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Optimality Conditions and Duality for Nonsmooth Minimax Programming Problems under Generalized Invexity

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Abstract. In this paper, we consider a class of nonsmooth minimax programming problems in which functions are locally Lipschitz. Sufficient optimality conditions are discussed under locally Lipschitz generalized (Φ, ρ)-invex functions. Moreover, usual duality results are proved under the said assumptions.

1. Introduction

Minimax problems occur frequently in many important areas like game theory, Chebychev approximation, economics, financial planning and facility location [11]. Some of the basic results of minimax problems can be found in [10]. Chew [5] Minimax problems under the assumptions of pseudolinear function is studied in [5]. Tanimoto [17] derived duality theorems for some minimax type problems involving convex functions. Mond and Weir [14] discussed optimality conditions and duality for minimax problem under pseudoconvexity. Antczak [2] established sufficiency and duality results for minimax problems under (p, r)-invexity.

Convexity plays an important role in many aspects of mathematical programming including optimality conditions, duality theorems and alternative theorems. But, due to insufficiency of convexity notion in many mathematical models used in decision science, economics, engineering, etc., there has been an increasing interest in relaxing convexity assumptions in connection with sufficiency and duality theorems. One of the most lively generalizations of convexity is due to Hanson [12], which was named as invexity by Craven [7]. Since many practical problems encountered in economics, engineering design and management science, etc., can be described only by nonsmooth functions; consequently, the theory of nonsmooth optimization using locally Lipschitz functions was put forward by Francis Clarke in 1980's (see [6]). He extended the properties of convex functions to the case of locally Lipschitz functions by suitably defining a generalized derivative and a subdifferential. Later on, the notion of invexity was extended to locally Lipschitz functions in [8], by replacing the derivative with Clarke generalized gradient. Reiland [15] pointed out that under the invexity assumption the Kuhn-Tucker conditions also assures the optimality in nondifferentiable programming involving locally Lipschitz functions.

The definition of (Φ, ρ)-invexity notion has been introduced by Caristi et al. [4] for differentiable function and established sufficient optimality conditions and duality results for differentiable optimization problems. Stefanescu and Stefanescu [16] used the (Φ, ρ)-invexity to discuss the optimality conditions and

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duality results for differentiable minimax programming problem. Recently, Antczak [3] generalize the definition of (Φ, ρ)invexity notion introduced by Caristi et al. [4] for differentiable optimization problems to the case of mathematical programming problems with locally Lipschitz functions and established sufficient optimality conditions and Mond-Weir duality results for a new class of nonconvex nonsmooth mathematical programming problems.

The purpose of this article is to discuss the application of locally Lipschitz generalized (Φ, ρ)-invexity for a class of nonsmooth minimax programming problems. We discuss the sufficient optimality conditions and duality results for a minimax programming problem.

2. Preliminaries

Throughout this section, X is a nonempty open subset of R^n.

A function $f : R^n \to R$ is said to *locally Lipschitz at a point* $\bar{x} \in R^n$ if there exist scalars $\zeta > 0$ and $\epsilon > 0$ such that

$$|f(x^1) - f(x^2)| \leq \zeta \|x^1 - x^2\|, \quad \text{for all } x^1, x^2 \in \bar{x} + \epsilon B$$

where $\bar{x} + \epsilon B$ is the open ball of radius ϵ around $\bar{x}$, and $\|\cdot\|$ being any norm in R^n.

The *Clarke generalized directional derivative* [6] of a locally Lipschitz function $f : R^n \to R$ at $\bar{x}$ in the direction $v \in R^n$, denoted by $f^\circ(\bar{x}; v)$, is defined as

$$f^\circ(\bar{x}; v) = \lim_{t \downarrow 0} \sup_{y \to \bar{x}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in R^n.

