Filomat 30:5 (2016), 1219–1232
DOI 10.2298/FIL1605219S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bernstein-Kantorovich Operators on Multidimensional Cube

G. Stan[a]

[a] *"Transilvania" University of Braşov, Faculty of Mathematics and Informatics, Iuliu Maniu 50, Braşov 500091, România*

Abstract. The aim of this paper is to introduce a multidimensional Bernstein-Kantorovich operators for vector functions and study its convergence and Voronovskaja-type results .

1. Introduction

Kantorovich type operators have been obtained and studied by many authors by modifying some known operators in the unidimensional case, see [1],[3],[5],[6],[7],[9],[17],[18].

In this paper we generalize Bernstein-Kantorovich unidimensional operators in a multidimensional cube.

For any $n \in \mathbb{N}$ and $f \in C[0,1]$,the Bernstein-Kantorovich unidimensional operators are defined as:

$$K_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ f \in L_1[0,1],\ x \in [0,1],\ n \in \mathbb{N}. \tag{1}$$

The Bernstein operators in the multidimensional cube were studied by Hildebrandt and Schoenberg [8], Lorentz [10] , Schnabl [16].

For any $\overline{n} = (n_1, n_2, ..., n_k) \in \mathbb{N}^k$ and $f \in C\left([0,1]^k\right)$, the multidimensional Bernstein operators are defined as:

$$\begin{aligned} B_{\overline{n}}(f,\overline{x}) &= B_{(n_1,n_2,...,n_k)}(f,\overline{x}) = \sum_{\substack{0 \le i_s \le n_s \\ s=\overline{1,k}}} \left[f\left(\frac{i_1}{n_1}, \frac{i_2}{n_2}, ..., \frac{i_k}{n_k}\right) \prod_{j=\overline{1,k}} p_{n_j,i_j}(x_j) \right] \\ &= \sum_{i_1=0}^{n_1} \cdots \sum_{i_k=0}^{n_k} \left[f\left(\frac{i_1}{n_1}, \frac{i_2}{n_2}, ..., \frac{i_k}{n_k}\right) \prod_{j=\overline{1,k}} p_{n_j,i_j}(x_j) \right], \end{aligned} \tag{2}$$

where $\overline{x} = (x_1, x_2, ..., x_k) \in [0,1]^k$ and $p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$, $i = \overline{0,n}$, $x \in [0,1]$.

2010 *Mathematics Subject Classification*. Primary 41A36; Secondary 41A35, 41A10
Keywords. Kantorovich type operators, Bernstein-Kantorovich multidimensional operators, Voronovskaja type theorem
Received: 29 March 2014; Accepted: 04 September 2015
Communicated by Dragan S. Djordjević
Email address: gabriel.stan@unitbv.ro (G. Stan)

Properties of convergence and Voronovskaja-type results for multidimensional Bernstein-Kantorovich operators will be investigated in our paper. Properties of convergence and Voronovskaja-type results of linear and positive operators have been studied by many authors in unidimensional case, see [2],[4],[11],[12],[13],[14],[15].

2. Definition. Preliminary Results.

Through the paper the following notations will be considered:
$H_k = \left(C\left([0,1]^k\right)\right)^k$,
$H_k^+ = \left\{\psi = (\psi_1, ...\psi_k) \in H_k,\ \psi_j(\overline{x}) \geq 0,\ \overline{x} \in [0,1]^k,\ j = \overline{1,k}\right\}$.
H_k is a normed space with the norm given by

$$\|\psi\| = \max_{i=\overline{1,k}} \|\psi_i\|_\infty, \text{ where } \psi = (\psi_1, ..., \psi_k).$$

Let the set

$$H_k^e = \left\{\psi = (\psi_1, ..., \psi_k) \in H_k, \text{ so that } \frac{\partial \psi_i}{\partial x_j} = \frac{\partial \psi_j}{\partial x_i}, 1 \leq\ i < j \leq k\right\}$$

Definition 2.1. *For any $\overline{n} = (n_1, n_2, ..., n_k) \in \mathbb{N}^k$ we define the operators $\widetilde{K_{\overline{n}}} : H_k^e \to H_k$, given by*

$$\widetilde{K_{\overline{n}}}(\psi, \overline{x}) = \left(\widetilde{K_{\overline{n},1}}(\psi_1, \overline{x}), \widetilde{K_{\overline{n},2}}(\psi_2, \overline{x}), ..., \widetilde{K_{\overline{n},k}}(\psi_k, \overline{x})\right), \tag{3}$$

where

$$\begin{aligned}\widetilde{K_{\overline{n},j}}(\varphi, \overline{x}) &= n_j \sum_{\substack{0 \leq i_s \leq n_s \\ s=\overline{1,k},\ s\neq j}} \left[\left(\prod_{l=\overline{1,k},\ l\neq j} p_{n_l,i_l}(x_l)\right) \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right) \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \varphi\left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_k}{n_k}\right) dt\right] \\ &= n_j \sum_{i_1=0}^{n_1} \cdots \sum_{i_{j-1}=0}^{n_{j-1}} \sum_{i_{j+1}=0}^{n_{j+1}} \cdots \sum_{i_k=0}^{n_k} \left[\left(p_{n_1,i_1}(x_1) \cdots p_{n_{j-1},i_{j-1}}\left(x_{j-1}\right) p_{n_{j+1},i_{j+1}}\left(x_{j+1}\right) \cdots p_{n_k,i_k}(x_k)\right)\right. \\ &\quad \left. \cdot \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right) \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \varphi\left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_k}{n_k}\right) dt\right],\end{aligned} \tag{4}$$

for any $j = \overline{1,k}$, $\overline{x} = (x_1, x_2, ..., x_k) \in [0,1]^k$ and $\varphi \in C\left([0,1]^k\right)$.

We will refer to them as multidimensional Bernstein-Kantorovich operators.
In Corollary 2.1, we will show that $\widetilde{K_{\overline{n}}}(\psi) \in H_k^e$ for $\psi \in H_k^e$.
For these operators we have the following positivity property.

