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PPF Dependent Fixed Point Results for Hybrid Rational and Suzuki-Edelstein Type Contractions in Banach Spaces

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Abstract. In this paper we introduce new notions of hybrid rational Geraghty and Suzuki-Edelstein type contractive mappings and investigate the existence and uniqueness of *PPF* dependent fixed point for such mappings in the Razumikhin class, where domain and range of the mappings are not the same. As an application of our *PPF* dependent fixed point results, we deduce corresponding *PPF* dependent coincidence point results in the Razumikhin class. Our results extend and improve the results of Sintunavarat and Kumam [J. Nonlinear Anal. Optim.: Theory Appl., Vol. 4, (2013), 157–162], Bernfeld, Lakshmikantham and Reddy [Applicable Anal., 6(1977), 271–280] and others. As an application of our results, we establish *PPF* dependent solution of a periodic boundary value problem.

1. Introduction and Preliminaries

In 1997, Bernfeld et al. [2] introduced the concept of a fixed point for mappings that have different domains and ranges, which is called PPF dependent fixed point or the fixed point with PPF dependence. Furthermore, they gave the notion of Banach type contraction for a non-self mapping and also proved the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contractive mappings. Very recently, Agarwal et al. [1], Ćirić et al. [3], Hussain et al. [8] and Sintunavarat and Kumam established the existence and uniqueness of PPF dependent fixed point for different types of contraction mappings and generalized some results of Bernfeld et al. [2] (Also, see [10]).

As a generalization of the Banach contraction principle, Geraghty [7] proved the following.

Theorem 1.1 (Geraghty [7]). *Let (X, d) be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition*

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to +\infty.$$

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If T satisfies the following inequality

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y), \text{ for all } x,y \in X, \tag{1}$$

then T has a unique fixed point.

Throughout this paper, we assume that $(E, \|\cdot\|_E)$ is a Banach space, I denotes a closed interval $[a,b]$ in $\mathbb{R}$ and $E_0 = C(I,E)$ denotes the set of all continuous E-valued functions on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E.$$

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in E_0 is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \ \|\phi\|_{E_0} = \|\phi(c)\|_E\}.$$

Clearly, every constant function from I to E belongs to $\mathcal{R}_c$.

Remark 1.2. *Let $\mathcal{R}_c$ be the Razumikhin class, then*

(i) the class $\mathcal{R}_c$ is algebraically closed with respect to difference, that is, $\phi - \xi \in \mathcal{R}_c$ when $\phi, \xi \in \mathcal{R}_c$;

(ii) the class $\mathcal{R}_c$ is topologically closed if it is closed with respect to the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Definition 1.3 ([2]). *A mapping $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.*

Definition 1.4 ([13]). *Let $S : E_0 \to E_0$ and $T : E_0 \to E$. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point or a coincidence point with PPF dependence of S and T if $T\phi = (S\phi)(c)$ for some $c \in I$.*

Definition 1.5 ([2]). *The mapping $T : E_0 \to E$ is called a Banach type contraction if there exists $k \in [0,1)$ such that,*

$$\|T\phi - T\xi\|_E \leq k\|\phi - \xi\|_{E_0}$$

for all $\phi, \xi \in E_0$.

In this paper, we introduce the notions of hybrid rational Geraghty and Suzuki-Edelstein type contractive mappings and study the existence and uniqueness of PPF dependent fixed point for such mappings in the Razumikhin class. As an application of our PPF dependent fixed point results we deduce corresponding PPF dependent coincidence point results in the Razumikhin class. These results extend and generalize some known results in the literature. An application to periodic boundary value problem is provided.

2. PPF Dependent Fixed Point Results

Let $\mathcal{F}$ denote the class of all functions $\beta : [0,+\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to +\infty. \tag{2}$$

Definition 2.1. *The mapping $T : E_0 \to E$ is called a hybrid rational Geraghty type contraction if there exists $\beta \in \mathcal{F}$ and $c \in I$ such that,*

$$\|T\phi - T\xi\|_E \leq \beta(\|\phi - \xi\|_{E_0})M(\phi,\xi) + \gamma(\|\phi - \xi\|_{E_0})N(\phi,\xi)$$

for all $\phi, \xi \in E_0$ where $\gamma : [0,\infty) \to [0,\infty)$ is a bounded function and

$$M(\phi,\xi) = \max\left\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}$$

and

$$N(\phi,\xi) = \min\Big\{\|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E, \|\phi(c) - T\xi\|_E, \|\xi(c) - T\phi\|_E\Big\}.$$

Now, we state and prove the following result.

