FILOMAT (Niš) 7 (1993), 105–107

ASYMPTOTICAL CHARACTERISTICS OF THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

J.Knežević-Miljanović

Let us consider differential equations:

$$y' + \sum_{k=0}^{m} p_k(x)y^k = 0 \tag{1}$$

$$y' = \frac{\sum_{k=0}^{m} p_k(x)y^k}{\sum_{k=0}^{m-1} q_k(x)y^k} \tag{2}$$

$$\frac{d}{dx}\left(|y'|^p y'\right) + Q(x)y = 0 \tag{3}$$

Equation (1) is considered in [1],[2], where $p_k(x)$ are continuos functions in $[x_0, +\infty)$. In [3] equation (2) has been considered, where $p_k(x)$,$q_k(x)$ are continuous and $q_0(x) \neq 0$ in $[x_0, +\infty)$. Equation (3) is considered in the theory of approximations ([4],[5]) and its solutions do have different characteristics. In this work $p = const$, $Q(x)$ is a continuos function in $[x_0, +\infty)$. We shall investigate prolongation of solutions for these equations, i.e. solutions which are defined in $[x_0, +\infty)$.

Theorem 1. *If $p_0(x) \leq 0$ in $[x_0, +\infty)$, $p_m(x) > 0$, $x \in [x_0, +\infty)$ then each solution of equation (1), such that $y(x_1) \geq 0$ for $x_0 \leq x_1 < \infty$ is prolongative in $[x_1, +\infty)$ and non negative in $(x_1, +\infty)$.*

PROOF. Since the angular coefficient of field directions of equation (1) is non negative on the axis $y = 0, x > 0$ the solution $y_1(x)$, such that $y_1(x_1) \geq 0$ for $x > x_1$, satisfies the inequality $y_1(x) \geq 0$ for $x > x_1$. Let X_1 be positive, such that $x_1 < X_1$. Let us show that $y_2(x)$ is prolongative on $[x_1, X_1]$. Due to arbitrary nature of x_1 it will mean prolongations of $y_1(x)$ in $[x_1, +\infty)$. Let us consider

$$m_k = \max_{x_1 \leq x \leq X_1} p_k(x) \qquad a_m = \min_{x_1 \leq x \leq X_1} p_m(x)$$

Received 19.11.1993
1991 *Mathematics Subject Classification*: 34A35
Supported by Grant 0401A of RFNS through Math. Inst. SANU

and select the number l so great, that

$$a_m l^m + \sum_{i=0}^{m-1} m_i l^i > 0 \qquad \text{and} \qquad 1 > y(x_1).$$

In such a case the field of directions of equation (1) on the straight line $y = 1$ for $x_1 \le x \le X_1$ is negative and accordingly $y_1(x) \le l < l+1$ for $x > X_1$. Since the solution is prolongative up to the limit of the colosed field: $-1 \le y \le l+1, x_1 \le x \le X_1$ then $y_1(x)$ is prolongative up to $x = x_1$. Thus the theorem is proven.

Theorem 2. *If in equation (2)*

$$q_{m-2}\frac{p_m}{q_{m-1}} - p_{m-1} > 0 \tag{4}$$

then there exists a solution $y(x)$ which is prolongative in $[o, +\infty)$.

PROOF. Inequality

$$\frac{\sum_{k=0}^{m} p_k y^k}{\sum_{k=0}^{m-1} q_k y^k} < \frac{p_m}{q_{m-1}} y$$

is developed in to

$$\begin{aligned}\left[q_{m-2}\frac{p_m}{q_{m-1}} - p_{m-1}\right] y^{m-1} + \left[q_{m-3}\frac{p_m}{q_{m-1}} - p_{m-2}\right] y^{m-2} + \cdots \\ + \left[q_0\frac{p_m}{q_{m-1}} - p_1\right] y - p_0 > 0\end{aligned} \tag{5}$$