Lemma 2.1. *If $\psi \in H_k^+ \cap H_k^e$ then $\widetilde{K_{\overline{n}}} \in H_k^+$ for $\overline{n} = (n_1, n_2, ..., n_k) \in \mathbb{N}^k$.*

Proof. It follows from Definition 2.1. □

Further the following notation will be used:
(i) D is the differentiation operator,

$$D(f,\overline{x}) = \left(\frac{\partial f}{\partial x_1}(\overline{x}),...,\frac{\partial f}{\partial x_k}(\overline{x})\right),\ f \in C^1\left([0,1]^k\right), \overline{x} = (x_1,...,x_k) \in [0,1]^k,$$

(ii) I is the antiderivative operator,

$$I(\psi,\overline{x}) = \int_{\overline{0}}^{\overline{x}} \psi_1(\overline{y})\,dy_1 + \psi_2(\overline{y})\,dy_2 + ... + \psi_k(\overline{y})\,dy_k,$$

where $\psi \in H_k^e$, $\overline{x} = (x_1,...,x_k) \in [0,1]^k$, $\overline{y} = (y_1,...,y_k) \in [0,1]^k$.

Remark 2.1. *$D(f)$ is ∇f, the gradient of function f.*

Remark 2.2. *Because $\psi \in H_k^e$, we have*

$$I(\psi,\overline{x}) = \int_0^{x_1} \psi_1(t,0,...,0)\,dt + \int_0^{x_2} \psi_2(x_1,t,0,...,0)\,dt + ... + \int_0^{x_k} \psi_k(x_1,x_2,...,x_{k-1},t)\,dt,$$

for $\overline{x} = (x_1,...,x_k) \in [0,1]^k$.

Lemma 2.2. *Let $\overline{n} = (n_1,n_2,...,n_k) \in \mathbb{N}^k$. Then*
i) $(D \circ I)(\psi,\overline{x}) = \psi(\overline{x})$, for all $\psi \in H_k^e$, $\overline{x} = (x_1,...,x_k) \in \mathbb{R}^k$.
ii) $(I \circ D)(f,\overline{x}) = f(\overline{x})$, for all $f \in C^1\left([0,1]^k\right)$, such that $f\left(\overline{0}\right) = 0$.

Proof. i)

$$(D \circ I)(\psi,\overline{x}) = D\left(\sum_{i=1}^{k}\int_0^{x_i} \psi_i(x_1,x_2,...,x_{i-1},t,0,...,0)\,dt\right) = (g_1(\overline{x}),...,g_k(\overline{x})),$$

where

$$\begin{aligned} g_j(\overline{x}) &= \psi_j\left(x_1,...,x_j,0,...,0\right) + \sum_{i=j+1}^{k}\int_0^{x_i} \frac{\partial \psi_i}{\partial x_j}(x_1,x_2,...,x_{i-1},t,0,...,0)\,dt \\ &= \psi_j\left(x_1,...,x_j,0,...,0\right) + \sum_{i=j+1}^{k}\int_0^{x_i} \frac{\partial \psi_j}{\partial x_i}(x_1,x_2,...,x_{i-1},t,0,...,0)\,dt \\ &= \psi_j(\overline{x}), \end{aligned}$$

for all $j = \overline{1,k}$.

ii)

$$\begin{aligned} (I \circ D)(f,\overline{x}) &= I\left(\frac{\partial f}{\partial x_1}(\overline{x}),...,\frac{\partial f}{\partial x_k}(\overline{x})\right) \\ &= \int_0^{x_1} \frac{\partial f}{\partial x_1}(t,0,...,0)\,dt + \int_0^{x_2} \frac{\partial f}{\partial x_2}(x_1,t,0,...,0)\,dt + ... + \int_0^{x_k} \frac{\partial f}{\partial x_k}(x_1,x_2,...,x_{k-1},t)\,dt \\ &= f(\overline{x}). \end{aligned}$$

□

Theorem 2.1. *For any* $\overline{n} = (n_1, n_2, ..., n_k) \in \mathbb{N}^k$, $\overline{x} = (x_1, ..., x_k) \in \mathbb{R}^k$ *and* $\psi \in H_k^e$ *the following equality holds*

$$\widetilde{K_{\overline{n}}}(\psi, \overline{x}) = (D \circ B_{\overline{n}} \circ I)(\psi, \overline{x}). \tag{5}$$

Proof.

$$(D \circ B_{\overline{n}} \circ I)(\psi, \overline{x}) = D\left(\sum_{\substack{0 \le i_s \le n_s \\ s=\overline{1,k}}} p_{n_s,i_s}(x_s)\, I\left(\psi, \overline{u}_{\overline{n},\overline{i}}\right)\right) = (h_1(\overline{x}), ..., h_k(\overline{x})),$$

with $\overline{u}_{\overline{n},\overline{i}} = \left(\frac{i_1}{n_1}, \frac{i_2}{n_2}, ..., \frac{i_k}{n_k}\right)$, where

$$\begin{aligned} h_j(\overline{x}) &= \sum_{\substack{0 \le i_s \le n_s \\ s=\overline{1,k},\ s \ne j}} \left[\left(\prod_{l=\overline{1,k},\ l \ne j} p_{n_l,i_l}(x_l)\right) \sum_{i_j=0}^{n_j} \left(p_{n_j,i_j}\left(x_j\right)\right)' I\left(\psi, \overline{u}_{\overline{n},\overline{i}}\right)\right] \\ &= n_j \sum_{\substack{0 \le i_s \le n_s \\ s=\overline{1,k},\ s \ne j}} \left[\left(\prod_{l=\overline{1,k},\ l \ne j} p_{n_l,i_l}(x_l)\right) \sum_{i_j=0}^{n_j} \left(p_{n_j-1,i_j-1}\left(x_j\right) - p_{n_j-1,i_j}\left(x_j\right)\right) \cdot I\left(\psi, \overline{u}_{\overline{n},\overline{i}}\right)\right], \end{aligned}$$

provided that $p_{n,i}(x) = 0$, for $x \in [0,1]$, $i < 0$ and $i > n$.