Theorem 2.2. *Let $T : E_0 \to E$ be a hybrid rational Geraghty contractive mapping. Assume, $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference. Then, T has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of T be defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.*

Proof. Let $\phi_0 \in \mathcal{R}_c$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that,

$$x_1 = \phi_1(c).$$

Continuing this process, by induction, we can build a sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E$ such that,

$$T\phi_{n-1} = \phi_n(c), \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Since $\mathcal{R}_c$ is algebraically closed with respect to difference, it follows

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E, \quad \text{for all } n \in \mathbb{N}.$$

In view of the fact that T is a hybrid rational Geraghty contractive mapping we have

$$\begin{aligned} \|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E = \|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \beta(\|\phi_{n-1} - \phi_n\|_{E_0})M(\phi_{n-1}, \phi_n) + \gamma(\|\phi_{n-1} - \phi_n\|_{E_0})N(\phi_{n-1}, \phi_n). \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned} \|\phi_{n-1} - \phi_n\|_{E_0} &\leq M(\phi_{n-1}, \phi_n) \\ &= \max\left\{\|\phi_{n-1} - \phi_n\|_{E_0}, \frac{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E \|\phi_n(c) - T\phi_n\|_E}{1 + \|T\phi_{n-1} - T\phi_n\|_E}\right\} \\ &= \max\left\{\|\phi_{n-1} - \phi_n\|_{E_0}, \frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E \|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_n(c) - \phi_{n+1}(c)\|_E}\right\} \\ &\leq \max\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - \phi_n(c)\|_E\} \\ &= \max\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_{n-1} - \phi_n\|_{E_0}\} \\ &= \|\phi_{n-1} - \phi_n\|_{E_0}. \end{aligned}$$

This implies, $M(\phi_{n-1}, \phi_n) = \|\phi_{n-1} - \phi_n\|_{E_0}$. Also,

$$\begin{aligned} N(\phi_{n-1}, \phi_n) &= \min\Big\{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E, \|\phi_n(c) - T\phi_n\|_E, \\ &\qquad \|\phi_{n-1}(c) - T\phi_n\|_E, \|\phi_n(c) - T\phi_{n-1}\|_E\Big\} \\ &= \min\Big\{\|\phi_{n-1}(c) - \phi_n(c)\|_E, \|\phi_n(c) - \phi_{n+1}(c)\|_E, \\ &\qquad \|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E, \|\phi_n(c) - \phi_n(c)\|_E\Big\}, \end{aligned}$$

which implies, $N(\phi_{n-1}, \phi_n) = 0$.

From (4) we obtain,

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq \beta(\|\phi_{n-1} - \phi_n\|_{E_0})\|\phi_{n-1} - \phi_n\|_{E_0} < \|\phi_{n-1} - \phi_n\|_{E_0} \tag{5}$$

for all $n \in \mathbb{N}$. This implies that the sequence $\{\|\phi_n - \phi_{n+1}\|_{E_0}\}$ is decreasing in $\mathbb{R}_+$. Then there exists $r \geq 0$ such that $\lim_{n\to+\infty} \|\phi_n - \phi_{n+1}\|_{E_0} = r$. Assume that, $r > 0$. Now, by taking limit as $n \to +\infty$ in (4) we get,

$$r \leq \lim_{n\to+\infty} \beta(\|\phi_{n-1} - \phi_n\|_{E_0})\, r$$

which implies, $1 \leq \lim_{n\to+\infty} \beta(\|\phi_{n-1} - \phi_n\|_{E_0})$. So,

$$\lim_{n\to+\infty} \beta(\|\phi_{n-1} - \phi_n\|_{E_0}) = 1,$$

and since $\beta \in \mathcal{F}$, $\lim_{n\to+\infty} \|\phi_{n-1} - \phi_n\|_{E_0} = 0$ which is a contradiction. Hence, $r = 0$. That is,

$$\lim_{n\to+\infty} \|\phi_{n-1} - \phi_n\|_{E_0} = 0. \tag{6}$$

Now, we prove that the sequence $\{\phi_n\}$ is Cauchy in $\mathcal{R}_c$. If not, then we get

$$\lim_{m,n\to+\infty} \|\phi_m - \phi_n\|_{E_0} > 0. \tag{7}$$

Using the triangular inequality and since T is a hybrid rational Geraghty type contractive mapping, we have

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{m+1}\|_{E_0} + \|\phi_{m+1} - \phi_m\|_{E_0} \\
&\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \beta(\|\phi_n - \phi_m\|_{E_0})M(\phi_n, \phi_m) \\
&+ \gamma(\|\phi_n - \phi_m\|_{E_0})N(\phi_n, \phi_m) + \|\phi_{m+1} - \phi_m\|_{E_0}.
\end{aligned}$$

Taking limit $m, n \to \infty$ in the above inequality and applying (6) we have

$$\begin{aligned}
&\lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} \\
&\leq \lim_{m,n\to+\infty} \beta(\|\phi_n - \phi_m\|_{E_0}) \lim_{m,n\to+\infty} M(\phi_n, \phi_m) \\
&+ \lim_{m,n\to+\infty} \gamma(\|\phi_n - \phi_m\|_{E_0}) \lim_{m,n\to+\infty} N(\phi_n, \phi_m).
\end{aligned} \tag{8}$$

