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NON-NORMAL TILINGS DERIVED FROM ISOHEDRAL TILINGS BY EDGE BIFURCATION

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Abstract. *After establishing connection between edge colorings of isohedral tilings and tilings containing digons, which can be obtained by edge bifurcation, all such tilings are derived.*

The 93 types of isohedral tilings were classified by B.N.Delone [1,2], H.Heesch [3], B.Grčnbaum and G.C.Shephard [4]. Replacement of an edge by a digon with the same vertices and preserved symmetry is called the edge bifurcation [4].

In this paper we will reduce the derivation of all tilings which can be obtained from the isohedral ones by the edge bifurcation to the enumeration of all edge colorings of isohedral tilings. This can be done if every edge bifurcation is denoted by the coloring of corresponding edge.

For the enumeration of all non-equivalent edge colorings of isohedral tilings we will use the theory of antisymmetry [5,6,7] and the antisymmetric characteristic method [8,9].

To every tiling can be asociated the corresponding incidence symbol [4]. For denoting pairs of adjacent edges, the adjacency of edges a^+ and b^+ will be denoted by +ab, adjacency of a^+ and b^- by -ab, and adjacency of a and b by ab. Two pairs of adjacent edges we will cal equivalent if there is an automorphism of the set of edges E of tiling T preserving the type of the tiling and transforming the first pair of edges into the second.

For the enumeration of all diferent edge colorings we will use adapted definition of antisymmetric characteristic:

Definition: Let the set of pairs of adjacent edges of some tiling T be divided on the subsets consisting of equivalent pairs of edges. The resulting system is called the edge antisymmetric characteristic of the tiling T (eAC(T)).

All the facts known from the theory of simple and multiple antisymmetry can be simply transfered and used in the further discusion of edge colorings.

For example, for the isohedral tiling IH24, given by incidence symbol $[a^+b^+c^+d^+e^+; a^-e^+c^+d^+b^+]$, we have three equivalence classes of edge pairs:$+cc \sim +dd$, -aa and +be, so the eAC(IH24)={+cc,+dd}{-aa}{+be}. Certainly, adjacent edges must be

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colored simultaneously, so there are eleven different edge colorings:
1) $\{\underline{+cc}, +dd\}\{-aa\}\{+be\} = \{+cc, \underline{+dd}\}\{-aa\}\{+be\}$, 2) $\{+cc, +dd\}\{\underline{-aa}\}\{+be\}$, 3) $\{+cc, +dd\}\{-aa\}\{\underline{+be}\}$, 4) $\{\underline{+cc}, \underline{+dd}\}\{-aa\}\{+be\}$, 5) $\{+cc, +dd\}\{-aa\}\{+be\} = \{+cc, +dd\}\{-aa\}\{+be\}$, 6) $\{+cc, +dd\}\{-aa\}\{+be\} = \{+cc, +dd\}\{-aa\}\{+be\}$, 7) $\{+cc, +dd\}\{-aa\}\{+be\}$, 8) $\{+cc, +dd\}\{-aa\}\{+be\}$, 9) $\{+cc, +dd\}\{-aa\}\{+be\}$, 10) $\{+cc, +dd\}\{-aa\}\{+be\} = \{+cc, +dd\}\{-aa\}\{+be\}$, 11) $\{+cc, +dd\}\{-aa\}\{+be\}$, where the pairs of colored edges are underlined. Using the results of the theory of simple and multiple antisymmetry we may conclude that eAC(IH24) is isomorphic to the antisymmetric characteristic {A,B }{C}{D}, belonging to the AC- isomorphism equivalence class 4.2 [9]. Hence, $N_1 = 11$, $N_2 = 126$, $N_3 = 1344$, $N_4 = 10080$.

The eleven edge-colored tilings derived from IH24 are illustrated by Figure 1. If we replace the colored edges by digons, as the result we have all non-normal tilings derived from the isohedral tiling IH24 by edge bifurcation (Figure 2).