If we write the sum which is right hand side of above equality as follows

$$\sum_{i_j=0}^{n_j} \left(p_{n_j-1,i_j-1}\left(x_j\right) - p_{n_j-1,i_j}\left(x_j\right)\right) I\left(\psi, \overline{u}_{\overline{n},\overline{i}}\right) = \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)[J_1 + J_2],$$

where

$$J_1 = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \psi_j\left(\frac{i_1}{n_1}, \frac{i_2}{n_2}, ..., \frac{i_{j-1}}{n_{j-1}}, t, 0, ..., 0\right) dt$$

and

$$\begin{aligned} J_2 &= \sum_{q=j+1}^{k} \int_0^{\frac{i_q}{n_q}} \left[\psi_q\left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, \frac{i_j+1}{n_j}, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_{q-1}}{n_{q-1}}, t, 0, ..., 0\right) - \psi_q\left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, \frac{i_j}{n_j}, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_{q-1}}{n_{q-1}}, t, 0, ..., 0\right)\right] dt \\ &= \sum_{q=j+1}^{k} \int_0^{\frac{i_q}{n_q}} \left[\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \frac{\partial \psi_q}{\partial x_j}\left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, u, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_{q-1}}{n_{q-1}}, t, 0, ..., 0\right) du\right] dt \\ &= \sum_{q=j+1}^{k} \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} du \int_0^{\frac{i_q}{n_q}} \frac{\partial \psi_j}{\partial x_q}\left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, u, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_{q-1}}{n_{q-1}}, t, 0, ..., 0\right) dt \end{aligned}$$

$$= \sum_{q=j+1}^{k} \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \left[\psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, u, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_{q-1}}{n_{q-1}}, \frac{i_q}{n_q}, 0, ..., 0 \right) - \psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, u, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_{q-1}}{n_{q-1}}, 0, ..., 0 \right) \right] du$$

$$= \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \left[\psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, u, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_k}{n_k} \right) - \psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, u, 0, ..., 0 \right) \right] du,$$

we obtain

$$J_1 + J_2 = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_k}{n_k} \right) dt.$$

Hence $\widetilde{K_{\overline{n}}} (\psi, \overline{x}) = (D \circ B_{\overline{n}} \circ I) (\psi, \overline{x})$. □

Corollary 2.1. *If $\psi \in H_k^e$ then $\widetilde{K_{\overline{n}}} (\psi) \in H_k^e$.*

Proof. It immediately follows from Theorem 2.1 and Theorem Schwartz on the equality of mixted derivatives. □

3. Convergence Properties and Voronovskaja-Type Results

Theorem 3.1. *Let $\psi \in H_k^e$. Then*

$$\lim_{\overline{n} \to \infty} \left\| \widetilde{K_{\overline{n}}} (\psi) - \psi \right\| = 0, \tag{6}$$

where the symbol $\overline{n} \to \infty$ denotes $n_i \to \infty$, for any $i = \overline{1, k}$.

Proof. We prove for each $1 \le j \le k$ that

$$\lim_{\overline{n} \to \infty} \widetilde{K}_{\overline{n}, j} \left(\psi_j, \overline{x} \right) = \psi_j (\overline{x}), \ \overline{x} = (x_1, ..., x_k) \in [0, 1]^k, \text{ uniformly.} \tag{7}$$

If we consider the integral right hand side of (4), we have

$$n_j \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_k}{n_k} \right) dt$$

$$= \psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, \frac{i_j}{n_j}, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_k}{n_k} \right) + n_j \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \left[\psi_j \left(\frac{i_1}{n_1}, ..., \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, ..., \frac{i_k}{n_k} \right) \right.$$

$$-\psi_j\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},\frac{i_j}{n_j},\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)\Bigg]dt=\psi_j\left(u_{\overline{n},\overline{i}}\right)+n_jR_{i_j,n_j},$$

where

$$R_{i_j,n_j}=\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}}\left[\psi_j\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)-\psi_j\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},\frac{i_j}{n_j},\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)\right]dt.$$

We obtain

$$\left|R_{i_j,n_j}\right|\le\frac{1}{n_j}\omega\left(\psi_j\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},\bullet,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right),\frac{1}{n_j}\right),$$

where ω is the modulus of continuity of the first order.

For $\phi\in C\left([0,1]^k\right)$ we define the modulus

$$\omega^{x_j}\left(\phi,h\right)=\sup_{\left(x_1,...,x_{j-1},x_{j+1},...,x_k\right)\in[0,1]^{k-1}}\omega\left(\phi\left(x_1,...,x_{j-1},\cdot,x_{j+1},...,x_k\right),h\right).$$

Whence

$$\left|R_{i_j,n_j}\right|\le\frac{1}{n_j}\omega^{x_j}\left(\psi_j,\frac{1}{n_j}\right).$$

It follows

$$\left|n_j\sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\ne j}}\left[\left(\prod_{l=\overline{1,k},\ l\ne j}p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1}p_{n_j-1,i_j}\left(x_j\right)\cdot R_{i_j,n_j}\right]\right|$$

$$\le n_j\sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\ne j}}\left[\left(\prod_{l=\overline{1,k},\ l\ne j}p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1}p_{n_j-1,i_j}\left(x_j\right)\cdot\left|R_{i_j,n_j}\right|\right]$$

$$\le n_j\sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\ne j}}\left[\left(\prod_{l=\overline{1,k},\ l\ne j}p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1}p_{n_j-1,i_j}\left(x_j\right)\cdot\frac{1}{n_j}\omega^{x_j}\left(\psi_j,h\right)\right]$$

$$\le\omega^{x_j}\left(\psi_j,\frac{1}{n_j}\right).$$

In view of above expressions we get

$$\left\|\widetilde{K}_{\overline{n},j}\left(\psi_j\right)-B_{\left(n_1,...,n_{j-1},n_j-1,n_{j+1},...,n_k\right)}\left(\psi_j\right)\right\|<\omega^{x_j}\left(\psi_j,\frac{1}{n_j}\right).$$

Since ψ_j is uniform continuous on $[0,1]^k$, it follows

$$\lim_{n_j\to\infty}\omega^{x_j}\left(\psi_j,\frac{1}{n_j}\right)=0.$$

Whence

$$\lim_{\overline{n}\to\infty}\left\|\widetilde{K}_{\overline{n},j}\left(\psi_j\right)-B_{\left(n_1,\ldots,n_{j-1},n_j-1,n_{j+1},\ldots,n_k\right)}\left(\psi_j\right)\right\|=0. \tag{8}$$

On the other hand it is well known that (see [2],[16]),

$$\lim_{\overline{n}\to\infty}B_{\left(n_1,\ldots,n_{j-1},n_j-1,n_{j+1},\ldots,n_k\right)}\left(\psi_j,\overline{x}\right)=\psi_j\left(\overline{x}\right), \tag{9}$$

converges uniformly on $[0,1]^k$ for $j=\overline{1,k}$.