Also we have,

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq M(\phi_n, \phi_m) \\
&= \max\left\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_m(c) - T\phi_m\|_E}{1 + \|T\phi_n - T\phi_m\|_E}\right\} \\
&= \max\left\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_m(c) - \phi_{m+1}(c)\|_E}{1 + \|\phi_{n+1}(c) - \phi_{m+1}(c)\|_E}\right\} \\
&= \max\left\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_m - \phi_{m+1}\|_{E_0}}{1 + \|\phi_{n+1} - \phi_{m+1}\|_{E_0}}\right\}.
\end{aligned}$$

Letting $m, n \to \infty$ in the above inequality and applying (6), we get

$$\lim_{m,n\to+\infty} M(\phi_n, \phi_m) = \lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}. \tag{9}$$

Also,

$$\begin{aligned}
&\lim_{m,n\to+\infty} N(\phi_n, \phi_m) \\
&= \lim_{m,n\to+\infty} \min\left\{\|\phi_n(c) - T\phi_n\|_E, \|\phi_m(c) - T\phi_m\|_E, \|\phi_n(c) - T\phi_m\|_E, \|\phi_m(c) - T\phi_n\|_E\right\} \\
&= \lim_{m,n\to+\infty} \min\left\{\|\phi_n(c) - \phi_{n+1}(c)\|_E, \|\phi_m(c) - \phi_{m+1}(c)\|_E, \|\phi_n(c) - \phi_{m+1}(c)\|_E, \|\phi_m(c) - \phi_{n+1}(c)\|_E\right\} \\
&= \lim_{m,n\to+\infty} \min\left\{\|\phi_n - \phi_{n+1}\|_{E_0}, \|\phi_m - \phi_{m+1}\|_{E_0}, \|\phi_n - \phi_{m+1}\|_{E_0}, \|\phi_m - \phi_{n+1}\|_{E_0}\right\} = 0.
\end{aligned} \tag{10}$$

Hence, from (8), (9) and (10), we obtain

$$\limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} \leq \limsup_{m,n\to+\infty} \beta(\|\phi_n - \phi_m\|_{E_0}) \limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}$$

and so by (7) we get, $1 \leq \limsup_{m,n\to+\infty} \beta(\|\phi_n - \phi_m\|_{E_0})$. That is,

$$\lim_{m,n\to+\infty} \beta(\|\phi_m - \phi_n\|_{E_0}) = 1$$

and since $\beta \in \mathcal{F}$, we deduce

$$\lim_{m,n\to+\infty} \|\phi_m - \phi_n\|_{E_0} = 0$$

which is a contradiction. Consequently,

$$\lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} = 0$$

and hence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. Completeness of E_0 yields that $\{\phi_n\}$ converges to a point $\phi^* \in E_0$, that is, $\phi_n \to \phi^*$, as $n \to +\infty$. Since, $\mathcal{R}_c$ is topologically closed, we deduce, $\phi^* \in \mathcal{R}_c$. Now, since T is a hybrid rational Geraghty type contractive mapping, we have

$$\begin{aligned}
&\|T\phi^* - \phi^*(c)\|_E \\
&\leq \|T\phi^* - \phi_n(c)\|_E + \|\phi_n(c) - \phi^*(c)\|_E \\
&= \|T\phi^* - T\phi_{n-1})\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\leq \beta(\|\phi^* - \phi_{n-1}\|_{E_0}) M(\phi^*, \phi_{n-1}) + \gamma(\|\phi^* - \phi_{n-1}\|_{E_0}) N(\phi^*, \phi_{n-1}) + \|\phi_n - \phi^*\|_{E_0}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
\|T\phi^* - \phi^*(c)\|_E \quad &\leq \lim_{n\to\infty} \beta(\|\phi^* - \phi_{n-1}\|_{E_0}) \lim_{n\to\infty} M(\phi^*, \phi_{n-1}) \\
&+ \lim_{n\to\infty} \gamma(\|\phi^* - \phi_{n-1}\|_{E_0}) \lim_{n\to\infty} N(\phi^*, \phi_{n-1}).
\end{aligned} \tag{11}$$

But,

$$\begin{aligned}
\lim_{n\to\infty} M(\phi^*, \phi_{n-1}) \quad &= \lim_{n\to\infty} \max\left\{\|\phi^* - \phi_{n-1}\|_{E_0}, \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|T\phi^* - T\phi_{n-1}\|_E}\right\} \\
&= \lim_{n\to\infty} \max\left\{\|\phi^* - \phi_{n-1}\|_{E_0}, \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|T\phi^* - \phi_n(c)\|_E}\right\} \\
&= \lim_{n\to\infty} \max\left\{\|\phi^* - \phi_{n-1}\|_{E_0}, \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1} - \phi_n\|_{E_0}}{1 + \|T\phi^* - \phi_n(c)\|_E}\right\} = 0
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} N(\phi^*, \phi_{n-1}) \\
&= \lim_{n\to\infty} \min\left\{\|\phi^*(c) - T\phi^*\|_E, \|\phi_{n-1}(c) - T\phi_{n-1}\|_E, \|\phi^*(c) - T\phi_{n-1}\|_E, \|\phi_{n-1}(c) - T\phi^*\|_E\right\} \\
&= \lim_{n\to\infty} \min\left\{\|\phi^*(c) - T\phi^*\|_E, \|\phi_{n-1}(c) - \phi_n(c)\|_E, \|\phi^*(c) - \phi_n(c)\|_E, \|\phi_{n-1}(c) - T\phi^*\|_E\right\} \\
&= \lim_{n\to\infty} \min\left\{\|\phi^*(c) - T\phi^*\|_E, \|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi^* - \phi_n\|_{E_0}, \|\phi_{n-1}(c) - T\phi^*\|_E\right\} = 0.
\end{aligned} \tag{13}$$