The *Clarke generalized gradient* [6] of $f : R^n \to R$ at $\bar{x}$, denoted by $\partial f(\bar{x})$, is defined as

$$\partial f(\bar{x}) = \{\xi \in R^n : f^\circ(\bar{x}; v) \geq \xi^\top v, \ \forall\, v \in R^n\}.$$

It follows that for any $v \in R^n$, $f^\circ(\bar{x}; v) = \max\{\xi^T v : \xi \in \partial f(\bar{x})\}$.

The following result was given by Clarke [6].

Theorem 2.1. *If a locally Lipschitz function $f : X \to R$ attains a local minimum or maximum at $\bar{x}$, then $0 \in \partial f(\bar{x})$.*

In the definitions below, Φ is a real valued locally Lipschitz function defined on $X \times X \times R^{n+1}$ such that $\Phi(x, \bar{x}, .)$ is convex on R^{n+1} and $\Phi(x, \bar{x}, (0, a)) \geq 0$ for every $x \in X$ and any $\bar{a} \in R_+$, and a real number ρ.

Definition 2.1. *[3]. f is said to be locally Lipschitz (Φ, ρ)-invex at $\bar{x}$ on X if*

$$f(x) - f(\bar{x}) \geq \Phi(x, \bar{x}, (\xi, \rho))$$

holds for any $\xi \in \partial f(\bar{x})$ and all $x \in X$.

Now, we introduce the definition of locally Lipschitz pseudo(Φ, ρ)-invex and locally Lipschitz quasi(Φ, ρ)-invex functions.

Definition 2.2. *f is said to be locally Lipschitz pseudo(Φ, ρ)-invex at $\bar{x}$ on X if for any $\xi \in \partial f(\bar{x})$ and all $x \in X$*

$$\Phi(x, \bar{x}, (\xi, \rho)) \geq 0 \Rightarrow f(x) - f(\bar{x}) \geq 0.$$

Definition 2.3. *f is said to be locally Lipschitz quasi(Φ, ρ)-invex at $\bar{x}$ on X if for any $\xi \in \partial f(\bar{x})$ and all $x \in X$*

$$f(x) - f(\bar{x}) \leq 0 \Rightarrow \Phi(x, \bar{x}, (\xi, \rho)) \leq 0.$$

In the Definition 2.3, if the inequalities hold as strict inequalities, then f is said to be locally Lipschitz semistrict quasi(Φ, ρ)-invex at $\bar{x}$

3. Nonsmooth Minimax Programming

We consider the following minimax programming problem:

$$(\mathrm{P}) \quad \min_{x\in X}\max_{1\le i\le k} f_i(x)$$

subject to

$$g_j(x) \le 0,\ j = 1, 2, \cdots, m, \tag{1}$$

where $f_i : X \to R,\ i = 1, 2, \ldots, k,\ g_j : X \to R,\ \ j = 1, 2, \ldots, m$ are locally Lipschitz functions, and X is a non-empty open subset of R^n. Let $D = \{x \in X : g_j(x) \le 0,\ j = 1, 2, \ldots, m\}$ be set of all feasible solutions of (P). Let $J(x) = \{j : g_j(\bar{x}) = 0\}$ be the set of active constraint at $\bar{x} \in D$.

It is well known (see example [9]) that the problem (P) is equivalent to the following problem (EP) in the sense of the Lemma 3.1 and 3.2 given further

$$(\mathrm{EP}) \quad \min v$$

$$\text{subject to } f(x) \le ve \tag{2}$$

$$g_j(x) \le 0,\ j = 1, 2, \cdots, m, \tag{3}$$

$$(x, v) \in X \times R. \tag{4}$$

Here e is the k-dimensional vector with entries 1 and $f(x) = (f_1(x), f_2(x), \cdots, f_k(x))$. Let $A = \{(x, v) \in X \times R : f(x) \le ve, g_j(x) \le 0,\ j = 1, 2 \ldots, m\}$ be set of all feasible solutions of (EP).