From relations (8) and (9), we obtain (7). □

Remark 3.1. *If we note $I(\psi)=f$ then Theorem 3.1 essentially says that the partial derivatives of $B_{(n_1,n_2,\ldots,n_k)}(f)$ uniformly approximate the partial derivatives of f.*

In the sequence we will study Voronovskaja properties for these operators. First we consider the moments.

Let the unity function $e_0:\mathbb{R}^k\to\mathbb{R}$, $e_0(\overline{x})=1$ and the projections $\Pi_q:\mathbb{R}^k\to\mathbb{R}$, $\Pi_q(\overline{x})=x_q$, for $q=\overline{1,k}$ and $\overline{x}=(x_1,x_2,\ldots,x_k)\in\mathbb{R}^k$.

Lemma 3.1. *For $j,q,q_1,q_2=\overline{1,k}$ and $\overline{x}=(x_1,x_2,\ldots,x_k)\in[0,1]^k$ we have:*

i) $\widetilde{K}_{\overline{n},j}(e_0,\overline{x})=1;$

ii) $$\widetilde{K}_{\overline{n},j}\left(\Pi_q,\overline{x}\right)=\begin{cases}x_q, & q\neq j\\ \frac{n_j-1}{n_j}x_j+\frac{1}{2n_j}, & q=j;\end{cases}$$

iii) $$\widetilde{K}_{\overline{n},j}\left(\Pi_q^2,\overline{x}\right)=\begin{cases}\frac{(n_q-1)}{n_q}x_q^2+\frac{1}{n_q}x_q, & q\neq j\\ \frac{(n_j-1)(n_j-2)}{n_j^2}x_j^2+\frac{2(n_j-1)}{n_j^2}x_j+\frac{1}{3n_j^2}, & q=j;\end{cases}$$

iv) $$\widetilde{K}_{\overline{n},j}\left(\Pi_{q_1}\Pi_{q_2},\overline{x}\right)=\begin{cases}x_{q_1}x_{q_2}, & q_1,q_2\neq j \text{ and } q_1\neq q_2\\ x_{q_1}\left(\frac{n_j-1}{n_j}x_j+\frac{1}{2n_j}\right), & q_2=j \text{ and } q_1\neq j;\end{cases}$$

v) $$\widetilde{K}_{\overline{n},j}\left(\Pi_q^3,\overline{x}\right)=\begin{cases}\frac{(n_q-1)(n_q-2)}{n_q^2}x_q^3+\frac{3(n_q-1)}{n_q^2}x_q^2+\frac{1}{n_q^2}x_q, & q\neq j\\ \frac{(n_j-1)(n_j-2)(n_j-3)}{n_j^3}x_j^3+\frac{9(n_j-1)(n_j-2)}{2n_j^3}x_j^2+\frac{7(n_j-1)}{2n_j^3}x_j+\frac{1}{4n_j^3}, & q=j;\end{cases}$$

vi) $$\widetilde{K}_{\overline{n},j}\left(\Pi_q^4,\overline{x}\right)=\begin{cases}\frac{(n_q-1)(n_q-2)(n_q-3)}{n_q^3}x_q^4+\frac{6(n_q-1)(n_q-2)}{n_q^3}x_q^3+\frac{7(n_q-1)}{n_q^3}x_q^2+\frac{1}{n_q^3}x_q, & q\neq j\\ \frac{(n_j-1)(n_j-2)(n_j-3)(n_j-4)}{n_j^4}x_j^4+\frac{8(n_j-1)(n_j-2)(n_j-3)}{n_j^4}x_j^3+\frac{15(n_j-1)(n_j-2)}{n_j^4}x_j^2+\frac{6(n_j-1)}{n_j^4}x_j+\frac{1}{5n_j^4}, & q=j.\end{cases}$$

Proof. We prove this lemma using (4) and the fact that $\sum_{i=0}^{n} p_{n,i}(x) = 1$, $\sum_{i=0}^{n} p_{n,i}(x)\frac{i}{n} = x$, $\sum_{i=0}^{n} p_{n,i}(x)\left(\frac{i}{n}\right)^2 = \frac{n-1}{n}x^2 + \frac{1}{n}x$, $\sum_{i=0}^{n} p_{n,i}(x)\left(\frac{i}{n}\right)^3 = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n^2}x^2 + \frac{1}{n^2}x$, $\sum_{i=0}^{n} p_{n,i}(x)\left(\frac{i}{n}\right)^4 = \frac{(n-1)(n-2)(n-3)}{n^3}x^4 + \frac{6(n-1)(n-2)}{n^3}x^3 + \frac{7(n-1)}{n^3}x^2 + \frac{1}{n^3}x$, for $n \in \mathbb{N}$, $x \in [0,1]$.

i) It is obvious.

ii) For $q \neq j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_q\left(\frac{i_1}{n_1}, \dots, \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, \dots, \frac{i_k}{n_k}\right) dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \frac{i_q}{n_q} dt = \frac{1}{n_j} \cdot \frac{i_q}{n_q}.$$

Replacing the above equality in (4) we obtain

$$\widetilde{K}_{\overline{n},j}\left(\Pi_q, \overline{x}\right) = \sum_{\substack{0 \le i_s \le n_s \\ s=\overline{1,k},\ s \neq j}} \left[\left(\prod_{l=\overline{1,k},\ l \neq j} p_{n_l,i_l}(x_l)\right) \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right) \cdot \frac{i_q}{n_q}\right] = \sum_{i_q=0}^{n_q} p_{n_q,i_q}\left(x_q\right) \frac{i_q}{n_q} = x_q.$$