Therefore, from (11), (12) and (13), we deduce

$$\|T\phi^* - \phi^*(c)\|_E = 0,$$

that is,

$$T\phi^* = \phi^*(c)$$

which implies that ϕ^* is a *PPF* dependent fixed point of T in $\mathcal{R}_c$.

Suppose that ϕ^* and φ^* be two *PPF* dependent fixed points of T in $\mathcal{R}_c$ such that $\phi^* \neq \varphi^*$. So,

$$\begin{aligned}\|\phi^* - \varphi^*\|_{E_0} &= \|\phi^*(c) - \varphi^*(c)\|_E \\ &= \|T\phi^* - T\varphi^*\|_E \leq \beta(\|\phi^* - \varphi^*\|_{E_0})M(\phi^*, \varphi^*) + \gamma(\|\phi^* - \varphi^*\|_{E_0})N(\phi^*, \varphi^*)\end{aligned}$$

where,

$$M(\phi^*, \varphi^*) = \max\left\{\|\phi^* - \varphi^*\|_{E_0}, \frac{\|\phi^*(c) - T\phi^*\|_E\|\varphi^*(c) - T\varphi^*\|_E}{1 + \|T\phi^* - T\varphi^*\|_E}\right\} = \|\phi^* - \varphi^*\|_{E_0}$$

and $N(\phi^*, \varphi^*) = 0$. Therefore, we have,

$$\|\phi^* - \varphi^*\|_{E_0} \leq \beta(\|\phi^* - \varphi^*\|_{E_0})\|\phi^* - \varphi^*\|_{E_0} < \|\phi^* - \varphi^*\|_{E_0}$$

which is a contradiction. Hence, $\phi^* = \varphi^*$. Then, T has a unique *PPF* dependent fixed point in $\mathcal{R}_c$. □

If in Theorem 2.2, we take $\beta(t) = r$ and $\gamma(t) = L$, where $0 \leq r < 1$ and $L \geq 0$, then we deduce the following corollary.

Corollary 2.3. *Let $T : E_0 \to E$ be a non-self mapping such that,*

$$\|T\phi - T\xi\|_E \leq rM(\phi, \xi) + LN(\phi, \xi)$$

for all $\phi, \xi \in E_0$ where $0 \leq r < 1$, $L \geq 0$, $c \in I$ and

$$M(\phi, \xi) = \max\left\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}$$

and

$$N(\phi, \xi) = \min\left\{\|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E, \|\phi(c) - T\xi\|_E, \|\xi(c) - T\phi\|_E\right\}.$$

Assume that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference. Then, T has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of T be defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

If in Corollary 2.3 we take $L = 0$, then we obtain the following result.

Corollary 2.4. *Let $T : E_0 \to E$ be a non-self mapping such that,*

$$\|T\phi - T\xi\|_E \leq r\max\left\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}$$

for all $\phi, \xi \in E_0$ where $0 \leq r < 1$ and $c \in I$. Assume that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference. Then, T has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$.

Corollary 2.5. *(Theorem 3.2 of [13]) Let $T : E_0 \to E$ be a non-self mapping such that,*

$$\|T\phi - T\xi\|_E \leq a\|\phi - \xi\|_{E_0} + b\frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{14}$$

for all $\phi, \xi \in E_0$ where $a, b \geq 0$, $0 \leq a + b < 1$ and $c \in I$. Assume, $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference. Then, T has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of T be defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Proof. Since,

$$a\|\phi-\xi\|_{E_0}+b\frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|T\phi-T\xi\|_E}\leq(a+b)\max\left\{\|\phi-\xi\|_{E_0},\frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|T\phi-T\xi\|_E}\right\},$$

then from (14) we have,

$$\|T\phi-T\xi\|_E\leq r\max\left\{\|\phi-\xi\|_{E_0},\frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|T\phi-T\xi\|_E}\right\}$$

where $r=a+b$. Hence, all the conditions of Corollary 2.4 hold and T has a unique *PPF* dependent fixed point $\phi^*\in\mathcal{R}_c$. □

In 1962, Edelstein [6] proved an interesting version of Banach contraction Principle. In 2009, Suzuki [14] proved certain remarkable results to improve the results of Banach and Edelstein (see also [9, 11, 12, 15]).

Denote with Ψ the family of all nondecreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<+\infty$ for all $t>0$, where ψ^n is the n-th iterate of ψ.

The following Remark is obvious.

