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HEMISYMMORPHIC AND ASYMMORPHIC SPACE GROUPS OF SIMPLE AND MULTIPLE COLORED ANTISYMMETRY

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Abstract. *For all nontrivial cases of assigning to points of three-dimensional Euclidean space colored by p colors signs + or -, survey of complete derivation of junior hemisymmorphic and asymmorphic groups of colored simple and multiple antisymmetry of different patterns, is given.*

I. The idea of colored antisymmetry of different patterns (simple colored antisymmetry or (p,2)-symmetry [2,4]) and l-multiple colored antisymmetry ((p,2,...,2)-symmetry [2,4], or $(p,2^1)$- symmetry [5]), is the synthesis of colored symmetry ((p)-symmetry [2]) and antisymmetry of diferent patterns [3] ((2,2,...,2)-symmetry where the number 2 is repeated l times, or shortly (2^l)-symmetry [5]).

Till now, beginning from the two-dimensional plane groups and their subgroups, the three-dimensional point, line and layer groups, as well as the symmorphic space groups of $(p,2^1)$-symmetry are examinated. For completing the scheme of crystallographic groups of colored multiple antisymmetry we need to derive only the hemisymmorphic and asymmorphic space groups $G_3^{1,p}$ of $(p,2^1)$- symmetry.

The purpose of the present work is to give a survey of the complete derivation of hemisymmorphic and asymmorphic space groups $G_3^{1,p}$ of $(p,2^1)$-symmetry for all nontrivial cases of assigning to points of three-dimensional Euclidean space colored by p colors l signs + or -.

1. Hemisymmorphic space groups of colored simple and multiple antisymmetry

II. Refering the reader for the necessary explanations to the paper [5], let us mention that the present paper is its logical continuation. In [5] they are given the basic notions of the theory of $(p,2^1)$-symmetry and the survey of complete derivation of junior symmorphic groups of $(p,2^1)$-symmetry $G_3^{1,p}$ of theM^m- type for all nontrivial cases of assigning to points of three- dimensional space colored by p colors l signs + or -.

According to [5] the mentioned groups can be derived from 238 p-generating hemisymmorphic groups ([2], Appendix P2) by Shubnikov-Zamorzaev method: by

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replacing l or more generators of a p-generating group by the corresponding transformations of colored antisymmetry of different patterns, among which they are l independent. After this, in the obtained groups of $(p,2^l)$- symmetry we need to recognize the equal ones and eliminate groups containing $(p,2^l)$-identity transformations; the same can be done by the use of antisymmetric characteristics (AC) of p-generating hemisymmorphic groups.

Since every initial hemisymmorphic group G_3^p is given by a finite set of generators, for given p and l the number of different possible replacements of the generators of p-generating group by the corresponding $(p,2^l)$-symmetry transformations is limited. Hence, the number of all junior hemisymmorphic space groups $G_3^{l,p}$ is finite as well.

By use of two proposed independent derivation methods for generalizing 238 p-generating hemisymmorphic groups to colored multiple antisymmetry, the following numbers Nm(p) of different groups of the p-M^m-type are obtained: $N1^{(3)}=115$, $N2^{(3)}=672$, $N3^{(3)}=4368$, $N4^{(3)}=22680$, $Nl^{(3)}=0$ for $l \geq 5$; $N1^{(4)}=596$, $N2^{(4)}=4290$, $N3^{(4)}=31626$, $N4^{(4)}=181440$, $Nl^{(4)}=0$ for $l \geq 5$; $N1^{(6)}=787$, $N2^{(6)}=6384$, $N3^{(6)}=53256$, $N4^{(6)}=340200$, $Nl^{(6)}=0$ for $l \geq 5$.

The numbers $(Nm^{(p)})$ of hemisymmorphic space groups of the complete $(p,2^l)$-symmetry of the type p-M^m are: $(Nl^{(3)})=Nl^{(3)}$, $(N1^{(4)})=500$, $(N2^{(4)})=2790$, $(N3^{(4)})=12096$, $(Nl^{(4)})=0$ for $l \geq 4$; $(N1^{(6)})=672$, $(N2^{(6)})=4368$, $(N3^{(6)})=22680$, $(Nl^{(6)})=0$ for $l \geq 4$.

III. In order to explain the method by which the mentioned results are obtained, we are giving the example of the derivation of $(p,2^l)$-symmetry groups of the type p-M^m directly from the hemisymmorphic p-generating groups of the family 2h and the complete survay of complete and uncomplete $(p,2^l)$-symmetry groups derived from them.