An edge coloring we may call connected or disconnected, depending on that the set of the colored edges is connected or disconnected. In the case of connected edge coloring, an information starting from some initial tile can be extended to the all tiling by colored edge-net (Figure 1: edge colorings 9,10,11). From this may arise some interesting questions about minimality conditions for connected edge colorings. As the table results we are giving the list of 93 isohedral tilings together with their topological type, incidence symbol [4], edge antisymmetric characteristic and the number of AC- isomorphism equivalence class [9] (Table 1). The numbers of simple and multiple antisymmetry edge colorings are given in Table 2.

Table 1

$IH1$	$[3^6]$	$[a^+b^+c^+d^+e^+f^+; d^+e^+f^+a^+b^+c^+]$	$(+ad, +be, +cf)$	3.5
$IH2$		$[a^+b^+c^+d^+e^+f^+; b^-a^-f^+e^-d^-c^+]$	$\{-ab, -de\}\{+cf\}$	3.2
$IH3$		$[a^+b^+c^+d^+e^+f^+; c^-e^+a^-f^-b^+d^+]$	$\{-ac, -df\}\{+be\}$	3.2
$IH4$		$[a^+b^+c^+d^+e^+f^+; a^+e^+c^+d^+b^+f^+]$	$\{[+aa, +ff], [+cc, +dd]\}\{+be\}$	4.12x1.1
$IH5$		$[a^+b^+c^+d^+e^+f^+; a^+e^+d^-c^-b^+f^+]$	$\{+aa, +ff\}\{+be\}\{-cd\}$	4.2
$IH6$		$[a^+b^+c^+d^+e^+f^+; a^+e^-c^+f^-b^-d^-]$	$\{+aa, +cc\}\{-be\}\{-df\}$	4.2
$IH7$		$[a^+b^+c^+d^+e^+f^+; b^+a^+d^+c^+f^+e^+]$	$(+ab, +cd, +ef)$	3.5
$IH8$		$[a^+b^+c^+a^+b^+c^+; a^+b^+c^+]$	$(+aa, +bb, +cc)$	3.5
$IH9$		$[a^+b^+c^+a^+b^+c^+; a^+c^-b^-]$	$\{+aa\}\{-bc\}$	2.1
$IH10$		$[a^+b^+a^+b^+a^+b^+; b^+a^+]$	$\{+ab\}$	1.1
$IH11$		$[a^+a^+a^+a^+a^+a^+; a^+]$	$\{+aa\}$	1.1
$IH12$		$[ab^+c^+dc^-b^-; dc^-b^-a]$	$\{ad\}\{-bc\}$	2.1
$IH13$		$[ab^+c^+dc^-b^-a; db^+c^+a]$	$\{+bb, +cc\}\{ad\}$	3.2
$IH14$		$[a^+b^+c^+c^-b^-a^-; c^-b^-a^-]$	$\{-ac\}\{-bb\}$	2.1
$IH15$		$[a^+b^+c^+c^-b^-a^-; a^+b^-c^+]$	$\{+aa, +cc\}\{-bb\}$	3.2
$IH16$		$[a^+b^+c^+c^-b^-a^-; a^-c^+b^+]$	$\{-aa\}\{+bc\}$	2.1
$IH17$		$[ab^+b^-ab^+b^-; ab^+]$	$\{aa\}\{+bb\}$	2.1
$IH18$		$[ababab; ba]$	$\{ab\}$	1.1
$IH19$		$[a^+a^-a^+a^-a^+a^-; a^-]$	$\{-aa\}$	1.1