Remark 2.6. *If $\psi\in\Psi$, then $\psi(t)<t$ for all $t>0$.*

Now, we are ready to prove the following Suzuki-Edelstein type theorem for nonlinear contractions in Razumikhin class.

Theorem 2.7. *Let $T:E_0\to E$ be a mapping. Suppose that there exists $\psi\in\Psi$ such that,*

$$\frac{1}{2}\|\phi(c)-T\phi\|_E\leq\|\phi-\xi\|_{E_0}\Longrightarrow\|T\phi-T\xi\|_E\leq\psi(\|\phi-\xi\|_{E_0})\tag{15}$$

for all $\phi,\xi\in E_0$. Assume, $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference. Then, T has a unique PPF dependent fixed point $\phi^*\in\mathcal{R}_c$. Moreover, for a fixed $\phi_0\in\mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of T be defined by $T\phi_{n-1}=\phi_n(c)$ for all $n\in\mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Proof. Let $\phi_0\in\mathcal{R}_c$. Since $T\phi_0\in E$, there exists $x_1\in E$ such that $T\phi_0=x_1$. Choose $\phi_1\in\mathcal{R}_c$ such that,

$$x_1=\phi_1(c).$$

Continuing this process, by induction, we can build a sequence $\{\phi_n\}$ in $\mathcal{R}_c\subseteq E$ such that,

$$T\phi_{n-1}=\phi_n(c),\quad\text{for all } n\in\mathbb{N}.\tag{16}$$

If there exists $n_0\in\mathbb{N}$ such that $\phi_{n_0}=\phi_{n_0+1}$ then, $\phi_{n_0}(c)=T\phi_{n_0}$ and so we have no thing for prove. Hence, for all $n\in\mathbb{N}$ we assume, $\|\phi_{n-1}-\phi_n\|_{E_0}>0$. Since $\mathcal{R}_c$ is algebraically closed with respect to difference, it follows

$$\|\phi_{n-1}-\phi_n\|_{E_0}=\|\phi_{n-1}(c)-\phi_n(c)\|_E,\quad\text{for all } n\in\mathbb{N}.$$

Now, we have,

$$\frac{1}{2}\|\phi_{n-1}(c)-T\phi_{n-1}\|_E=\frac{1}{2}\|\phi_{n-1}(c)-\phi_n(c)\|_E=\frac{1}{2}\|\phi_{n-1}-\phi_n\|_{E_0}\leq\|\phi_{n-1}-\phi_n\|_{E_0}$$

and so by (15) we get,

$$\|\phi_n-\phi_{n+1}\|_{E_0}=\|\phi_n(c)-\phi_{n+1}(c)\|_E=\|T\phi_{n-1}-T\phi_n\|_E\leq\psi(\|\phi_{n-1}-\phi_n\|_{E_0})\tag{17}$$

and then,

$$\|\phi_n-\phi_{n+1}\|_{E_0}\leq\psi^n(\|\phi_0-\phi_1\|_{E_0})$$

for all $n \in \mathbb{N}$.

Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n(\|\phi_0 - \phi_1\|_{E_0}) < \epsilon.$$

Let $m, n \in \mathbb{N}$ with $m > n \geq N$. Then by triangular inequality we get

$$\|\phi_n - \phi_m\|_{E_0} \leq \sum_{k=n}^{m-1} \|\phi_k - \phi_{k+1}\|_{E_0} \leq \sum_{n \geq N} \psi^n(\|\phi_0 - \phi_1\|_{E_0}) < \epsilon.$$

Consequently, $\lim_{m,n \to +\infty} \|\phi_n - \phi_m\|_{E_0} = 0$. Hence, $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. Completeness of E_0 yields that $\{\phi_n\}$ converges to a point $\phi^* \in E_0$, that is, $\phi_n \to \phi^*$ as $n \to \infty$. Since, $\mathcal{R}_c$ is topologically closed, we deduce, $\phi^* \in \mathcal{R}_c$. Therefore from (17) we get,

$$\|\phi_n - \phi_{n+1}\|_{E_0} < \|\phi_{n-1} - \phi_n\|_{E_0} \tag{18}$$

for all $n \in \mathbb{N}$. Suppose that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{1}{2}\|\phi_{n_0}(c) - T\phi_{n_0}\|_E > \|\phi_{n_0-1} - \phi^*\|_{E_0}$$

and

$$\frac{1}{2}\|\phi_{n_0+1}(c) - T\phi_{n_0+1}\|_E > \|\phi_{n_0} - \phi^*\|_{E_0}$$