Hemisymmorphic space group 2h (or Bb, or {a,b,(a+c)/2}(b/2m)) possesses the reduced AC {b/2m,(a+c)/2b/2m} [6], and defines five junior p-symmetry groups for p=3,4,6: 1) $\{a,b^{(3)},(a+c)/2\}(b/2m(-3))$, 2) $\{a,b^{(2)},(a+c)/2\}(b/2m^{(4)})$, 3) $\{a^{(2)},b,(a+c)/2^{(4)}\}(b/2m)$, 4) $\{a,b^{(3)},(a+c)/2\}(b/2m^{(6)})$, 5) $\{a^{(3)},b,(a+c)/2^{(6)}\}(b/2m)$. According to [5], it is possible to conclude that the AC of the (3)-symmetry group 1) is the same as the AC of its generating classical Fedorov symmetry group 2h. Hence, the derivation of $(3,2^l)$-symmetry groups of the type 3-M^m is identical to the derivation of (2^l)-symmetry groups from the classical symmetry group 2h, resulting in two groups $\{a,b^{(3)},(a+c)/2\}(b/2\underline{m}^{(-3)})$ and $\{a,b^{(3)},(a+c)/\underline{2}\}(b/2m(-3))$ of (3,2)-symmetry of the type 3-M^1, three groups $\{a,b^{(3)},(a+c)/\underline{2}\}(b/2{}^*m(-3))$, $\{a,b^{(3)},{}^*(a+c)/2\}(b/2{}^*\underline{m}^{(-3)})$ and $\{a,b^{(3)},{}^*(a+c)/\underline{2}\}(b/2\underline{m}^{(-3)})$ of $(3,2^2)$-symmetry of the type 3-M^3, and no one group of $(3,2^l)$-symmetry of the type 3-M^l for $l \geq 3$. Analogously, AC remains the same in transition from classical to (4)-symmetry in the case of the group 2), resulting in the same derivation of $(4,2^l)$-symmetry groups of the type 4-M^m from the 4- generating group 2). From $\{a,b^{(2)},(a+c)/2\}(b/2m^{(4)})$ we derive two groups $\{a,b^{(2)},(a+c)/2\}(b/2\underline{m}^{(4)})$ and $\{a,b^{(2)},(a+c)/\underline{2}\}(b/2m^{(4)})$, the first of uncomplete and the second of complete (4,2)-symmetry of the type 4-M^1; three groups $\{a,b^{(2)},(a+c)/\underline{2}\}(b/2{}^*m^{(4)})$,

$\{a,b^{(2)},{}^*(a+c)/2\}(b/2\underline{m}^{(4)})$ and $\{a,b^{(2)},{}^*(a+c)/\underline{2}\}$ $(b/2\underline{m}^{(4)})$ of uncomplete $(4,2^2)$-symmetry of the type 4-M^2, and no one group of $(4,2^l)$-symmetry of the type 4-M^l for $l \geq 3$.

Since the reduced AC of the 4-generating group 3) is $\{(a+c)/2\}\{b/2m\}$, according to [6], the group in question belongs to the AC-isomorphism class XIX represented by the group 23s [6,7]. Hence, the number of the groups of $(4,2^l)$-symmetry which can be derived from the group $\{a^{(2)},b,(a+c)/2^{(4)}\}(b/2m)$ will be the same as the number of (2^l)-symmetry groups of the M^m-type derived from the group 23s. In this way, from the group $\{a^{(2)},b,(a+c)/2^{(4)}\}(b/2m)$ they are derived three groups $\{a^{(2)},b,(a+c)/2^{(4)}\}(b/2\underline{m})$, $\{a^{(2)},b,(a+c)/\underline{2}^{(4)}\}(b/2\underline{m})$ and $\{a^{(2)},b,(a+c)/\underline{2}^{(4)}\}$ $(b/2m)$ of the type 4-M^1, the first and second of complete, and the third of uncomplete $(4,2^2)$-symmetry; six groups $\{a^{(2)},b,(a+c)/\underline{2}^{(4)}\}(b/2{}^*m)$, $\{a^{(2)},b,(a+c)/\underline{2}^{(4)}\}$ $(b/2{}^*\underline{m})$, $\{a^{(2)},b,{}^*(a+c)/2^{(4)}\}$ $(b/2\underline{m})$, $\{a^{(2)},b,{}^*(a+c)/2^{(4)}\}$ $(b/2{}^*\underline{m})$, $\{a^{(2)},b,{}^*(a+c)/\underline{2}^{(4)}\}$ $(b/2\underline{m})$ and $\{a^{(2)},b,{}^*(a+c)/\underline{2}^{(4)}\}$ $(b/2{}^*m)$ of the type 4-M^2 of uncomplete $(4,2^2)$-symmetry, and no one group of $(4,2^l)$-symmetry of the type 4-M^l for $l \geq 3$.