$IH20$	$[aaaaaa;a]$	$\{aa\}$	1.1
$IH21$ $[3^4.6]$	$[a^+b^+c^+d^+e^+;e^+c^+b^+d^+a^+]$	$\{+ae\}\{+bc\}\{+dd\}$	3.1
$IH22$ $[3^3.42]$	$[a^+b^+c^+d^+e^+;a^-e^+d^-c^-b^+]$	$\{-aa\}\{+be\}\{-cd\}$	3.1
$IH23$	$[a^+b^+c^+d^+e^+;a^+e^+c^+d^+b^+]$	$\{+cc,+dd\}\{+aa\}\{+be\}$	4.2
$IH24$	$[a^+b^+c^+d^+e^+;a^-e^+c^+d^+b^+]$	$\{+cc,+dd\}\{-aa\}\{+be\}$	4.2
$IH25$	$[a^+b^+c^+d^+e^+;a^+e^+d^-c^-b^+]$	$\{+aa\}\{+be\}\{-cd\}$	3.1
$IH26$	$[ab^+c^+c^-b^-;ab^-c^+]$	$\{aa\}\{-bb\}\{+cc\}$	3.1
$IH27$ $[3^2.4.3.4]$	$[a^+b^+c^+d^+e^+;a^+d^-e^-b^-c^-]$	$\{-bd,-ce\}\{+aa\}$	3.2
$IH28$	$[a^+b^+c^+d^+e^+;a^+c^+b^+e^+d^+]$	$\{+bc,+de\}\{+aa\}$	3.2
$IH29$	$[ab^+c^+c^-b^-;ac^+b^+]$	$\{aa\}\{+bc\}$	2.1
$IH30$ $[3.4.6.4]$	$[a^+b^+c^+d^+;a^-b^-d^+c^+]$	$\{-aa,-bb\}\{+cd\}$	3.2
$IH31$	$[a^+b^+c^+d^+;b^+a^+d^+c^+]$	$\{+ab,+cd\}$	2.2
$IH32$	$[a^+b^+b^-a^-;a^-b^-]$	$\{-aa\}\{-bb\}$	2.1
$IH33$ $[3.6.3.6.]$	$[a^+b^+c^+d^+;d^+c^+b^+a^+]$	$\{+ad,+bc\}$	2.2
$IH34$	$[a^+b^+a^+b^+;b^+a^+]$	$\{+ab\}$	1.1
$IH35$	$[a^+b^+b^-a^-;a^-b^-]$	$\{-aa,-bb\}$	2.2
$IH36$	$[a^+b^+b^-a^-;b^-a^-]$	$\{-ab\}$	1.1
$IH37$	$[a^+a^+a^+a^+;a^-]$	$\{-aa\}$	1.1
$IH38$ $[3.12^2]$	$[a^+b^+c^+;a^-c^+b^+]$	$\{-aa\}\{+bc\}$	2.1
$IH39$	$[a^+b^+c^+;a^+c^+b^+]$	$\{+aa\}\{+bc\}$	2.1
$IH40$	$[ab^+b^-;ab^-]$	$\{aa\}\{-bb\}$	2.1
$IH41$ $[4^4]$	$[a^+b^+c^+d^+;c^+d^+a^+b^+]$	$\{+ac,+bd\}$	2.2
$IH42$	$[a^+b^+c^+d^+;c^+b^-a^+d^-]$	$\{-bb,-dd\}\{+ac\}$	3.2
$IH43$	$[a^+b^+c^+d^+;c^-d^+a^-b^+]$	$\{-ac\}\{+bd\}$	2.1
$IH44$	$[a^+b^+c^+d^+;b^-a^-d^-c^-]$	$\{-ab,-cd\}$	2.2
$IH45$	$[a^+b^+c^+d^+;c^-b^-a^-d^-]$	$\{-bb,-dd\}\{-ac\}$	3.2
$IH46$	$[a^+b^+c^+d^+;a^+b^+c^+d^+]$	$(+aa,+bb,+cc,+dd)$	4.14
$IH47$	$[a^+b^+c^+d^+;c^+b^+a^+d^+]$	$\{+bb,+dd\}\{+ac\}$	3.2
$IH48$	$[a^+b^+c^+d^+;a^-b^-c^-d^-]$	$(-aa,-bb,-cc,-dd)$	4.14
$IH49$	$[a^+b^+c^+d^+;a^-b^+c^-d^+]$	$\{-aa,-cc\}\{+bb,+dd\}$	4.6
$IH50$	$[a^+b^+c^+d^+;c^+b^-a^+d^+]$	$\{+ac\}\{-bb\}\{+dd\}$	3.1
$IH51$	$[a^+b^+c^+d^+;c^-b^+a^-d^+]$	$\{+bb,+dd\}\{-ac\}$	3.2
$IH52$	$[a^+b^+c^+d^+;c^-d^-a^-b^-]$	$\{-ac,-bd\}$	2.2
$IH53$	$[a^+b^+c^+d^+;b^-a^-c^+d^+]$	$\{+cc,+dd\}\{-ab\}$	3.2
$IH54$	$[a^+b^+c^+d^+;a^-b^-c^-d^+]$	$\{-aa,-cc\}\{-bb\}\{+dd\}$	4.2
$IH55$	$[a^+b^+c^+d^+;b^+a^+d^+c^+]$	$\{+ab,+cd\}$	2.2
$IH56$	$[a^+b^+c^+d^+;b^+a^+c^-d^-]$	$\{-cc,-dd\}\{+ab\}$	3.2
$IH57$	$[a^+b^+a^+b^+;a^+b^+]$	$\{+aa,+bb\}$	2.2
$IH58$	$[a^+b^+a^+b^+;a^-b^+]$	$\{-aa\}\{+bb\}$	2.1
$IH59$	$[a^+b^+a^+b^+;b^-a^-]$	$\{-ab\}$	1.1
$IH60$	$[a^+b^+a^+b^+;a^-b^-]$	$\{-aa,-bb\}$	2.2
$IH61$	$[a^+b^+a^+b^+;b^+a^+]$	$\{+ab\}$	1.1
$IH62$	$[a^+a^+a^+a^+;a^+]$	$\{+aa\}$	1.1
$IH63$	$[a^+a^+a^+a^+;a^-]$	$\{-aa\}$	1.1
$IH64$	$[ab^+cb^-;cb^-a]$	$\{ac\}\{-bb\}$	2.1

