beta} + Z^x_{|\gamma} T_{\beta}{}^{\gamma}{}_{A} + Z^x_{|\overline{\gamma}} T_{\beta}{}^{\overline{\gamma}}{}_{A}+\\ &Z^x\big|_D T_{\beta}{}^{D}{}_{A} + Z^x\big|_{\overline{D}} T_{\beta}{}^{\overline{D}}{}_{A}\big)(X^A Y^\beta - Y^A X^\beta),\end{aligned} \tag{4.8}$$

($R^x(\beta,\overline{A})$, $R^x(\overline{\beta},A)$, $R^x(\overline{\beta},\overline{A})$ are obtained if in (4.8) (β,A) is substituted by $(\beta,\overline{A})$, $(\overline{\beta},A)$ and $(\overline{\beta},\overline{A})$ respectively),

$$\begin{aligned}R^x(B,A) =& \big(Z^x\big|_B\big|_A - Z^x\big|_A\big|_B + Z^x_{|\gamma} T_{B}{}^{\gamma}{}_{A} + Z^x_{|\overline{\gamma}} T_{B}{}^{\overline{\gamma}}{}_{A}+\\ &Z^x\big|_D T_{B}{}^{D}{}_{A} + Z^x\big|_{\overline{D}} T_{B}{}^{\overline{D}}{}_{A}\big)(X^A Y^B - Y^A X^B).\end{aligned} \tag{4.9}$$

($R^x(B,\overline{A})$, $R^x(\overline{B},A)$, $R^x(\overline{B},\overline{A})$ are obtained if in (4.9) (B,A) is substituted by $(B,\overline{A})$, $(\overline{B},A)$ and $(\overline{B},\overline{A})$ respectively).

In (4.7) - (4.9) T is the torsion tensor and its components are determined by (6.1) - (6.4) in [4].

Comparation of (1.16) - (1.19) with (4.6) - (4.9) gives

Theorem 4.1. *The Ricci equations expressed in the basis $\overline{B}$ of $T(E)$ (under condition (1.12)) are*

$$\begin{aligned}&Z^x_{|\beta|\alpha} - Z^x_{|\alpha|\beta} + Z^x_{|\gamma} T_{\beta}{}^{\gamma}{}_{\alpha} + Z^x_{|\overline{\gamma}} T_{\beta}{}^{\overline{\gamma}}{}_{\alpha} + Z^x\big|_D T_{\beta}{}^{D}{}_{\alpha} + Z^x\big|_{\overline{D}} T_{\beta}{}^{\overline{D}}{}_{\alpha}\\ &= R_{\chi}{}^{x}{}_{\beta\alpha} Z^\chi + R_{\overline{\chi}}{}^{x}{}_{\beta\alpha} Z^{\overline{\chi}} + R_{E}{}^{x}{}_{\beta\alpha} Z^E + R_{\overline{E}}{}^{x}{}_{\beta\alpha} Z^{\overline{E}}.\end{aligned} \tag{4.10}$$

(4.10) *is valid if* (β,α) *is substituted by* $(\beta,\overline{\alpha})$ *or* $(\overline{\beta},\alpha)$ *or* $(\overline{\beta},\overline{\alpha})$.

$$\begin{aligned}(4.11)\qquad & Z^x_{|\beta}\big|_A - Z^x\big|_{A|\beta} + Z^x_{|\gamma}T_\beta{}^\gamma{}_A + Z^x_{|\overline{\gamma}}T_\beta{}^{\overline{\gamma}}{}_A + Z^x\big|_D T_\beta{}^D{}_A + Z^x\big|_{\overline{D}}T_\beta{}^{\overline{D}}{}_A \\ & = P_\chi{}^x{}_{\beta A}Z^\chi + P_{\overline{\chi}}{}^x{}_{\beta A}Z^{\overline{\chi}} + P_E{}^x{}_{\beta A}Z^E + P_{\overline{E}}{}^x{}_{\beta A}Z^{\overline{E}}.\end{aligned}$$