Because AC of the group 4) coincides to AC of the group 2), from the 6-generating group $\{a,b^{(3)},(a+c)/2\}(b/2m^{(6)})$ we derive two groups $\{a,b^{(3)},(a+c)/\underline{2}\}$ $(b/2m^{(6)})$ and $\{a,b^{(3)},(a+c)/2\}(b/2\underline{m}^{(6)})$ of the type 6-M^1, the first of complete and the second of uncomplete $(6,2^2)$-symmetry; three groups $\{a,b^{(3)},(a+c)/\underline{2}\}$ $(b/2{}^*m^{(6)})$, $\{a,b^{(3)},{}^*(a+c)/\underline{2}\}(b/2{}^*m^{(6)})$ and $\{a,b^{(3)},{}^*(a+c)/\underline{2}\}(b/2\underline{m}^{(6)})$ of the type 6-M^2 of uncomplete $(6,2^2)$- symmetry, and no one group of $(6,2^l)$-symmetry of the type 6-M^l for $l \geq 3$.

Since AC of the group 5) coincides to AC of the group 3), from the 6-generating group $\{a^{(2)},b,(a+c)/2^{(6)}\}(b/2m)$ are derived three groups $\{a^{(2)},b,(a+c)/2^{(6)}\}$ $(b/2\underline{m})$, $\{a^{(2)},b,(a+c)/\underline{2}^{(6)}\}(b/2\underline{m})$ and $\{a^{(2)},b,(a+c)/\underline{2}^{(6)}\}(b/2m)$ of the type 6-M^1, the first and second of complete and the third of uncomplete (6,2)-symmetry; six groups $\{a^{(2)},b,(a+c)/\underline{2}^{(6)}\}(b/2{}^*m)$, $\{a^{(2)},b,(a+c)/\underline{2}^{(6)}\}$ $(b/2{}^*\underline{m})$, $\{a^{(2)},b,{}^*(a+c)/2^{(6)}\}(b/2\underline{m})$, $\{a^{(2)},b,{}^*(a+c)/2^{(6)}\}(b/2{}^*\underline{m})$, $\{a^{(2)},b,{}^*(a+c)/\underline{2}^{(6)}\}$ $(b/2\underline{m})$ and $\{a^{(2)},b,{}^*(a+c)/\underline{2}^{(6)}\}(b/2{}^*m)$ of the type 6-M^2 of uncomplete $(6,2^2)$-symmetry, and no one group of $(6,2^l)$-symmetry of the type 6-M^l for $l \geq 3$.

These examples illustrate that even the derivation of hemisymmorphic groups of $(p,2^1)$-symmetry of only one type p-M^1 is a large problem, but knowing the AC of every p-generating hemisymmorphic space group it is possible to find the number of $(p,2^1)$-symmetry groups of the type p-M^l for every l. Hence, for the complete enumeration of hemisymmorphic space groups of the $(p,2^1)$-symmetry of the type p-M^l we only need to know the AC of every p-generating group.

According to this is derived the list of all p-generating hemisymmorphic groups and their AC. They are distributed into the families ([2], Appendix P2), given by the symbol of Fedorov group, its AC and the number of the corresponding AC-isomorphism equivalence class. The same data about AC are given for all p-generating groups. Every hemisymmorphic Fedorov space group is given by three different symbols: Fedorov, international and Zamorzaev symbol [3,P1].

From the list obtained it is clear that except for only one, all AC of p-generating hemisymmorphic space groups are isomorphic to the already obtained and investigated AC [6,7], given in the partial catalogue of AC of the classical-symmetry Fedorov groups G_3 [7, Appendix]. The only exception is the AC of the 4- generating group 5) $\{a^{(2)},b^{(2)},c^{(2)}\}(2c/2m^{(4)})$ belonging to the family 5h, which is non isomorphic with any AC of 230 Fedorov groups. Because its AC is isomorphic to the AC 4.12 [8], we may use these data for this exceptional p-generating hemisymmorphic group.

Because from groups with isomorphic AC we derive the same number of $(p,2^1)$-symmetry groups of the M^m-type for l fixed, to solve the posed problem it is sufficient:

1) to distribute all p-generating hemisymmorphic groups in the equivalence classes according to the isomprphism of AC;

2) to find the number of classes obtained;

3) to find the number of p-generating groups belonging to each class;

4) to find the number of p-junior groups of $(p,2^1)$-symmetry groups derived from each hemisymmorphic p-symmetry group G_3^p for all possible values of l.

By forming the sets consisting of groups with isomorphic AC, all p-generating groups from the list obtained are distributed into the disjoint equivalence classes. As the representative of each class, except for the group 5h.5, can be used the corresponding Fedorov group G_3, for which the numbers of (2^l)- symmetry groups of the M^m-type derived from it are already computed for all the values of l [6,7]. Because groups with isomorphic AC posses the same properties and structure, we may conclude that from every p-generating group belonging to the same AC-isomorphism equivalence class will be derived the number of $(p,2^1)$-symmetry groups of the type p-M^m same as the number of (2^l)-symmetry groups of the M^m-type derived from the Fedorov group G^3 taken for the representative of equivalence class. All such data about Fedorov groups are given in [6,7], and for the exceptional group 5h.5 in [8]. Therefore, we have all relevant data necessary for solving the problem mentioned in the point 4) of the proposed method.