For $q = j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_j\left(\frac{i_1}{n_1}, \dots, \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, \dots, \frac{i_k}{n_k}\right) dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} t dt = \frac{2i_j + 1}{2n_j^2}.$$

Replacing the equality in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_j, \overline{x}\right) &= \sum_{\substack{0 \le i_s \le n_s \\ s=\overline{1,k},\ s \neq j}} \left[\left(\prod_{l=\overline{1,k},\ l \neq j} p_{n_l,i_l}(x_l)\right) \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right) \frac{2i_j + 1}{2n_j}\right] \\ &= \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right) \frac{2i_j+1}{2n_j} = \frac{n_j - 1}{n_j} x_q + \frac{1}{2n_j}.\end{aligned}$$

iii) For $q \neq j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_q^2\left(\frac{i_1}{n_1}, \dots, \frac{i_{j-1}}{n_{j-1}}, t, \frac{i_{j+1}}{n_{j+1}}, \dots, \frac{i_k}{n_k}\right) dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \left(\frac{i_q}{n_q}\right)^2 dt = \frac{1}{n_j} \cdot \frac{i_q^2}{n_q^2}.$$

If we replace the equality in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_q^2, \overline{x}\right) &= \sum_{\substack{0 \le i_s \le n_s \\ s=\overline{1,k},\ s \neq j}} \left[\left(\prod_{l=\overline{1,k},\ l \neq j} p_{n_l,i_l}(x_l)\right) \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right) \frac{i_q^2}{n_q^2}\right] \\ &= \sum_{i_q=0}^{n_q} p_{n_q,i_q}\left(x_q\right) \frac{i_q^2}{n_q^2} = \frac{\left(n_q - 1\right)}{n_q} x_q^2 + \frac{1}{n_q} x_q.\end{aligned}$$

For $q = j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_j^2\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} t^2 dt = \frac{3i_j^2+3i_j+1}{3n_j^3}.$$

Taking this into consideration in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_j^2,\overline{x}\right) &= \left[\sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\neq j}}\left(\prod_{l=\overline{1,k},\ l\neq j} p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{3i_j^2+3i_j+1}{3n_j^2}\right]\\ &= \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{3i_j^2+3i_j+1}{3n_j^2} = \frac{\left(n_j-1\right)\left(n_j-2\right)}{n_j^2}x_j^2+\frac{2\left(n_j-1\right)}{n_j^2}x_j+\frac{1}{3n_j^2}.\end{aligned}$$

iv) For $q_1, q_2 \neq j$ and $q_1 \neq q_2$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_{q_1}\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)\Pi_{q_2}\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \frac{i_{q_1}}{n_{q_1}}\cdot\frac{i_{q_2}}{n_{q_2}}dt = \frac{1}{n_j}\cdot\frac{i_{q_1}}{n_{q_1}}\cdot\frac{i_{q_2}}{n_{q_2}}.$$

Replacing the above equality in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_{q_1}\Pi_{q_2},\overline{x}\right) &= \sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\neq j}}\left[\left(\prod_{l=\overline{1,k},\ l\neq j} p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{i_{q_1}i_{q_2}}{n_{q_1}n_{q_2}}\right]\\ &= \sum_{i_{q_1}=0}^{n_{q_1}} p_{n_{q_1},i_{q_1}}\left(x_{q_1}\right)\frac{i_{q_1}}{n_{q_1}}\sum_{i_{q_2}=0}^{n_{q_2}} p_{n_{q_2},i_{q_2}}\left(x_{q_2}\right)\frac{i_{q_2}}{n_{q_2}} = x_{q_1}x_{q_2}.\end{aligned}$$

For $q_2 = j$ and $q_1 \neq j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_{q_1}\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)\Pi_j\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \frac{i_{q_1}}{n_{q_1}}t dt = \frac{i_{q_1}}{n_{q_1}}\cdot\frac{2i_j+1}{2n_j^2}.$$

Taking this into consideration in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_{q_1}\Pi_j,\overline{x}\right) &= \left[\sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\neq j}}\left(\prod_{l=\overline{1,k},\ l\neq j} p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{i_{q_1}}{n_{q_1}}\cdot\frac{2i_j+1}{2n_j^2}\right]\\ &= \sum_{i_{q_1}=0}^{n_{q_1}} p_{n_{q_1},i_{q_1}}\left(x_{q_1}\right)\frac{i_{q_1}}{n_{q_1}}\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{2i_j+1}{2n_j^2}\\ &= x_{q_1}\left(\frac{n_j-1}{n_j}x_j+\frac{1}{2n_j}\right).\end{aligned}$$

v) For $q \neq j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_q^3\left(\frac{i_1}{n_1},\ldots,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},\ldots,\frac{i_k}{n_k}\right)dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}}\left(\frac{i_q}{n_q}\right)^3 dt = \frac{1}{n_j}\cdot\left(\frac{i_q}{n_q}\right)^3.$$

Replacing the above equality in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_q^3,\overline{x}\right) &= \sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\neq j}}\left[\left(\prod_{l=\overline{1,k},\ l\neq j} p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\cdot\left(\frac{i_q}{n_q}\right)^3\right]\\ &= \sum_{i_q=0}^{n_q} p_{n_q,i_q}\left(x_q\right)\left(\frac{i_q}{n_q}\right)^3 = \frac{\left(n_q-1\right)\left(n_q-2\right)}{n_q^2}x_q^3 + \frac{3\left(n_q-1\right)}{n_q^2}x_q^2 + \frac{1}{n_q^2}x_q.\end{aligned}$$

For $q = j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_j^3\left(\frac{i_1}{n_1},\ldots,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},\ldots,\frac{i_k}{n_k}\right)dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} t^3 dt = \frac{4i_j^3+6i_j^2+4i_j+1}{4n_j^4}.$$