(4.11) is valid if (β,A) *is substituted by* $(\beta,\overline{A})$ *or* $(\overline{\beta},A)$ *or* $(\overline{\beta},\overline{A})$.

$$\begin{aligned}(4.12)\qquad & Z^x\big|_B\big|_A - Z^x\big|_A\big|_B + Z^x_{|\gamma}T_B{}^\gamma{}_A + Z^x_{|\overline{\gamma}}T_B{}^{\overline{\gamma}}{}_A + Z^x\big|_D T_B{}^D{}_A + Z^x\big|_{\overline{D}}T_B{}^{\overline{D}}{}_A \\ & = S_\chi{}^x{}_{BA}Z^\chi + S_{\overline{\chi}}{}^x{}_{BA}Z^{\overline{\chi}} + S_E{}^x{}_{BA}Z^E + S_{\overline{E}}{}^x{}_{BA}Z^{\overline{E}}.\end{aligned}$$

(4.12) *is valid if* (B,A) *is substituted by* $(B,\overline{A})$ *or* $(\overline{B},A)$ *or* $(\overline{B},\overline{A})$.
In formulae (4.10) - (4.12) $x\in\{\delta,\overline{\delta},C,\overline{C}\}$.

The direct consequence of Theorem 4.1. is

Theorem 4.2. *For the torsion free connection* ∇ *defined by* (5.1) *in* [4] *the Ricci equations have the form*

$$(4.10')\qquad Z^x_{|\beta|\alpha} - Z^x_{|\alpha|\beta} = R_\chi{}^x{}_{\beta\alpha}Z^\chi + R_{\overline{\chi}}{}^x{}_{\beta\alpha}Z^{\overline{\chi}} + R_E{}^x{}_{\beta\alpha}Z^E + R_{\overline{E}}{}^x{}_{\beta\alpha}Z^{\overline{E}}.$$

$$(4.11')\qquad Z^x_{|\beta}\big|_A - Z^x\big|_{A|\beta} = P_\chi{}^x{}_{\beta A}Z^\chi + P_{\overline{\chi}}{}^x{}_{\beta A}Z^{\overline{\chi}} + P_E{}^x{}_{\beta A}Z^E + P_{\overline{E}}{}^x{}_{\beta A}Z^{\overline{E}}.$$

$$(4.12')\qquad Z^x\big|_B\big|_A - Z^x\big|_A\big|_B = S_\chi{}^x{}_{BA}Z^\chi + S_{\overline{\chi}}{}^x{}_{BA}Z^{\overline{\chi}} + S_E{}^x{}_{BA}Z^E + S_{\overline{E}}{}^x{}_{BA}Z^{\overline{E}}.$$

$x\in\{\delta,\overline{\delta},C,\overline{C}\}$.

(4.10′) - (4.12′) *is valid if one or two indices from the set* $\{\alpha,\beta,A,B\}$ *are overlined on the both sides of equations. In these formulae the curvature tensors are determined by* (1.20) - (1.26), *but the connection ceofficients are torsion free.*

Compareing (3.4) with (4.5) - (4.9) and using (1.14) and (2.1) we get

Theorem 4.3. *The covariant derivatives of the vector field* $Z\in T(E)$ *in the bases* B *and* B' *are related by*

$$(4.13)\qquad (Z^k_{|j|i} - Z^k_{|i|j} + T_j{}^l{}_i Z^k_{|l} + T_j{}^d{}_i Z^k\big|_d)B^j_\beta B^i_\alpha = \overline{R}^\delta(\beta,\alpha)B^k_\delta + \overline{R}^{\overline{\delta}}(\beta,\alpha)B^k_{\overline{\delta}}$$

$$(4.14)\qquad (Z^c_{|j|i} - Z^c_{|i|j} + T_j{}^l{}_i Z^c_{|l} + T_j{}^d{}_i Z^c\big|_d)B^j_\beta B^i_\alpha = \overline{R}^C(\beta,\alpha)B^c_C + \overline{R}^{\overline{C}}(\beta,\alpha)B^c_{\overline{C}}$$