All data for the steps 1), 2) and 4) of the proposed method: number of the AC-isomorphism equivalence class of p-generating group G and the corresponding numbers Nl(G) of all p-junior groups of the $(p,2^1)$-symmetry which can be derived from G, are given in Table 1.

Table 1

G	N1(G)	N2(G)	N3(G)	N4(G)
IV	4	15	42	
VI	5	24	84	
XIII	11	126	1344	10080
XIV	9	108	1260	10080
XIX	3	6		
XX	7	42	168	
XXI	2	3		
XXII	8	75	714	5040

XXIII	15	210	2520	20160
XXIV	1			
XXVII	5	39	357	2520
5s.5	6	57	630	

Among hemisymmorphic (p,2^1)-symmetry groups of the type p- M^m, derived from p-generating groups by the use of the proposed method, there are groups of complete (p,2^1)-symmetry, as well as the groups of uncomplete (p,2^1)-symmetry. For p=3, as it is prooved in [5], all 3-junior (3,2^l)-symmetry groups are groups of complete (3,2^l)-symmetry, but for p=4,6 there occur also the groups of uncomplete (p,2^1)-symmetry. In Table 2 formed analogously to Table 1, are given the numbers (Nm(G)) of all complete (p,2^1)-symmetry groups of the type p-M^m which can be derived from every p-generating hemisymmorphic group G belonging to certain AC-isomorphism equivalence class for p=4,6 and l=1,2,3.

Table 2

G	(N1(G))	(N2(G))	(N3(G))
IV	3	6	
VI	4	12	
VIII	8	60	336
XIV	8	84	672
XIX	2		
XX	6	24	
XXI	1		
XXII	7	54	336
XXIII	14	168	1344
XXVII	4	27	168
XXVIII	2	3	
5h.5	5	42	336

In Table 3 is given the distribution of all 238 p-generating hemisymmorphic groups according to the AC-isomorphism equivalence classes they belong.

Table 3

G	p=3	p=4	p=6	p=4, 6	p=3, 4, 6
IV	1	4	4	9	8
VI	5	24	16	45	40
XIII	2	9	16	27	25
XIV			1	1	1
XIX	3	8	12	23	20
XX	5	35	46	86	81
XXI	3	3	3	9	6
XXII			1	1	1
XXIII		4	8	12	12
XXIV	6	4	6	16	10

XXVII	1	2	1	4	3
XXVIII	1	2	1	4	3
	27	96	115	238	211

From Table 1,2,3 we may calculate the numbers Nl(p) of all hemisymmorphic groups of $(p,2^1)$-symmetry of the type p-M^m, and the numbers (Nl(p)) of the hemisymmorphic groups of complete $(p,2^1)$-symmetry of the type p-M^m.

Multiplying the number of the groups belonging to certain AC-isomorphism equivalence class by the corresponding number Nm(G) or (Nm(G)), and adding the products obtained, we have the numbers Nm and (Nm) of hemisymmorphic space groups of $(p,2^1)$- symetry (p=3,4,6).

$N1=115^{(3)}+596^{(4)}+787^{(6)}=1498$
$(N1)=115^{(3)}+500^{(4)}+672^{(6)}=1287$
$N2=672^{(3)}+4290^{(4)}+6384^{(6)}=11346$
$(N2)=672^{(3)}+2790^{(4)}+4368^{(6)}=7830$
$N3=4368^{(3)}+31626^{(4)}+53256^{(6)}=89250$
$(N3)=4368^{(3)}+12096^{(4)}+22680^{(6)}=39144$
$N4=22680^{(3)}+181440^{(4)}+340200^{(6)}=544320$
$(N4)=22680^{(3)}=22680$

2. Asymmorphic space groups of colored simple and multiple antisymmetry

IV. The investigation of asymmorphic space groups of colored simple and multiple antisymmetry is the concluding chapter of the generalization of space groups of colored symmetry G_3^p (or p- symmetry [2]) still realized for the symmorphic [5] and hemisymmorphic groups , resulting in the groups of colored antisymmetry of different patterns- colored simple antisymmetry G31,p (or (p,2)-symmetry [4]) and colored l-multiple antisymmetry (or $(p,2^1)$-symmetry [5]).

Till now, beginning from the two-dimensional plane groups and their subgroups, the three-dimensional point, line and layer groups, the symmorphic and asymmorphic space groups of $(p,2^1)$- symmetry are examinated. For completing the scheme of crystallographic groups of colored multiple antisymmetry we need to derive only the asymmorphic space groups $G_3^{1,p}$ of $(p,2^1)$- symmetry.

The purpose of the present work is to give a survey of the complete derivation of asymmorphic space groups $G_3^{1,p}$ of $(p,2^1)$- symmetry for all nontrivial cases of assigning to points of three-dimensional Euclidean space colored by p colors l signs + or -.