Lemma 3.1. *If a point (x, v) is feasible for (EP), then x is a feasible point for (P). Moreover, if a point x is feasible for (P) then there exists $v \in R$ such that (x, v) is feasible point for (EP).*

Lemma 3.2. *A point $\bar{x}$ is an optimal solution for (P) with the corresponding value of the objective function of (P) equal to $\bar{v}$ if and only if a point $(\bar{x}, \bar{v})$ is an optimal solution of (EP) with the corresponding optimal value of the objective function of (EP) equal to $\bar{v}$.*

Note that (EP) is a nonsmooth minimization nonlinear programming problem. The following necessary optimality conditions are the nonsmooth version of the necessary optimality conditions given in [2].

Theorem 3.1. *(Kuhn-Tucker type necessary optimality conditions). Let $\bar{x} \in D$ be an optimal solution of (P) with the corresponding optimal value for (P) equal to $\bar{v}$ and a Slater's constraint qualification is satisfied at $\bar{x}$. Then, there exist $\lambda \in R^k_+,\ \mu \in R^m_+$ such that $(\bar{x}, \bar{v}, \lambda, \mu)$ satisfies the following conditions:*

$$0 \in \sum_{i=1}^{k} \lambda_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \partial g_j(\bar{x}) \tag{5}$$

$$\lambda f(\bar{x}) = \bar{v} \tag{6}$$

$$f(\bar{x}) \le \bar{v}e \tag{7}$$

$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, 2, ..., m \tag{8}$$

$$\lambda e = 1. \tag{9}$$

If we set $P(\bar{x}) = \{i : f_i(\bar{x}) = \max_{1\le l\le p} f_l(\bar{x})\}$, then $(\bar{x}, \lambda, \mu) \in D \times R^k_+ \times R^m_+$ verifies (8) and the conditions

$$0 \in \sum_{i\in P(\bar{x})} \lambda_i \partial f_i(\bar{x}) + \sum_{j\in J(\bar{x})} \mu_j \partial g_j(\bar{x}) \tag{10}$$

$$\sum_{i\in P(\bar{x})} \lambda_i = 1. \tag{11}$$

Now, we prove that Kuhn-Tucker conditions are satisfied by solutions of (P) (or (EP)) if these problems also satisfies some locally Lipschitz generalized (Φ, ρ)-invexity conditions.

Theorem 3.2. *Let $(\bar{x}, \bar{v})$ an optimal solution of (EP). Moreover, we assume that the functions $g_j, j \in J(\bar{x})$ is locally Lipschtiz semistrict quasi(Φ, ρ_{g_j})-invex at $\bar{x}$ for some $\rho_{g_j} \geq 0$. Then there exist $\lambda \in R^k_+$ and $\mu \in R^m_+$ such that (5)-(9) are verified by $((\bar{x}, \bar{v}), \lambda, \mu)$.*

Proof. The Fritz John necessary conditions claim the existence of non-negative multipliers $w \in R, \alpha \in R^k, \beta \in R^m$ such that

$$w - \alpha e = 0, 0 \in \sum_{i=1}^{k} \alpha_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \beta_j \partial g_j(\bar{x}), \tag{12}$$

$$f(\bar{x}) \leq \bar{v}e,\ \alpha f(\bar{x}) - \alpha \bar{v} e = 0, \beta_j g_j(\bar{x}) = 0, j = 1, 2, \cdots, m, \tag{13}$$

$$w + \sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{m} \beta_j > 0. \tag{14}$$

we have to prove that $w + \sum_{i=1}^{k} \alpha_i > 0$.

Otherwise, if $w = 0$ and $\alpha = 0$, it follows from (14) that $\beta_0 = \sum_{j=1}^{m} \beta_j > 0$. Setting $\beta_j^{'} = \frac{\beta_j}{\beta_0}, j = 1, 2, ..., m,$ equation (5) becomes $\sum_{j \in J(\bar{x})} \beta_j^{'} \zeta_j = 0; \zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x})$. If then follows from the property of Φ that

$$0 \leq \Phi(x, \bar{x}, (\sum_{j \in J(\bar{x})} \beta_j^{'} \zeta_j, \sum_{j \in J(\bar{x})} \beta_j^{'} \rho_{g_j})) \leq \sum_{j \in J(\bar{x})} \beta_j^{'} \Phi(x, \bar{x}; (\zeta_j, \rho_{g_j})) \text{ for every } x \in D.$$