Replacing the equality in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_j^3,\overline{x}\right) &= \sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\neq j}}\left[\left(\prod_{l=\overline{1,k},\ l\neq j} p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{4i_j^3+6i_j^2+4i_j+1}{4n_j^3}\right]\\ &= \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{4i_j^3+6i_j^2+4i_j+1}{4n_j^3}\\ &= \frac{\left(n_j-1\right)\left(n_j-2\right)\left(n_j-3\right)}{n_j^3}x_j^3 + \frac{9\left(n_j-1\right)\left(n_j-2\right)}{2n_j^3}x_j^2 + \frac{7\left(n_j-1\right)}{2n_j^3}x_j + \frac{1}{4n_j^3}.\end{aligned}$$

vi) For $q \neq j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_q^4\left(\frac{i_1}{n_1},\ldots,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},\ldots,\frac{i_k}{n_k}\right)dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}}\left(\frac{i_q}{n_q}\right)^4 dt = \frac{1}{n_j}\cdot\left(\frac{i_q}{n_q}\right)^4.$$

Replacing the above equality in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_q^4,\overline{x}\right) &= \sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\ s\neq j}}\left[\left(\prod_{l=\overline{1,k},\ l\neq j} p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\cdot\left(\frac{i_q}{n_q}\right)^4\right] = \sum_{i_q=0}^{n_q} p_{n_q,i_q}\left(x_q\right)\left(\frac{i_q}{n_q}\right)^4\\ &= \frac{\left(n_q-1\right)\left(n_q-2\right)\left(n_q-3\right)}{n_q^3}x_q^4 + \frac{6\left(n_q-1\right)\left(n_q-2\right)}{n_q^3}x_q^3 + \frac{7\left(n_q-1\right)}{n_q^3}x_q^2 + \frac{1}{n_q^3}x_q.\end{aligned}$$

For $q = j$ we have

$$\int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} \Pi_j^4\left(\frac{i_1}{n_1},...,\frac{i_{j-1}}{n_{j-1}},t,\frac{i_{j+1}}{n_{j+1}},...,\frac{i_k}{n_k}\right)dt = \int_{\frac{i_j}{n_j}}^{\frac{i_j+1}{n_j}} t^4 dt = \frac{5i_j^4+10i_j^3+10i_j^2+5i_j+1}{5n_j^5}.$$

Replacing the equality in (4) we obtain

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\Pi_j^4,\overline{x}\right) &= \sum_{\substack{0\le i_s\le n_s\\ s=\overline{1,k},\, s\neq j}}\left[\left(\prod_{l=\overline{1,k},\, l\neq j} p_{n_l,i_l}(x_l)\right)\sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{5i_j^4+10i_j^3+10i_j^2+5i_j+1}{5n_j^4}\right]\\ &= \sum_{i_j=0}^{n_j-1} p_{n_j-1,i_j}\left(x_j\right)\frac{5i_j^4+10i_j^3+10i_j^2+5i_j+1}{5n_j^4}\\ &= \frac{\left(n_j-1\right)\left(n_j-2\right)\left(n_j-3\right)\left(n_j-4\right)}{n_j^4}x_j^4+\frac{8\left(n_j-1\right)\left(n_j-2\right)\left(n_j-3\right)}{n_j^4}x_j^3\\ &+ \frac{15\left(n_j-1\right)\left(n_j-2\right)}{n_j^4}x_j^2+\frac{6\left(n_j-1\right)}{n_j^4}x_j+\frac{1}{5n_j^4}.\end{aligned}$$

□

Lemma 3.2. *For $j,q,q_1,q_2 = \overline{1,k}$ and $\overline{x} = (x_1,x_2,...,x_k) \in [0,1]^k$ we have:*

i) $\widetilde{K}_{\overline{n},j}\left(\Pi_q - x_q e_0,\overline{x}\right) = \begin{cases} 0, & q\neq j\\ \frac{1-2x_j}{2n_j}, & q=j;\end{cases}$

ii) $\widetilde{K}_{\overline{n},j}\left(\left(\Pi_q - x_q e_0\right)^2,\overline{x}\right) = \begin{cases} \frac{x_q\left(1-x_q\right)}{n_q}, & q\neq j\\ \frac{n_j-2}{n_j^2}x_j\left(1-x_j\right)+\frac{1}{3n_j^2}, & q=j;\end{cases}$

iii) $\widetilde{K}_{\overline{n},j}\left(\left(\Pi_{q_1} - x_{q_1} e_0\right)\left(\Pi_{q_2} - x_{q_2} e_0\right),\overline{x}\right) = 0;$

iv) $\widetilde{K}_{\overline{n},j}\left(\left(\Pi_q - x_q e_0\right)^4,\overline{x}\right) = \begin{cases} \frac{3\left(n_q-2\right)}{n_q^3}x_q^2\left(1-x_q\right)^2+\frac{1}{n_q^3}x_q\left(1-x_q\right), & q\neq j\\ \frac{3n_j^2-26n_j+24}{n_j^4}x_j^2\left(1-x_j\right)^2+\frac{5n_j-6}{n_j^4}x_j\left(1-x_j\right)+\frac{1}{5n_j^4}, & q=j.\end{cases}$

Proof. It immediately follows from Lemma 3.1 and the linearity property of the operators. □

Theorem 3.2. *Let $\varphi : [0,1]^k \to \mathbb{R}$ be a function which admits second order partial derivatives on $[0,1]^k$ and these are continuous in a fixed point $\overline{x} = (x_1,x_2,...,x_k) \in [0,1]^k$ and $n_1 = n_2 = ... = n_k = n$. Then*

$$\lim_{n\to\infty} n\left(\widetilde{K}_{\overline{n},j}(\varphi,\overline{x}) - \varphi(\overline{x})\right) = \frac{1-2x_j}{2}\frac{\partial\varphi}{\partial x_j}(\overline{x}) + \frac{1}{2}\sum_{q=1}^{k} x_q\left(1-x_q\right)\frac{\partial^2\varphi}{\partial x_q^2}(\overline{x}). \tag{10}$$