V. In accordance with the theoretical background given in [5] all groups in question can be derived from the 263 p- generating groups listed in the Appendix P2 of the monograph [2] by Shubnikov-Zamorzaev method: by replacing l or more generators of a p-generating group by the corresponding transformations of colored antisymmetry of different patterns, among which they are l independent. After this, in the obtained groups of $(p,2^1)$- symmetry we need to recognize the equal ones and eliminate the groups containing (p,2^1)-identity transformations; the same can be done by the use of antisymmetric characteristics (AC) of p- generating asymmorphic groups. The AC method is already efficiently used for the analogous derivation of symmorphic [5] and hemisymmorphic colored multiple antisymmetry space groups.

From the symbols of p-generating groups G_3^p listed in [2, Appendix P2), we may conclude that every initial asymmorphic group G_3^p is given by a finite set of generators, for given p and l the number of different possible replacements of the generators of p-generating group by the corresponding (p,2^1)-symmetry transformations is limited. Hence, the number of all junior asymmorphic space groups $G_3^{1,p}$ of any family is finite as well.

By the parallel use of two proposed independent derivation methods, as the generalization of 263 p-generating asymmorphic groups to colored multiple antisymmetry, the following numbers Nm(p) of different groups of the p-M^m-type are obtained: N1$^{(3)}$=99, N2$^{(3)}$=340, N3$^{(3)}$=1393, N4$^{(3)}$=5040, Nl$^{(3)}$=0 for $l \geq 5$; N1$^{(4)}$=660, N2$^{(4)}$=4045, N3$^{(4)}$=26530, N4$^{(4)}$=141120, Nl$^{(4)}$=0 for $l \geq 5$; N1$^{(6)}$=439, N2$^{(6)}$=2413, N3$^{(6)}$=14791, N4$^{(6)}$=75600, Nl$^{(6)}$=0 for ≥ 5.

The numbers (Nm(p)) of asymmorphic space groups of the complete (p,2^1)-symmetry of the type p-M^m are: (Nl$^{(3)}$)=Nl$^{(3)}$, (N1$^{(4)}$)=533, (N2$^{(4)}$)=2446, (N3$^{(4)}$)=9408, (Nl$^{(4)}$)=0 for $l \geq 4$; (N1$^{(6)}$)=340, (N2$^{(6)}$)=1393, (N3$^{(6)}$)=5040, (Nl$^{(6)}$)=0 for $l \geq 4$.

VI. As the illustration of the generalization of 263 p- generating asymmorphic groups resulting in the mentioned numbers Nm(p) and (Nm(p)) we are giving few examples. In this order we will consider the examples of the derivation of (p,2^1)-symmetry groups of the type p-M^m directly from the p-generating groups of the family 30a, and the complete survay of complete and uncomplete (p,2^1)-symmetry groups derived.

Asymmorphic space group 30a (or P41, or {a,b,c}(c/44)) ([3], Appendix P1) possesses the reduced AC {c/44,ac/44} [6], and defines five junior p-symmetry groups for p=3,4,6 (1,Appendix P2): 1) {a,b,c$^{(3)}$}(c/44$^{(3)}$), 2) {a,b,c}(c/44$^{(4)}$), 3) {a$^{(2)}$,b$^{(2)}$,c}(c/44$^{(4)}$), 4) {a,b,c$^{(3)}$}(c/44(-6)), 5) {a$^{(2)}$,b$^{(2)}$,c$^{(3)}$}(c/44$^{(3)}$). It is clear that the AC of the (3)-symmetry group 1), constructed according to the rules given in [5], is the same as the AC of its generating classical Fedorov symmetry group 30a. Therefore, the derivation of (3,2^l)-symmetry groups of the type 3-M^m coincides with the derivation of (2^l)- symmetry groups from the classical symmetry group 30a, resulting in two groups {a,b,c$^{(3)}$}(c/4$\underline{4}^{(3)}$) and {$\underline{a}$, $\underline{b}$,c$^{(3)}$}(c/44$^{(3)}$) of (3,2)-symmetry of the type 3-M^1, three groups of (3,2^2)-symmetry of the type 3-M^3, {$\underline{a}$, $\underline{b}$,c$^{(3)}$}(c/4*4$^{(3)}$), {*a,*b,c$^{(3)}$}(c/4$\underline{4}^{(3)}$) and {*$\underline{a}$, *$\underline{b}$,c$^{(3)}$}(c/4$\underline{4}^{(3)}$) and no one group of (3,2^l)-symmetry of the type 3-M^l for $l \geq 3$. Analogously, AC remains the same in transition from classical to (4)-symmetry in the case of the groups 2), 3) resulting in the same derivation of (4,2^l)-symmetry groups of the type 4-M^m from the 4-generating groups 2), 3). From {a,b,c}(c/44$^{(4)}$) we derive two groups {a,b,c}(c/4$\underline{4}^{(4)}$) and {$\underline{a}$, $\underline{b}$,c}(c/44$^{(4)}$), the first of uncomplete and the second of complete (4,2)-symmetry of the type 4-M^1; three groups {$\underline{a}$, $\underline{b}$,c}(c/4*4$^{(4)}$), {*a,*b,c}(c/4$\underline{4}^{(4)}$) and {*$\underline{a}$,*$\underline{b}$,c}(c/4$\underline{4}^{(4)}$) of uncomplete $(4,2^2)$-symmetry of the type 4-M^2, and no one group of $(4,2^l)$-symmetry of the type 4-M^l for $l \geq 3$. From {$a^{(2)}$,$b^{(2)}$,c}(c/$44^{(4)}$) we derive two groups {$a^{(2)}$,$b^{(2)}$,c}(c/4$\underline{4}^{(4)}$) and {$\underline{a}^{(2)}$,$\underline{b}^{(2)}$,c}(c/$44^{(4)}$), the first of uncomplete and the second of complete (4,2)-symmetry of the type 4-M^1; three groups {$\underline{a}^{(2)}$,$\underline{b}^{(2)}$,c}(c/4*$4^{(4)}$), {*$a^{(2)}$,*$b^{(2)}$,c}(c/4$\underline{4}^{(4)}$) and {*$\underline{a}^{(2)}$,*$\underline{b}^{(2)}$,c}(c/4$\underline{4}^{(4)}$) of uncomplete (4,22)-symmetry of the type 4-M2, and no one group of $(4,2^l)$-symmetry of the type 4-M^l for $l \geq 3$.