Let $x^* \in D$ satisfies Slater's conditions $g_j(x^*) < 0, \forall j \in J(\bar{x})$. Then because each g_j is locally Lipschitz semistrict quasi$(\phi, \rho - g_j)$-invex, we have

$$\sum_{j \in J(\bar{x})} \beta_j^{'} \phi(x^*, \bar{x}, (\zeta_j, \rho_{g_j})) < 0, \zeta_j \in \partial g_j(\bar{x}),$$

so that we have reached a contradiction.

Now, observe that the above inequality together (12) say that $\lambda e = 1$ where $\lambda_i = \frac{\alpha_i}{w}$. Finally, set $\mu_j = \frac{\beta_j}{w}, j = 1, 2, ..., m$ and the proof is complete. □

The next two results concern the sufficiency of Kuhn-Tucker conditions when locally Lipschitz generalized (ϕ, ρ)-invexity is assumed.

Theorem 3.3. *Let $(\bar{x}, \bar{v}, \lambda, \mu) \in D \times R \times R^k_+ \times R^m_+$ satisfying relations (5)-(9). Moreover, we assume that $\sum_{i=1}^{k} \lambda_i f_i$ is locally Lipschitz pseudo(Φ, ρ_f)-invex at $\bar{x}, g_j, j \in J(\bar{x})$ is locally Lipschitz quasi(Φ, ρ_{g_j})-invex at $\bar{x}$ on D and $\rho_f + \sum_{j \in J(\bar{x})} \mu_j \rho_{g_j} \geq 0$. Then $\bar{x}$ is an optimal solution for (P) with the corresponding value equal to $\bar{v}$.*

Proof. By (7), $(\bar{x}, \bar{v})$ is a feasible solution of (EP) . Let (x^*, v) be an arbitrary feasible solution of (EP). Two situations are considered (i) $\lambda = 0$ (ii) $\lambda \neq 0$. From (5) it is clear that there exists $\xi_i \in \partial f_i(\bar{x}), \zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x})$, such that

$$\sum_{i=1}^{k}\lambda_i\xi_i + \sum_{j\in J(\bar{x})}\mu_j\zeta_j = 0.$$

In case (i) we have

$$\Phi(x,\bar{x},(\sum_{i=1}^{k}\lambda_i\xi_i,\rho_f)) \geq 0. \tag{15}$$

If case (ii) holds, let $w = 1 + \sum_{j\in J(\bar{x})}\mu_j$, so that we have

$$\frac{1}{w}\Phi(x,\bar{x},(\sum_{i=1}^{k}\lambda_i\xi_i,\rho_f)) + \sum_{j\in J(\bar{x})}\frac{\mu_j}{w}\Phi(x,\bar{x},(\zeta_j,\rho_{g_j})) \geq \Phi(x,\bar{x},(\frac{1}{w}\sum_{i=1}^{k}\lambda_i\xi_i + \sum_{j\in J(\bar{x})}\frac{\mu_j}{w}\zeta_j, \frac{\rho_f}{w} + \sum_{j\in J(\bar{x})}\frac{\mu_j\rho_j}{w})) \geq 0.$$

Since, g_j are locally Lipschitz quasi(Φ,ρ)-invex at $\bar{x}$ and $g_j(x) \leq g_j(\bar{x})$ for each $j \in J(\bar{x})$, we have

$$\sum_{j\in J(\bar{x})}\frac{\mu_j}{w}\Phi(x,\bar{x},(\nabla g_j(\bar{x}),\rho_{gj}) \leq 0.$$

we arrive again to (15).
Now by (15), locally Lipschitz pseudo(Φ,ρ_f)-invexity implies the inequality

$$\sum_{i=1}^{k}\lambda_i f_i x - \sum_{i=1}^{k}\lambda_i f_i(\bar{x}) \geq 0.$$

But $f_i(x) \leq v$, $\sum_{i=1}^{k}\lambda_i f_i(\bar{x}) = \bar{v}$ and $\lambda e = 1$. Therefore, the inequality above implies the inequality $v - \bar{v} \geq 0$. □