Let us considered the solution $y(x)$ which satisfies the initial condition $y_1(0) = b_1$. For example, let m be an even number. Since the field of directions of the equation (2) on the straight line $y = b_1$ is positive, due to (5) and condition (4), then $y_1(x) > b_1$ for $x > 0$. By analogy, the solution $y_2(x)$, which satisfies condition $y_2(0) = -b_1$, satisfies condition $y_2(x) < b_1$ for $x > 0$. Acoording to the Schauder principle, there exists a solution $|y| < b_1$ whose graph lies in the half-axis $0 < x < \infty$, i.e. it is prolongative.

Theorem 3. *Let $0 < p < 1$, $Q(X) \ge 0$, and for some $q < p$*

$$\int_0^\infty t^q Q(t)dt = +\infty$$

In such a case all prolongative solutions of equation (3) are oscillatory.

PROOF. Let us assume the contrary. In such a case there exists a solution $y(x) > 0$ of the equation (3), such that $y'(x) > 0$ for $x > \overline{x}$. Transforming (3):

$$y^{-1}x^q((y')^p)' + Q(x)x^q = 0 \tag{6}$$

By partial integration of (6) from $\overline{x}$ to x we obtain:

$$(7)\quad y^{-1}x^q(y')^p + \int_{\overline{x}}^{x} y^{-1}x^q(y')^{p+1}dx + \int_{\overline{x}}^{x} Q(x)x^q dx = q \int_{\overline{x}}^{x} y^{-1}x^{q-1}(y')^p dx + C$$

Let us evaluate the integral on the right-hand side of the equation (7) by means on Holder inequality:

$$(8)\quad \begin{aligned} &\int_{\overline{x}}^{x} y^{-1}x^{q-1}(y')^p dx \le \\ &\le \left(\int_{\overline{x}}^{x} y^{-2}x^q(y')^{p+1}dx\right)^{\frac{p}{p+1}} \cdot \left(\int_{\overline{x}}^{x} y^{p-1}x^{q-p-1}dx\right)^{\frac{1}{p+1}} \le \\ &\le \frac{p}{p+1}\int_{\overline{x}}^{x} y^{-2}x^q(y')^{p+1}dx + \frac{1}{p+1}\int_{\overline{x}}^{x} y^{p-1}x^{q-p-1}dx \end{aligned}$$

Since $q < p$ and $y(x)$ is increasing, the last integral in (8) converges. Therefore from (7) ans (8) we obtain:

$$y^{-1}x^q(y')^p + \frac{1}{p+1}\int_{\overline{x}}^{x} y^{-1}x^q(y')^{p+1}dx + \int_{\overline{x}}^{x} Q(x)x^q dx \le C$$

which is contradicts Theorem 3.

A similar equation to the above is considered in the theory of differential equations,i.e., the equation of Edmen-Fowler ([4],[5])

$$\frac{d^2}{dx^2}\,y + Q(X)\,|y|^{p-1}y = 0.$$

REFERENCES

[1] J. Knežević, *On construction periodical solution of ordinary differential equation*, Diff. Uravnenija, 19 (1983), 901–903 (Russian).

[2] J. Knežević-Miljanović, *On prolongation of solution of nonlinear differential equation first order*, Diff. Uravnenija, 26 (1990), 1162–1165 (Russian).

[3] M. Bertolino, *Inegalites differentielles lineaires de Tchapliguine de l' ordre arbitrarie dans l'intervalle infini*, Annali di Mathematica Pura ed Aplicata, 85 (1970).

[4] I.T. Kiguradze, T.A. Canturija, *Asymptotical characteristic of the solutions of non autonomical differential equation*, Nauka, Moskva, (1990) (Russian).

[5] F.V. Atkinson, *On second ordere non-linear oscilations*, Pacif. J. Math., 5 (1955), 159–169.

Mathematical Faculty
University of Belgrade
11 000 Belgrade, Yugoslavia