Analogously, from the 6-generating group 4) we derive two groups {a,b,$c^{(3)}$}(c/4$\underline{4}^{(-6)}$) and {$\underline{a}$, $\underline{b}$,$c^{(3)}$}(c/$44^{(-6)}$), the first of uncomplete and the second of complete (6,2)-symmetry of the type 6-M^1; three groups {$\underline{a}$, $\underline{b}$,$c^{(3)}$}(c/4*4(-6)), {*a,*b,$c^{(3)}$}(c/4$\underline{4}^{(-6)}$) and {*$\underline{a}$, *$\underline{b}$,$c^{(3)}$}(c/$44^{(-6)}$) of uncomplete $(6,2^2)$-symmetry of the type 6-M^2, and no one group of $(4,2^l)$-symmetry of the type 6-M^l for $l \geq 3$.

It is possible to check that the reduced AC of the 6- generating group 5) is {a}{c/44}. According to [6,7], this group belongs to the AC-isomorphism class XIX represented by the group 23s. Hence, the number of the groups of $(6,2^l)$-symmetry which can be derived from the group {$a^{(2)}$,$b^{(2)}$,$c^{(3)}$}(c/$44^{(3)}$) will be the same as the number of (2^l)-symmetry groups of the M^m-type derived from the group 23s. In this way, from the group {$a^{(2)}$,$b^{(2)}$,$c^{(3)}$}(c/$44^{(3)}$) they are derived three groups {$\underline{a}^{(2)}$, $\underline{b}^{(2)}$,$c^{(3)}$}(c/$44^{(3)}$), {$\underline{a}^{(2)}$,$\underline{b}^{(2)}$,$c^{(3)}$}(c/4$\underline{4}^{(3)}$), {$a^{(2)}$,$b^{(2)}$,$c^{(3)}$}(c/4$\underline{4}^{(3)}$) of the type 6-M^1, the first and second of complete, and the third of uncomplete $(4,2^2)$-symmetry; six groups {$\underline{a}^{(2)}$,$\underline{b}^{(2)}$,$c^{(3)}$}(c/4*$4^{(3)}$), {$\underline{a}^{(2)}$,$\underline{b}^{(2)}$,$c^{(3)}$}(c/4 * $\underline{4}^{(3)}$), {*$a^{(2)}$,*$b^{(2)}$,$c^{(3)}$}(c/4$\underline{4}^{(3)}$), {*$a^{(2)}$,*$b^{(2)}$,$c^{(3)}$}(c/4 * $\underline{4}^{(3)}$), {*$\underline{a}^{(2)}$, *$\underline{b}^{(2)}$,$c^{(3)}$}(c/4$\underline{4}^{(3)}$) and {*$\underline{a}^{(2)}$, *$\underline{b}^{(2)}$,$c^{(3)}$}(c/4*$4^{(3)}$) of the type 6-M^2 of uncomplete $(6,2^2)$-symmetry, and no one group of $(6,2^l)$-symmetry of the type 6-M^l for $l \geq 3$.

These examples illustrate that the derivation of asymmorphic groups of $(p,2^1)$-symmetry of only one type p-M^1 is a very complicated, but knowing the AC of every p-generating asymmorphic space group it is possible to simply find the number of $(p,2^1)$- symmetry groups of the type p-M^l for every l. Hence, for the complete enumeration of asymmorphic space groups of the $(p,2^1)$- symmetry of the type p-M^l we only need to know the AC of every p- generating group.