Theorem 3.4. *Let $(\bar{x},\lambda,\mu)$ satisfying (3), (10) and (11). Moreover, we assume that f_i is locally Lipschitz semistrict quasi (Φ,ρ_{f_i})- invex at $\bar{x}$ for each $i \in P(\bar{x})$, g_j is locally Lipschitz quasi (Φ,ρ_{g_j})-invex at $\bar{x}$ for each $j \in J(\bar{x})$, and $\sum_{i\in P(\bar{x})}\lambda_i\rho_{f_i} + \sum_{j\in J(\bar{x})}\mu_j\rho_{g_j} \geq 0$. Then $\bar{x}$ is an optimal solution of (P).*

Proof. By Theorem 3.1, $\bar{x}$ is a feasible solution of (P). Put $\bar{v} = \max_{1\leq i\leq k} f_i(\bar{x})$. Thus, $f_i(\bar{x}) = \bar{v}$ for every $i \in P(\bar{x})$. Suppose that $\bar{x}$ is not optimal. Then there exists $x \in D$ such that $v = \max_{1\leq i\leq k} f_i(x) < \bar{v}$. From (10) , it is clear that there exist $\xi_i \in \partial f_i(\bar{x}), i \in P(\bar{x})$ and $\zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x})$ such that

$$\sum_{i\in P(\bar{x})}\alpha_i\xi_i + \sum_{j\in J(\bar{x})}\beta_j\zeta_j = 0,$$

where $\alpha_i = \frac{\lambda_i}{\lambda_0}, i \in P(\bar{x}), \beta_j = \frac{\mu_j}{\lambda_0}, j \in J(\bar{x}), \lambda_0 = 1 + \sum_{j\in J(\bar{x})}\mu_j$. Also $\sum_{i\in P(\bar{x})}\alpha_i\rho_{f_i} + \sum_{j\in J(\bar{x})}\beta_j\rho_{g_j} \geq 0$, then

$$\Phi(x,\bar{x},(\sum_{i\in P(\bar{x})}\alpha_i\xi_i + \sum_{j\in J(\bar{x})}\beta_j\zeta_j, \sum_{i\in P(\bar{x})}\alpha_i\rho_{f_i} + \sum_{j\in J(\bar{x})}\beta_j\rho_{g_j})) \geq 0.$$

Moreover,

$$\sum_{i\in P(\bar{x})}\alpha_i, \Phi(x,\bar{x},(\xi_i,\rho_{f_i})) + \sum_{j\in J(\bar{x})}\beta_j\Phi(x,\bar{x},(\zeta_j,\rho_{g_j})) \geq 0.$$

Hence

$$\sum_{i\in P(\bar{x})} \alpha_i, \Phi(x, \bar{x}, (\xi_i, \rho_{f_i})) \geq 0. \tag{16}$$

For $i \in P(\bar{x})$ we have $f_i(x) - f_i(\bar{x}) \leq v - \bar{v} < 0$, so that $\Phi(x, \bar{x}, (\xi_i, \rho_{f_i})) < 0$. Since $\alpha \in R^k_+$ and $\sum_{i=1}^{R} \alpha_i > 0$, we have

$$\sum_{i\in P(\bar{x})} \alpha_i, \Phi(x, \bar{x}, (\xi_i, \rho_{f_i})) < 0,$$

which contradicts inequality (16). □

4. Mond-Weir Type Duality

Making use of the first-order necessary conditions of Section 3, in the section, we present the following Mond weir dual and establish appropriate duality theorems under locally Lipschitz (Φ, ρ)- invexity assumptions

$$\text{(MD)} \max f(y) = (f_1(y), f_2(y), \dots f_k(y))$$

subject to

$$0 \in \sum_{i=1}^{k} \lambda_i \partial f_i(y) + \sum_{j=1}^{m} \mu_j \partial g_j(y) \tag{17}$$

$$\mu_j g_j(y) \geq 0, \quad j \in M \tag{18}$$

$\lambda \geq 0,\ \lambda e = 1,\ \mu \geq 0.$

The equivalence of (P) and (EP) allow us to refer to (MD) as to the dual of (P). Let U be the set of all feasible solution of (MD).