then from (18) we have,

$$\begin{aligned}
\|\phi_{n_0-1} - \phi_{n_0}\|_{E_0} &\leq \|\phi_{n_0-1} - \phi^*\|_{E_0} + \|\phi_{n_0} - \phi^*\|_{E_0} \\
&< \frac{1}{2}\|\phi_{n_0}(c) - T\phi_{n_0}\|_E + \frac{1}{2}\|\phi_{n_0+1}(c) - T\phi_{n_0+1}\|_E \\
&= \frac{1}{2}\|\phi_{n_0}(c) - \phi_{n_0+1}(c)\|_E + \frac{1}{2}\|\phi_{n_0+1}(c) - \phi_{n_0+2}(c)\|_E \\
&= \frac{1}{2}\|\phi_{n_0} - \phi_{n_0+1}\|_{E_0} + \frac{1}{2}\|\phi_{n_0+1} - \phi_{n_0+2}\|_{E_0} \\
&\leq \frac{1}{2}\|\phi_{n_0} - \phi_{n_0-1}\|_{E_0} + \frac{1}{2}\|\phi_{n_0} - \phi_{n_0-1}\|_{E_0} = \|\phi_{n_0} - \phi_{n_0-1}\|_{E_0}
\end{aligned}$$

which is a contradiction. Hence, either,

$$\frac{1}{2}\|\phi_n(c) - T\phi_n\|_E \leq \|\phi_{n-1} - \phi^*\|_{E_0}$$

or

$$\frac{1}{2}\|\phi_{n+1}(c) - T\phi_{n+1}\|_E \leq \|\phi_n - \phi^*\|_{E_0}$$

holds for all $n \in \mathbb{N}$. First, suppose that,

$$\frac{1}{2}\|\phi_n(c) - T\phi_n\|_E \leq \|\phi_{n-1} - \phi^*\|_{E_0}$$

holds for all $n \in \mathbb{N}$. Then from (15) we have,

$$\begin{aligned}
\|T\phi^* - \phi^*(c)\|_E &\leq \|T\phi^* - T\phi_n\|_E + \|T\phi_n - \phi^*(c)\|_E \\
&= \|T\phi^* - T\phi_n\|_E + \|\phi_{n+1}(c) - \phi^*(c)\|_{E_0} \\
&\leq \psi(\|\phi^* - \phi_n\|_{E_0}) + \|\phi_{n+1} - \phi^*\|_{E_0}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequality we get, $\|T\phi^* - \phi^*(c)\|_E = 0$. i.e.,

$$T\phi^* = \phi^*(c)$$

By a similar method we can deduce $T\phi^* = \phi^*(c)$ when

$$\frac{1}{2}\|\phi_{n+1}(c) - T\phi_{n+1}\|_E \leq \|\phi_n - \phi^*\|_{E_0}.$$

Hence, we proved that ϕ^* is a *PPF* dependent fixed point of T in $\mathcal{R}_c$.

Finally, Suppose that ϕ^* and φ^* be two *PPF* dependent fixed points of T in $\mathcal{R}_c$ such that $\phi^* \neq \varphi^*$. So,

$$\frac{1}{2}\|\phi^*(c) - T\phi^*\|_E = 0 \leq \|\phi^* - \varphi^*\|_{E_0},$$

then from (15) we get,

$$\|\phi^* - \varphi^*\|_{E_0} = \|\phi^*(c) - \varphi^*(c)\|_E = \|T\phi^* - T\varphi^*\|_E \leq \psi(\|\phi^* - \varphi^*\|_{E_0}) < \|\phi^* - \varphi^*\|_{E_0}$$

which is a contradictions. Hence, $\phi^* = \varphi^*$. □

If in Theorem 2.7 we take $\psi(t) = rt$ where $0 \leq r < 1$ then we deduce the following corollary.

Corollary 2.8. *Let $T : E_0 \to E$ be a mapping. Suppose that there exists $0 \leq r < 1$ such that,*

$$\frac{1}{2}\|\phi(c) - T\phi\|_E \leq \|\phi - \xi\|_{E_0} \Longrightarrow \|T\phi - T\xi\|_E \leq r\|\phi - \xi\|_{E_0} \tag{19}$$

for all $\phi, \xi \in E_0$. Assume, $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference. Then, T has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of T be defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

3. PPF Dependent Coincidence Point Results in the Razumikhin Class

Definition 3.1. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$. The ordered pair (T, S) is said to satisfy the condition of hybrid rational Geraghty contraction if there exists $\beta \in \mathcal{F}$ and $c \in I$ such that,*

$$\|T\phi - T\xi\|_E \leq \beta(\|S\phi - S\xi\|_{E_0})M^S(\phi, \xi) + \gamma(\|S\phi - S\xi\|_{E_0})N^S(\phi, \xi)$$

for all $\phi, \xi \in E_0$, where $\gamma : [0, \infty) \to [0, \infty)$ is a bounded function and

$$M^S(\phi, \xi) = \max\left\{\|S\phi - S\xi\|_{E_0}, \frac{\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}$$

and

$$N(\phi, \xi) = \min\Big\{\|(S\phi)(c) - T\phi\|_E, \|(S\xi)(c) - T\xi\|_E, \|(S\phi)(c) - T\xi\|_E, \|(S\xi)(c) - T\phi\|_E\Big\}.$$

Using Theorem 2.2 we deduce the following PPF dependent coincidence point Theorem.