According to this is derived the list of all p-generating asymmorphic groups and their AC. They are distributed into the families ([2], Appendix P2), given by the symbol of Fedorov group, its AC and the number of the corresponding AC-isomorphism equivalence class. The same data about AC are given for all p-generating groups [6,7]. Every asymmorphic Fedorov space group is given by three different symbols: Fedorov, international and Zamorzaev symbol [3,P1].

From this list we can conclude that AC of all 263 p- generating asymmorphic p-symmetry space groups are isomorphic to the already obtained and investigated AC [6,7], given in the partial catalogue of AC of the classical-symmetry Fedorov groups G_3 [7, Appendix]. Using this fact it is possible to immensely simplify the derivation of all asymmorphic space groups of $(p,2^1)$-symmetry of the type p-M^m, because from groups with isomorphic AC for l fixed we derive the same number of new groups [6,7].

Therefore, in order to solve the posed problem, it is necessary to make the following steps:

1) to distribute all p-generating asymmorphic groups in the equivalence classes according to the isomprphism of AC;

2) to find the number of classes obtained;

3) to find the number of p-generating groups belonging to each class;

4) to find the number of p-junior groups of $(p,2^1)$-symmetry groups derived from each asymmorphic p-symmetry group G_3^p for all possible values of l.

By forming the sets consisting of groups with isomorphic AC, all p-generating groups from the list obtained will be distributed into the disjoint equivalence classes. As the representative of each class can be used the corresponding Fedorov group G_3, for which the numbers of (2^l)-symmetry groups of the M^m-type derived from it are already computed for all the values of l [6,7]. Because groups with isomorphic AC posses the same properties and structure, we may conclude that from every p-generating group belonging to the same AC-isomorphism equivalence class will be derived the number of $(p,2^1)$-symmetry groups of the type p-M^m same as the number of (2^l)-symmetry groups of the M^m- type derived from the Fedorov group G_3 taken for the representative of equivalence class. All such data about Fedorov groups are given in [6,7]. Therefore, we have all data necessary for solving the problem mentioned in the point 4) of the proposed method [5].

All data for the steps 1), 2) and 4) of the proposed method: number of the AC-isomorphism equivalence class of p-generating group G and the corresponding numbers Nl(G) of all p-junior groups of the $(p,2^1)$-symmetry which can be derived from G, are given in Table 4.

Table 4

G	N1(G)	N2(G)	N3(G)	N4(G)
IV	4	15	42	
VI	5	24	84	
XIII	11	126	1344	10080
XIX	3	6		
XX	7	42	168	
XXI	2	3		
XXII	8	75	714	5040
XXIII	15	210	2520	20160
XXIV	1			
XXVIII	3	9		
21 XXXI	2	4	7	

Among asymmorphic $(p,2^1)$-symmetry groups of the type p-M^m, derived from p-generating groups by the use of the proposed method, there are groups of complete $(p,2^1)$-symmetry, as well as the groups of uncomplete $(p,2^1)$-symmetry. For p=3, as it is prooved in [5], all 3-junior $(3,2^l)$-symmetry groups are groups of complete $(3,2^l)$-symmetry, but for p=4,6 there occur also the groups of uncomplete $(p,2^1)$-symmetry. In Table 5 formed analogously to Table 4, are given the numbers (Nm(G)) of all complete $(p,2^1)$-symmetry groups of the type p-M^m which can be derived from every p-generating asymmorphic group G belonging to certain AC-isomorphism equivalence class for p=4,6 and l=1,2,3.

Table 5

G	(N1(G))	(N2(G))	(N3(G))
IV	3	6	
VI	4	12	
XIII	10	96	672
XIX	2		
XX	6	24	
XXI	1		
XXII	7	54	336
XXIII	14	168	1344
XXVIII	2	3	
XXXI	1	1	

In Table 6 is given the distribution of all 263 p-generating asymmorphic groups according to the AC-isomorphism equivalence classes they belong.

Table 6

G	p=3	p=4	p=6	p=4,6	p=3,4,6
IV		5	1	6	6
VI	6	30	18	54	48
XIII		7	4	11	11
XIX	10	35	35	80	70
XX	1	30	19	50	49
XXI	5	5	5	15	10
XXII	1	6	3	10	9
XXIII		2	1	3	3
XXIV	12	5	12	29	17
XXV	1			1	
XXVIII		1		1	1
XXXI	1	1	1	3	2
	37	127	99	263	226

From Table 4,5,6 we may calculate the numbers Nl(p) of all asymmorphic groups of $(p,2^1)$-symmetry of the type p-M^m, and the numbers (Nl(p)) of the asymmorphic groups of complete $(p,2^1)$- symmetry of the type p-M^m.

Multiplying the number of the groups belonging to certain AC-isomorphism equivalence class by the corresponding number Nm(G) or (Nm(G)), and adding the products obtained, we have the numbers Nm and (Nm) of asymmorphic space groups of $(p,2^1)$-symetry (p=3,4,6).