Proof. Using Taylor formula for function $\varphi : [0,1]^k \to \mathbb{R}$ in $\overline{x} = (x_1, x_2, ..., x_k) \in [0,1]^k$ for any $\overline{t} = (t_1, t_2, ..., t_k) \in [0,1]^k$ we have

$$\begin{aligned}
\varphi\left(\overline{t}\right) &= \varphi\left(\overline{x}\right) + \sum_{q=1}^{k} \frac{\partial \varphi}{\partial x_q}\left(\overline{x}\right)\left(t_q - x_q\right) + \frac{1}{2}\sum_{q=1}^{k} \frac{\partial^2 \varphi}{\partial x_q^2}\left(\overline{x}\right)\left(t_q - x_q\right)^2 \\
&+ \sum_{1 \le q_1 < q_2 \le k} \frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{x}\right)\left(t_{q_1} - x_{q_1}\right)\left(t_{q_2} - x_{q_2}\right) + R\left(\overline{t}, \overline{x}\right) \qquad (11)
\end{aligned}$$

where $\overline{\xi} = \overline{\xi}(t)$ belongs to the segment of end points $\overline{t}$ and $\overline{x}$, and $R\left(\overline{t}, \overline{x}\right)$ given by

$$\begin{aligned}
R\left(\overline{t}, \overline{x}\right) &= \frac{1}{2}\sum_{q=1}^{k} \left[\frac{\partial^2 \varphi}{\partial x_q^2}\left(\overline{\xi}\right) - \frac{\partial^2 \varphi}{\partial x_q^2}\left(\overline{x}\right)\right]\left(t_q - x_q\right)^2 \\
&+ \sum_{1 \le q_1 < q_2 \le k} \left[\frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{\xi}\right) - \frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{x}\right)\right]\left(t_{q_1} - x_{q_1}\right)\left(t_{q_2} - x_{q_2}\right).
\end{aligned}$$

Since φ admits continuous second order partial derivatives in $\overline{x}$ it follows that

$$\frac{\partial^2 \varphi}{\partial x_q^2}\left(\overline{\xi}\right) \underset{\overline{\xi} \to \overline{x}}{\longrightarrow} \frac{\partial^2 \varphi}{\partial x_q^2}\left(\overline{x}\right), \ \frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{\xi}\right) \underset{\overline{\xi} \to \overline{x}}{\longrightarrow} \frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{x}\right).$$

From the uniform continuity of the second order partial derivatives of φ on $[0,1]^k$, we obtain that for each $\varepsilon > 0$,there exists $r_\varepsilon > 0$ such that for any $\overline{y}$ with the property $\left\|\overline{y} - \overline{x}\right\| < r_\varepsilon$

$$\left|\frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{y}\right) - \frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{x}\right)\right| < \varepsilon, q_1, q_2 = \overline{1,k}.$$

We consider

$$M = \max_{1 \le q_1, q_2 \le k} \left\|\frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\right\|.$$

By using above fact, we obtain

$$\left|\frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{t}\right) - \frac{\partial^2 \varphi}{\partial x_{q_1} \partial x_{q_2}}\left(\overline{x}\right)\right| \le \varepsilon + \frac{2M}{r_\varepsilon}\left\|\overline{x} - \overline{t}\right\|^2, \ \overline{x}, \overline{t} \in [0,1]^k, \ q_1, q_2 = \overline{1,k}.$$

With the help of Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|R\left(\overline{t}, \overline{x}\right)\right| &\le \frac{\varepsilon + \frac{2M}{r_\varepsilon}\left\|\overline{x} - \overline{t}\right\|^2}{2}\sum_{q=1}^{k}\left(t_q - x_q\right)^2 + \left(\varepsilon + \frac{2M}{r_\varepsilon}\left\|\overline{x} - \overline{t}\right\|^2\right)\sum_{1 \le q_1 < q_2 \le k}\left|t_{q_1} - x_{q_1}\right| \cdot \left|t_{q_2} - x_{q_2}\right| \\
&\le \frac{\varepsilon + \frac{2M}{r_\varepsilon}\left\|\overline{x} - \overline{t}\right\|^2}{2}\left(\sum_{q=1}^{k}\left|t_q - x_q\right|\right)^2 \le \frac{\varepsilon + \frac{2M}{r_\varepsilon}\left\|\overline{x} - \overline{t}\right\|^2}{2} \cdot k \cdot \left\|\overline{x} - \overline{t}\right\|^2.
\end{aligned}$$

Whence

$$\left|R\left(\overline{t}, \overline{x}\right)\right| \le \frac{\varepsilon}{2} \cdot k \cdot \left\|\overline{x} - \overline{t}\right\|^2 + \frac{k \cdot M}{r_\varepsilon}\left\|\overline{x} - \overline{t}\right\|^4.$$

By using following expression $\left\|\overline{x}-\overline{t}\right\|^2 = \sum_{q=1}^{k}\left(t_q - x_q\right)^2$ and $\left\|\overline{x}-\overline{t}\right\|^4 = \left[\sum_{q=1}^{k}\left(t_q - x_q\right)^2\right]^2 \leq k\sum_{q=1}^{k}\left(t_q - x_q\right)^4$, we obtain

$$\left|R\left(\overline{t},\overline{x}\right)\right| \leq \frac{\varepsilon}{2}\cdot k\cdot\sum_{q=1}^{k}\left(t_q - x_q\right)^2 + \frac{k^2\cdot M}{r_\varepsilon}\sum_{q=1}^{k}\left(t_q - x_q\right)^4.$$

Finally we have

$$\left|\widetilde{K}_{\overline{n},j}\left(R\left(\cdot,\overline{x}\right),\overline{x}\right)\right| \leq \widetilde{K}_{\overline{n},j}\left(\left|R\left(\cdot,\overline{x}\right)\right|,\overline{x}\right) \leq \frac{\varepsilon}{2}\cdot k\cdot\widetilde{K}_{\overline{n},j}\left(\sum_{q=1}^{k}\left(t_q - x_q\right)^2,\overline{x}\right) + \frac{k^2\cdot M}{r_\varepsilon}\widetilde{K}_{\overline{n},j}\left(\sum_{q=1}^{k}\left(t_q - x_q\right)^4,\overline{x}\right).$$