$$(4.15)\qquad (Z^k_{|j}\big|_a - Z^k\big|_{a|j} + T_j{}^l{}_a Z^k_{|l} + T_j{}^d{}_a Z^k\big|_d)B^j_\beta B^a_A = \overline{R}^\delta(\beta,A)B^k_\delta + \overline{R}^{\overline{\delta}}(\beta,A)B^k_{\overline{\delta}}$$

$$(4.16)\quad \left(Z^c_{|j}\big|_a - Z^c\big|_{a|j} + T_j{}^l{}_a Z^c_{|l} + T_j{}^d{}_a Z^c\big|_d\right) B^j_\beta B^a_A = \overline{R}^C(\beta, A) B^c_C + \overline{R}^{\overline{C}}(\beta, A) B^c_{\overline{C}}$$

$$(4.17)\quad \left(Z^k\big|_b\big|_a - Z^k\big|_a\big|_b + T_b{}^l{}_a Z^k_{|l} + T_b{}^d{}_a Z^k\big|_d\right) B^b_B B^a_A = \overline{R}^\delta(B, A) B^k_\delta + \overline{R}^{\overline{\delta}}(B, A) B^k_{\overline{\delta}}$$

$$(4.18)\quad \left(Z^c\big|_b\big|_a - Z^c\big|_a\big|_b + T_b{}^l{}_a Z^c_{|l} + T_b{}^d{}_a Z^c\big|_d\right) B^b_B B^a_A = \overline{R}^C(B, A) B^c_C + \overline{R}^{\overline{C}}(B, A) B^c_{\overline{C}},$$

where $\overline{R}^x(\beta,\alpha)$, $\overline{R}^x(\beta, A)$, $\overline{R}^x(B, A)$, $x \in \{\delta, \overline{\delta}, C, \overline{C}\}$ *are determined by* (4.7), (4.8), (4.9) *but without the factors* $(X^\alpha Y^\beta - Y^\alpha X^\beta)$, $(X^A Y^\beta - Y^A X^\beta)$, $(X^A Y^B - Y^A X^B)$, *respectively.*

Formulae (4.13) - (4.18) *are valid if in them one or two indices from the set* $\{\alpha, \beta, A, B\}$ *are overlined on the both sides of the equations. The explicit form of* T*'s appeared in these formulae are given in* (3.4) *and also in* (3.5).

Using the above remarks we can (4.13) write in the form

$$\begin{aligned}
&[Z^k_{|j|i} - Z^k_{|i|j} + (F_j{}^l{}_i - F_i{}^l{}_j) Z^k_{|l} + (F_j{}^b{}_i - F_i{}^b{}_j + K_j{}^b{}_i) Z^k\big|_b] B^j_\beta B^i_\alpha = \\
&(Z^\delta_{|\beta|\alpha} - Z^\delta_{|\alpha|\beta} + T_\beta{}^\gamma{}_\alpha Z^\delta_{|\gamma} + T_\beta{}^{\overline{\gamma}}{}_\alpha Z^\delta_{|\overline{\gamma}} + T_\beta{}^D{}_\alpha Z^\delta\big|_D + T_\beta{}^{\overline{D}}{}_\alpha Z^\delta\big|_{\overline{D}}) B^k_\delta + \\
&(Z^{\overline{\delta}}_{|\beta|\alpha} - Z^{\overline{\delta}}_{|\alpha|\beta} + T_\beta{}^\gamma{}_\alpha Z^{\overline{\delta}}_{|\gamma} + T_\beta{}^{\overline{\gamma}}{}_\alpha Z^{\overline{\delta}}_{|\overline{\gamma}} + T_\beta{}^D{}_\alpha Z^{\overline{\delta}}\big|_D + T_\beta{}^{\overline{D}}{}_\alpha Z^{\overline{\delta}}\big|_{\overline{D}}) B^k_{\overline{\delta}}.
\end{aligned}$$

The other formulae can be obtained in the similar way.

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