Theorem 3.2. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$. Assume that (T, S) satisfy the condition of hybrid rational Geraghty type contraction such that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$. Suppose that $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference. Then T and S have a PPF dependent coincidence point.*

Proof. As $S : E_0 \to E_0$, so there exists $F_0 \subseteq E_0$ such that $S(F_0) = S(E_0)$ and $S\mid_{F_0}$ is one-to-one. Since $T(F_0) \subseteq T(E_0) \subseteq E$, we can define the mapping $\mathcal{A} : S(F_0) \to E$ by $\mathcal{A}(S\phi) = T\phi$ for all $\phi \in F_0$. Since $S\mid_{F_0}$ is one-to-one, then $\mathcal{A}$ is well-defined. Now, since (T, S) satisfy the condition of hybrid rational Geraghty contraction, we have,

$$\|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\|_E = \|T(\phi) - T(\xi)\|_E \leq \beta(\|S\phi - S\xi\|_{E_0})M^S(\phi, \xi) + \gamma(\|S\phi - S\xi\|_{E_0})N^S(\phi, \xi)$$

for all $\phi, \xi \in E_0$ where

$$M^S(\phi, \xi) = \max\left\{\|S\phi - S\xi\|_{E_0}, \frac{\|(S\phi)(c) - \mathcal{A}(S\phi)\|_E\|(S\xi)(c) - \mathcal{A}(S\xi)\|_E}{1 + \|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\|_E}\right\}$$

and

$$\begin{aligned} &N(\phi, \xi) \\ &= \min\left\{\|(S\phi)(c) - \mathcal{A}(S\phi)\|_E, \|(S\xi)(c) - \mathcal{A}(S\xi)\|_E, \|(S\phi)(c) - \mathcal{A}(S\xi)\|_E, \|(S\xi)(c) - \mathcal{A}(S\phi)\|_E\right\}. \end{aligned}$$

This shows that $\mathcal{A}$ is a hybrid rational Geraghty contractive mapping and all conditions of Theorem 2.2 hold. Then there exists unique *PPF* dependent fixed point $\varphi \in S(F_0)$ of $\mathcal{A}$, i.e., $\mathcal{A}\varphi = \varphi(c)$. Since $\varphi \in S(F_0)$ then there exists $\omega \in F_0$ such that,

$$T\omega = \mathcal{A}(S\omega) = \mathcal{A}\varphi = \varphi(c) = (S\omega)(c).$$

That is, ω is a *PPF* dependent coincidence point of S and T. □

Also, we can obtain the following corollaries.

Corollary 3.3. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two mappings. There exists $c \in I$ such that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$ and*

$$\|T\phi - T\xi\|_E \leq rM^S(\phi, \xi) + LN^S(\phi, \xi)$$

for all $\phi, \xi \in E_0$ where $\leq r < 1$, $L \geq 0$,

$$M^S(\phi, \xi) = \max\left\{\|S\phi - S\xi\|_{E_0}, \frac{\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}$$

and

$$N(\phi, \xi) = \min\left\{\|(S\phi)(c) - T\phi\|_E, \|(S\xi)(c) - T\xi\|_E, \|(S\phi)(c) - T\xi\|_E, \|(S\xi)(c) - T\phi\|_E\right\}.$$

Let $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference. Then, T and S have a PPF dependent coincidence point.

Corollary 3.4. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two mappings. There exists $c \in I$ such that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$ and*

$$\|T\phi - T\xi\|_E \leq r \max\left\{\|S\phi - S\xi\|_{E_0}, \frac{\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}$$

for all $\phi, \xi \in E_0$ where $\leq r < 1$. Let $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference. Then, T and S have a PPF dependent coincidence point.

Corollary 3.5. *(Theorem 4.3 of [13]) Let, $T : E_0 \to E$ and $S : E_0 \to E_0$ be two mappings. There exists $c \in I$ such that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$ and*

$$\|T\phi - T\xi\|_E \leq a\|S\phi - S\xi\|_{E_0} + b\frac{\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}$$

for all $\phi, \xi \in E_0$, where $a, b \geq 0$ and $0 \leq a + b < 1$. Let $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference. Then, T and S have a PPF dependent coincidence point.

Theorem 3.6. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$. Assume that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$ and (T, S) satisfy the condition*

$$\frac{1}{2}\|(S\phi)(c) - T\phi\|_E \leq \|S\phi - S\xi\|_{E_0} \Longrightarrow \|T\phi - T\xi\|_E \leq \psi(\|S\phi - S\xi\|_{E_0})$$

for all $\phi, \xi \in E_0$ where $\psi \in \Psi$. Let $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference. Then T and S have a PPF dependent coincidence point.

Corollary 3.7. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$. Assume that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$ and (T, S) satisfy the condition*

$$\frac{1}{2}\|(S\phi)(c) - T\phi\|_E \leq \|S\phi - S\xi\|_{E_0} \Longrightarrow \|T\phi - T\xi\|_E \leq r\|S\phi - S\xi\|_{E_0}$$

for all $\phi, \xi \in E_0$, where $0 \leq r < 1$. Let $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference. Then T and S have a PPF dependent coincidence point.