If we consider Lemma 3.2 and $n_1 = n_2 = ... = n_k = n$, we obtain

$$\widetilde{K}_{\overline{n},j}\left(\sum_{q=1}^{k}\left(t_q - x_q\right)^2,\overline{x}\right) = \frac{n-2}{n^2}x_j\left(1-x_j\right) + \frac{1}{3n^2} + \sum_{q=\overline{1,k},\ q\neq j}\frac{x_q\left(1-x_q\right)}{n} \leq \frac{3kn-2}{12n^2} \leq \frac{k}{4n}$$

and

$$\begin{aligned}\widetilde{K}_{\overline{n},j}\left(\sum_{q=1}^{k}\left(t_q - x_q\right)^4,\overline{x}\right) &= \frac{3n^2-26n+24}{n^4}x_j^2\left(1-x_j\right)^2 + \frac{5n-6}{n^4}x_j\left(1-x_j\right) + \frac{1}{5n^4}\\ &+ \sum_{q=\overline{1,k},\ q\neq j}\left[\frac{3\left(n-2\right)}{n^3}x_q^2\left(1-x_q\right)^2 + \frac{1}{n^3}x_q\left(1-x_q\right)\right] \leq \frac{3k}{16n^2}.\end{aligned}$$

We get

$$\left|\widetilde{K}_{\overline{n},j}\left(R\left(\cdot,\overline{x}\right),\overline{x}\right)\right| \leq \widetilde{K}_{\overline{n},j}\left(\left|R\right|,\overline{x}\right) < \frac{\varepsilon}{2}\cdot k^2\frac{1}{4n} + \frac{k^3\cdot M}{r_\varepsilon}\cdot\frac{3}{16n^2}$$

and thus

$$n\cdot\widetilde{K}_{\overline{n},j}\left(R,\overline{x}\right) \underset{n\to\infty}{\rightarrow} 0.$$

From Lemma 3.2 and (11) we have

$$\begin{aligned}n\left(\widetilde{K}_{\overline{n},j}\left(\varphi,\overline{x}\right)-\varphi\left(\overline{x}\right)\right) &= \frac{1-2x_j}{2}\frac{\partial\varphi}{\partial x_j}\left(\overline{x}\right) + \frac{1}{2}\left[\frac{n-2}{n}x_j\left(1-x_j\right) + \frac{1}{3n}\right]\frac{\partial^2\varphi}{\partial x_j^2}\left(\overline{x}\right)\\ &+ \frac{1}{2}\sum_{q=\overline{1,k},\ q\neq j}\frac{\partial^2\varphi}{\partial x_q^2}\left(\overline{x}\right)x_q\left(1-x_q\right) + n\widetilde{K}_{\overline{n},j}\left(R,\overline{x}\right).\end{aligned}$$

When $n\to\infty$, we obtain

$$n\left(\widetilde{K}_{\overline{n},j}\left(\varphi,\overline{x}\right)-\varphi\left(\overline{x}\right)\right) \to \frac{1-2x_j}{2}\frac{\partial\varphi}{\partial x_j}\left(\overline{x}\right) + \frac{1}{2}\sum_{q=1}^{k}x_q\left(1-x_q\right)\frac{\partial^2\varphi}{\partial x_q^2}\left(\overline{x}\right).$$

□

References

[1] D. Barbosu , Kantorovich-Stancu type operators, J. Inequal. Pure Appl. Math. 5 (2004) no. 3, article 53.
[2] D. Barbosu, O. T. Pop, On the Bernstein bivariate approximation formula, Carpathian J. Math. 24 (3) (2008) 293–298.
[3] J. de la Cal, A. M. Valle, A generalization of the Bernstein-Kantorovich operators, J. of Math. Anal. and Appl., 252 (2000) 750–766.
[4] A. Ciupa, A Voronovskaja type theorem for a positive linear operator, Internat. J. Math. & Math. Sci., (2006) Article ID 42368, 7 pages.
[5] Z. Ditzian and X. Zhou , Kantorovich-Bernstein polynomials, Constr. Approx. 6 (1990) 421–435.
[6] H. Gonska, M. Heilmann , Raşa,I., Kantorovich Operators of Order k, Numer. Funct. Anal. Optim., 32(7) (2011) 717–738.
[7] H. Gonska, R. Păltănea, General Voronovskaya and Asymptotic Theorems in simultaneous Approximation, Mediterr. J. Math. 7 (2010) 37–49.
[8] T.H. Hildebrandt, I.J. Schoenberg, On linear functional operations and moment problem, Ann. of Math, (2) 34 (1933) 317–328.
[9] L.V. Kantorovich, Sur certains developpements suivant les polynomes de la forme de S. Bernstein, I, II, C.R. Acad. URSS (1930) 563–568, 595–600.
[10] G.G. Lorentz, Bernstein polynomials, (2nd edn.) , Chelsea Publ. Co., New York (1986).
[11] N. I. Mahmudov, Approximation by Genuine -Bernstein-Durrmeyer Polynomials in Compact Disks in the Case $q > 1$, Abstr. Appl. Anal., (2014) Article ID 959586, 11 pages.
[12] N. I. Mahmudov, On q-Parametric Szasz-Mirakjan Operators, Mediterr. J. Math. 7 (2010) 297–311.
[13] N. I. Mahmudov, P. Sabancigil, Approximation Theorems for q-Bernstein-Kantorovich Operators, Filomat 27:4 (2013) 721–730.
[14] N. I. Mahmudov, M. Kara, Approximation Theorems for Generalized Complex Kantorovich-Type Operators, J. Appl. Math. (2012) Article ID 454579, 14 pages.
[15] O. T. Pop, About Some Linear Operators, Internat. J. Math. & Math. Sci., (2007) , Article ID 91781, 13 pages.
[16] R. Schnabl , Eine Verallgemeinerung der Bernstein polynome. Math. Ann., (1968) 179:74–82.
[17] I. G. Stan, On the Durrmeyer-Kantorovich type operator, Bull. Transilv. Univ. Braşov ser. III, vol. 6 (55) (2013) 33–42.
[18] S. Sucu, E. Ibikli, Approximation by means of Kantorovich-Stancu type operators, Numer. Funct. Anal. Optim., 34 (5) (2013) 557–575.