$IH65$	$[ab^+cb^-;ab^-c]$	$\{-bb,-cc\}\{aa\}$	3.2
$IH66$	$[ab^+cb^-;cb^+a]$	$\{ac\}\{+bb\}$	2.1
$IH67$	$[ab^+cb^-;ab^+c]$	$\{aa,cc\}\{+bb\}$	3.2
$IH68$	$[a^+b^+b^-a^-;b^-a^-]$	$\{-ab\}$	1.1
$IH69$	$[a^+b^+b^-a^-;a^+b^+]$	$\{+aa,+bb\}$	2.2
$IH70$	$[a^+b^+b^-a^-;a^-b^-]$	$\{-aa,-bb\}$	2.2
$IH71$	$[a^+b^+b^-a^-;b^+a^+]$	$\{+ab\}$	1.1
$IH72$	$[abab;ab]$	$\{aa,bb\}$	2.2
$IH73$	$[abab;ba]$	$\{ab\}$	1.1
$IH74$	$[a^+a^-a^+a^-;a^+]$	$\{+aa\}$	1.1
$IH75$	$[a^+a^-a^+a^-;a^-]$	$\{-aa\}$	1.1
$IH76$	$[aaaaa;a]$	$\{aa\}$	1.1
$IH77$ [4.6.12]	$[a^+b^+c^+;a^-b^-c^-]$	$\{-aa\}\{-bb\}\{-cc\}$	3.1
$IH78$ [4.8^2]	$[a^+b^+c^+;a^+b^-c^-]$	$\{-bb,-cc\}\{+aa\}$	3.2
$IH79$	$[a^+b^+c^+;a^+c^+b^+]$	$\{+aa\}\{+bc\}$	2.1
$IH80$	$[a^+b^+c^+;a^-b^-c^-]$	$\{-bb,-cc\}\{-aa\}$	3.2
$IH81$	$[a^+b^+c^+;a^-c^+b^+]$	$\{-aa\}\{+bc\}$	2.1
$IH82$	$[ab^+b^-;ab^-]$	$\{aa\}\{-bb\}$	2.1
$IH83$ [6^3]	$[a^+b^+c^+;b^-a^-c^-]$	$\{-ab\}\{-cc\}$	2.1
$IH84$	$[a^+b^+c^+;a^+b^+c^+]$	$(+aa,+bb,+cc)$	3.5
$IH85$	$[a^+b^+c^+;a^-b^+c^+]$	$\{+bb,+cc\}\{-aa\}$	3.2
$IH86$	$[a^+b^+c^+;b^-a^-c^+]$	$\{-ab\}\{+cc\}$	2.1
$IH87$	$[a^+b^+c^+;a^-b^-c^-]$	$(-aa,-bb,-cc)$	3.5
$IH88$	$[a^+b^+c^+;b^+a^+c^+]$	$\{+ab\}\{+cc\}$	2.1
$IH89$	$[a^+a^+a^+;a^-]$	$\{-aa\}$	1.1
$IH90$	$[a^+a^+a^+;a^+]$	$\{+aa\}$	1.1
$IH91$	$[ab^+b^-;ab^+]$	$\{aa\}\{+bb\}$	2.1
$IH92$	$[ab^+b^-;ab^-]$	$\{aa\}\{-bb\}$	2.1
$IH93$	$[aaa;a]$	$\{aa\}$	1.1