Theorem 4.1. *Let x be a feasible solution of (P) and (y, λ, μ) a feasible solution of (MD). Further, assume that f_i is locally Lipschtiz(Φ, ρ_{f_i})- invex at y for each $i \in \{1, 2, ..., k\}$ and g_j is locally Lipschtiz (Φ, ρ_{g_j})– invex at y for each $j \in \{1, 2, ..., m\}$. If $\sum_{i=1}^{k} \lambda_i \rho_{f_i} + \sum_{j=1}^{m} \mu_j \rho_{g_j} \geq 0$, then*

$$\sum_{i=1}^{k} \lambda_i f_i(y) \leq \max_{1\leq i\leq k} f_i(x).$$

Proof. Since $x \in S$ and $(y, \mu, \lambda) \in U$, we have

$$\mu_j g_j(x) \leq 0 \leq \mu_j g_j(y)$$

or

$$\mu_j g_j(x) - \mu_j g_j(y) \leq 0 \tag{19}$$

By (17), it is clear that there exist $\xi_i \in \partial f_i(y), \zeta_j \in \partial g_j(y)$ such that

$$\sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j = 0. \tag{20}$$

The locally Lipschtiz invexity of f_i and g_j imply

$$\lambda_i f_i(x) - \lambda_i f_i(y) \geq \lambda_i \Phi(x, y, (\xi_i, \rho_{f_i})),\ \xi_i \in \partial f_i(y)$$

and

$$\mu_j g_j(x) - \mu_j g_j(y) \geq \mu_j \Phi(x, y, (\zeta_j, \rho_{g_j})),\ \zeta_j \in \partial g_j(y).$$

Setting

$$\alpha_i = \frac{\lambda_i}{w}, \beta_j = \frac{\mu_j}{w} \text{ and } w = 1 + \sum_{j=1}^{m} \mu_j, \tag{21}$$

we have

$$\alpha_i f_i(x) - \alpha_i f_i(y) \geq \alpha_i \Phi(x, y, (\xi_i, \rho_{f_i})),\ \xi_i \in \partial f_i(y) \tag{22}$$

and

$$\beta_j g_j(x) - \beta_j g_j(y) \geq \beta_j \Phi(x, y, (\zeta_j, \rho_{g_j})),\ \zeta_j \in \partial g_j(y) \tag{23}$$

$$\beta_j g_j(x) - \beta_j g_j(y) \leq 0. \tag{24}$$

Adding (22) and (23), and using (24), we get

$$\sum_{i=1}^{k} \alpha_i (f_i(x) - f_i(y)) \geq \sum_{i=1}^{k} \alpha_i \Phi(x, y, (\xi_i, \rho_{f_i})) + \sum_{j=1}^{m} \beta_j \Phi(x, y, (\zeta_j, \rho_{g_j})), \tag{25}$$

for any $\xi_i \in \partial f_i(y)$ and $\zeta_j \in \partial g_j(y)$. By definition that $\Phi(x, y; .)$ is convex on R^{n+1}. Therefore

$$\sum_{i=1}^{k} \alpha_i \Phi(x, y, (\xi_i, \rho_{f_i})) + \sum_{j=1}^{m} \beta_j \Phi(x, y, (e_j \rho_{g_j}))$$

$$\geq \Phi(x, y(\sum_{i=1}^{k} \alpha_i \xi_i + \sum_{j=1}^{k} \beta_j \zeta_j, \sum_{i=1}^{k} \alpha_i \rho_{f_i} + \sum_{j=1}^{m} \beta_j \rho_{g_j}). \tag{26}$$