4. Application

In this section, we present an application of our Theorem 2.7 to establish PPF dependent solution to a periodic boundary value problem.

Consider the first-order periodic boundary value problem

$$\begin{cases} x'(t) = f(t, x(t), x_t), \\ x_0 = \phi_0 \in C[[-t, 0], \mathbb{R}] = C, \\ x(0) = x(T) = \phi_0(0), \end{cases} \tag{20}$$

where $t \in I = [0, T]$, $f \in C[[0, T] \times \mathbb{R} \times C, \mathbb{R}]$ and $x_t(s) = x(t + s)$ with $s \in [-t, 0]$.

Problem (20) can be rewritten as

$$\begin{cases} x'(t) + \lambda x(t) = f(t, x(t), x_t) + \lambda x(t), \\ x_0 = \phi_0 \in C[[-t, 0], \mathbb{R}] = C, \\ x(0) = x(T) = \phi_0(0). \end{cases}$$

Consider

$$\begin{cases} x'(t) + \lambda x(t) = \sigma(t) = F(t, x(t), x_t), \\ x_0 = \phi_0, \\ x(0) = x(T) = \phi_0(0), \end{cases}$$

where $t \in I$.

Using variation of parameters formula, we get,

$$x(t) = x(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\sigma(s)ds \tag{21}$$

which yields

$$x(T) = x(0)e^{-\lambda T} + \int_0^T e^{-\lambda(T-s)}\sigma(s)ds.$$

Since $x(0) = x(T)$, we get

$$x(0)[1 - e^{-\lambda T}] = e^{-\lambda T} \int_0^T e^{\lambda(s)} \sigma(s) ds$$

or

$$x(0) = \frac{1}{e^{\lambda T} - 1} \int_0^T e^{\lambda s} \sigma(s) ds.$$

Substituting the value of x(0) in (21) we arrive at

$$x(t) = \int_0^T G(t, s) \sigma(s) ds$$

where

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, 0 \le s \le t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, 0 \le t \le s \le T. \end{cases}$$

Let

$$\hat{E} = \{\hat{x} = (x_t)_{t \in I} : x_t \in C, x \in C[[0, T], \mathcal{R}], x(0) = x(T) = \phi_0(0), x_0 = \phi_0 \in C\}.$$

This means that $\hat{x} \in C[[-t, 0], \mathcal{R}]$.

Let

$$\|\hat{x} - \hat{y}\|_{\hat{E}} = \sup_{t \in I} \max_{-t \le s \le 0} |x_t(s) - y_t(s)| = \sup_{t \in I} \|x_t - y_t\|_C.$$

In [16], it is shown that $\hat{E}$ is complete.

Assume that there exists $\lambda > 0$ such that for all $x, y : I \to \mathbb{R}$ and $\phi, \xi \in C$ with, $\frac{1}{2} \|\phi(t) - \int_0^T G(t, s) F(s, x(s), \phi) ds\|_{\mathbb{R}} \le \|\phi - \xi\|_C$ we have,

$$\|[f(t, x(t), \phi) + \lambda x] - [f(t, y(t), \xi) + \lambda y]\| \le \lambda \psi(\|\phi - \xi\|_C).$$

Then the PBVP (20) has a unique solution in a Razumikhin class.

For this define operator $S : \hat{E} \to \mathbb{R}$ as

$$S\hat{x}(t) = \int_0^T G(t, s) F(s, x(s), x_s) ds.$$

Clearly, S is continuous. Further for $x, y : I \to \mathbb{R}$ and $\phi, \xi \in C$ with, $\frac{1}{2} \|\phi(t) - \int_0^T G(t, s) F(s, x(s), x_s) ds\|_{\mathbb{R}} \le \|\phi - \xi\|_C$. We have,

$$\begin{aligned} |S\hat{x}(t) - S\hat{y}(t)| &= \int_0^T G(t, s)[F(s, x(s), x_s) - F(s, y(s), y_s)] ds \\ &\le \int_0^T G(t, s) \lambda \psi(\|x_s - y_s\|_C) ds \\ &\le \lambda \psi(\|\hat{x} - \hat{y}\|_{\hat{E}}) \left[\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right] \\ &= \lambda \psi(\|\hat{x} - \hat{y}\|_{\hat{E}}) \left[\frac{1}{\lambda(e^{\lambda T} - 1)} \left(e^{\lambda(T+s-t)} \Big|_0^t + e^{\lambda(s-t)} \Big|_t^T \right) \right] \\ &= \lambda \psi(\|\hat{x} - \hat{y}\|_{\hat{E}}) \left[\frac{1}{\lambda(e^{\lambda T} - 1)} \left(e^{\lambda T} - e^{\lambda(T-t)} + e^{\lambda(T-t)} - 1 \right) \right] \\ &= \psi(\|\hat{x} - \hat{y}\|_{\hat{E}}). \end{aligned}$$

Hence, the hypotheses of Theorem 2.7 are satisfied and so, there exists a fixed point $\hat{x}^* \in \hat{E}$ such that $S\hat{x}^* = (x^*(t))_{t\in I}$.

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