$N1=99^{(3)}+660^{(4)}+439^{(6)}=1198$
$(N1)=99^{(3)}+533^{(4)}+340^{(6)}=972$

$N2=340^{(3)}+4045^{(4)}+2413^{(6)}=6798$
$(N2)=340^{(3)}+2446^{(4)}+1393^{(6)}=4179$
$N3=1393^{(3)}+26530^{(4)}+14791^{(6)}=42714$
$(N3)=1393^{(3)}+9408^{(4)}+5040^{(6)}=15841$
$N4=5040^{(3)}+141120^{(4)}+75600^{(6)}=221760$
$(N4)=5040^{(3)}=5040$

3. Conclusion

As the final result of [5] and the present work, dealing respectively with symmorphic, hemisymmorphic and asymmorphic space groups of $(p,2^1)$-symmetry, we have completed the derivation of all $(p,2^1)$-symmetry space groups of the type p-M^m derived from 817 p-generating groups listed in the monograph [2, Appendix P2].

The final data are summarized in Table 7,8,9, giving respectively:

1) the number Nm of all $(p,2^1)$-symmetry groups of the type p-M^m derived from each Belov's p-generating group (Table 7);

2) the number (Nm) of all complete $(p,2^1)$-symmetry groups of the type p-M^m derived from each Belov's p-generating group (Table 8);

3) the distribution of all 817 Belov's p-generating groups G_3^p into the AC-isomorphism equivalence classes (Table 9).

Table 7

G	N1(G)	N2(G)	N3(G)	N4(G)	N5(G)
I	1	1	1		
III	5	28	168	840	
IV	4	15	42		
V	5	34	266	1680	
VI	5	24	84		
VIII	9	84	756	5040	
X	3	10	28		
XII	11	186	3948	83160	1249920
XIII	11	126	1344	10080	
XIV	9	108	504	3360	
XVI	17	348	7812	166320	2499840
XVII	7	58	504	3360	
XIX	3	6			
XX	7	42	168		
XXI	2	3			
XXII	8	75	714	5040	
XXIII	15	210	2520	20160	
XXIV	1				
XXVII	5	39	357	2520	
XXVIII	3	9	21		
XXXI	2	4	7		
13s.5	23	570	14280	322560	4999680

13s.7	19	486	13104	312480	4999680
5h.5	6	57	630	5040	

Table 8

G	(N1(G))	(N2(G))	(N3(G))	(N4(G))
III	4	16	56	
IV	3	6		
V	4	22	112	
VI	4	12		
VIII	8	60	336	
X	2	4		
XII	10	156	2856	40320
XIII	10	96	672	
XIV	8	84	672	
XVI	16	300	5712	80640
XVII	6	40	224	
XIX	2			
XX	6	24		
XXI	1			
XXII	7	54	336	
XXIII	14	168	1344	
XXVII	4	27	168	
XXVIII	2	3		
XXXI	1	1		
13s.5	22	504	10752	161280
13s.7	18	432	10080	161280
5h.5	5	42	336	

Table 9

G	p=3	p=4	p=6	p=4,6	p=3,4,6
I	1			1	
III	1	2	3	6	5
IV	2	13	8	23	21
V	1		1	2	1
VI	16	69	49	134	118
VIII		1	2	3	3
X		1		1	1
XII	1	2	3	6	5
XIII	5	30	42	77	72
XIV	1	4	4	9	8
XVI		1	2	3	3
XVII			1	1	1
XIX	24	60	86	170	146
XX	8	84	90	182	174
XXI	14	10	14	38	24

XXII	1	13	6	20	19
XXIII		14	27	41	41
XXIV	31	12	31	74	43
XXV	2			2	
XXVII	1	2	1	4	3
XXVIII	1	3	1	5	4
XXXI	1	2	2	5	4
13s.5		2	4	6	6
13s.7		1	2	3	3
5h.5		1		1	1
	111	327	379	817	706

Multiplying the number of the groups belonging to certain AC-isomorphism equivalence class by the corresponding number Nm(G) or (Nm(G)), and adding the products obtained, we have the numbers Nm and (Nm) of all space groups of $(p,2^1)$-symetry (p=3,4,6).

$N1 = 379^{(3)}+2032^{(4)}+2429^{(6)}=4840$
$(N1)=379^{(3)}+1705^{(4)}+2050^{(6)}=4134$
$N2 = 2050^{(3)}+16562^{(4)}+22384^{(6)}=40996$
$(N2)=2050^{(3)}+11447^{(4)}+16234^{(6)}=29731$
$N3 = 16234^{(3)}+170289^{(4)}+267358^{(6)}=453881$
$(N3)=16234^{(3)}+90160^{(4)}+153720^{(6)}=260114$
$N4 = 153720^{(3)}+1997520^{(4)}+3555720^{(6)}=5706960$
$(N4)=153720^{(3)}+645120^{(4)}+1249920^{(6)}=2048760$
$N5 = 1249920^{(3)}+19998720^{(4)}+38747520^{(6)}=59996160$
$(N5)=1249920^{(3)}=1249920$

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