From the feasibility of dual problem and (21), it follows that

$$\sum_{i=1}^{k} \alpha_i \xi_i + \sum_{j=1}^{m} \beta_j \zeta_j = 0. \tag{27}$$

Combining (25),(26) and (27) we get

$$\sum_{i=1}^{k} \alpha_i f_i(x) - \sum_{i=1}^{k} \alpha_i f_i(y) \geq \Phi(x, y, (0, \sum_{i=1}^{k} \alpha_i \rho_{f_i} + \sum_{j=1}^{m} \beta_j \rho_{g_j})). \tag{28}$$

By Definition 2.1, it follows that $\Phi(x, y, (0, y)) \geq 0$. Since $\sum_{i=1}^{k} \alpha_i \rho_{f_i} + \sum_{j=1}^{m} \beta_j \rho_{g_j} \geq 0$, therefore

$$\Phi(x, y, (0, \sum_{i=1}^{R} \alpha_i \rho f_i + \sum_{j=1}^{m} \beta_j \rho_{gj})) \geq 0. \tag{29}$$

By (28) and (29),

$$\sum_{i=1}^{k} \alpha_i f_i(x) - \sum_{i=1}^{k} \alpha_i f_i(y) \geq 0.$$

Or

$$\sum_{i=1}^{k} \alpha_i f_i(x) \geq \sum_{i=1}^{k} \alpha_i f_i(y).$$

Hence (21) imply that

$$\sum_{i=1}^{k} \lambda_i f_i(y) \leq \sum_{i=1}^{k} \lambda_i f_i(x) \leq \max_{1\leq i\leq k} f_i(x).$$

□

Remark 4.1. *Under locally Lipschitz invexity assumptions if* (x, v) *is feasible solution of (EP) , then* $\sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j=1}^{m} \mu_j g_j(y) \leq v$.

Theorem 4.2. *Let* $\bar{x}$ *be an optimal solution of (P). Assume that Slater's constraints qualification holds at* $\bar{x}$. *Then there exists* $\lambda \in R^k_+, \mu \in R^m_+$ *such that* $(\bar{x} = \bar{y}, \lambda, \mu)$ *is feasible for (MD) and the objective functions of (P) and (MD) are equal at these points. If, also the hypotheses of the weak duality theorem hold, then* $(\bar{x}, \lambda, \mu)$ *is an optimal solution for (WD).*

Proof. By Theorem 3.1 and (11), there exist $\lambda \in R^k_+$ and $\mu \in R^m_+$ such that $(\bar{x}, \lambda, \mu)$ is a Kuhn-Tucker point of (P). Then $(\bar{x}, \lambda, \mu)$ is a feasible solution of (WD) and since $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0$, we have

$$\sum_{i=1}^{k} \lambda_i f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}) = \sum_{i\in P(\bar{x})} \lambda_i f_i(\bar{x}) = \max_{1\leq i\leq k} f_i(\bar{x}).$$

Then by the optimality of $(\bar{x}, \lambda, \mu)$ follows by Theorem 4.1. □

5. Conclusion

We have proved the several sufficient optimality conditions and duality results for a nonsmooth minimax programming problem under Lipschitz generalized (Φ, ρ)-invex functions. The results can be further generalize for a class of following nonsmooth fractional minimax programming

$$\text{(FP)} \quad \min_{x\in X} \max_{1\leq i\leq k} \frac{f_i(x)}{h_i(x)}$$

subject to

$$g_j(x) \leq 0, \ j = 1, 2, \cdots, m,$$

where $f_i : X \to R, h_i : X \to R, i = 1, 2, \ldots, R$ and $g_j : X \to R, j = 1, 2, \ldots, m$ are locally Lipschtiz functions, and X is a nonempty subset of R^n. We assume that $f_i(x) \geq 0$ and $h_i(x) > 0$, $i = 1, 2, \ldots, k$.

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