Table 2

1.1	1				
2.1	3	6			
2.2	2	3			
3.1	7	42	168		
3.2	5	24	84		
3.5	3	14	56		
4.2	11	126	1344	10080	
4.6	8	75	714	5040	
4.14	5	54	630	5040	
4.12x1.1	13	264	6636	156240	2499840

Theorem 1. *From 93 isohedral tilings can be derived $N_1=344$ different edge colorings. The same is the number of different non-normal tilings which can be derived from the isohedral ones by the edge bifurcation, among which 157 are 2-homeohedral [10], 124 3-homeohedral, 52 4-homeohedral, 10 5-homeohedral and 1 6- homeohedral.*

Theorem 2. *From 93 isohedral tilings can be derived $N_2=1999$, $N_3=18130$, $N_4=221760$ and $N_5=2499840$ multiple antisymmetry edge colorings of the type M^m.*

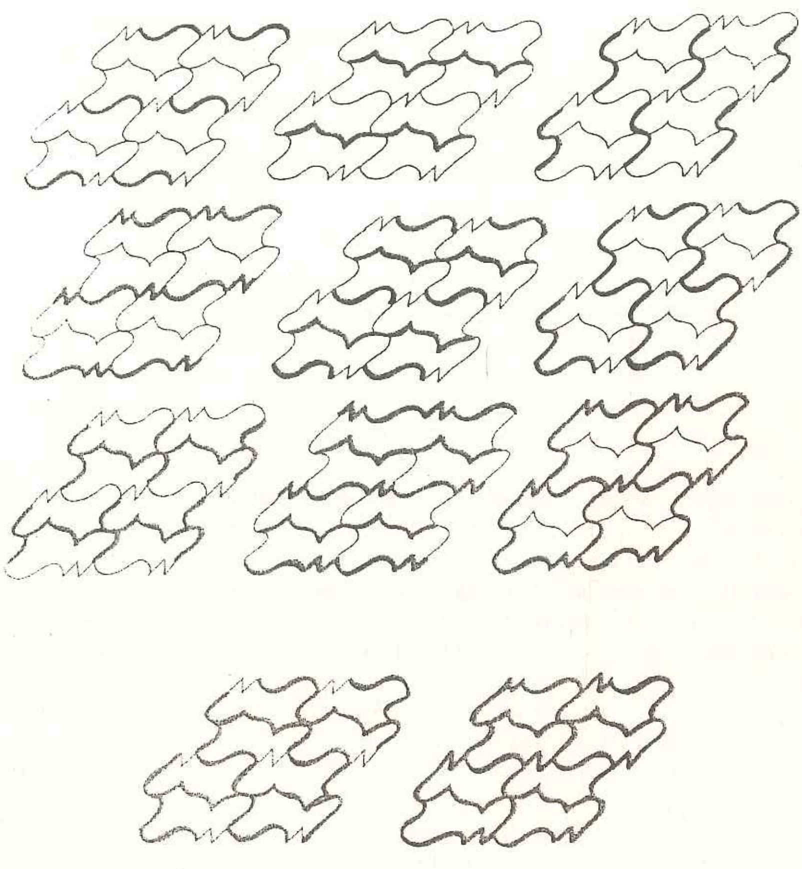

Figure 1

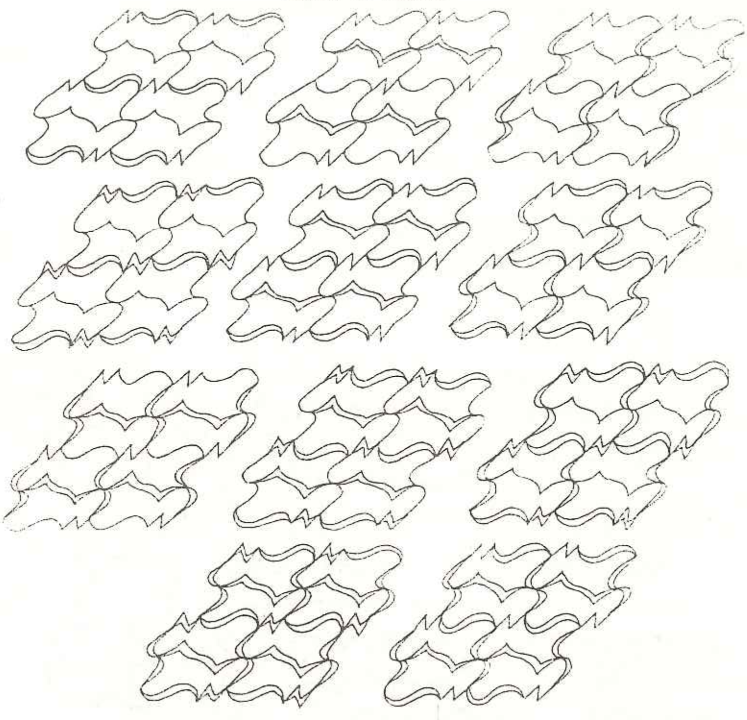

Figure 2

REFERENCES

[1] DELONE B.N., *Teoriya planigonov*, Izv. Akad. Nauk SSSR Ser. Mat. 23 (1959), 365-386.

[2] DELONE B.N., DOLBILIN N.P., STOGRIN M.I., *Combinatorial and Metric Theory of Planigons*, Trudy Matem. Inst. Steklov., 148 (1978), 111-141.

[3] HEESCH H, *Reguläre Parkettierungsproblem*, Westdeutscher Verlag, Cologne, 1968.

[4] GRUNBAUM B., SHEPHARD G.C., *Tilings and Patterns*, Freeman, New York, 1987.

[5] ZAMORZAEV A.M., *Teoriya prostoi i kratnoi antisimmetrii*, Shtiintsa, Kishinev, 1976.

[6] ZAMORZAEV A.M., PALISTRANT A.F., *Antisymmetry, its Generalizations and Geometrical Applications*, Z.Kristall., 151 (1980), 231-248.

[7] ZAMORZAEV A.M., *Generalized Antisymmetry*, Comput. Math. Applic., 16, 5-8 (1988), 555-561.

[8] JABLAN S.V., *A New Method for Generating Plane Groups of Simple and Multiple Antisymmetry*, Acta Cryst., A42 (1986), 209-212.

[9] JABLAN S.V., *Algebra of Antisymmetric Characteristics*, Publ. Inst. Math., 47 (61) (1990), 39-55.

[10] DELGADO O., HUSON D., ZAMORZAEVA E., *The Classification of 2-Isohedral Tilings of the Plane*, Preprint 90-056, Universitšt Bielefeld, 1990.

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