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The Linear Arboricity of Planar Graphs without 5-Cycles with Two Chords

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Abstract. The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests which partition the edges of G. In this paper, it is proved that for a planar graph G, $la(G)=\lceil(\Delta(G)/2)\rceil$ if $\Delta(G) \geq 7$ and G has no 5-cycles with two chords.

1. Introduction

In this paper, all graphs are finite, simple and undirected. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set, respectively. If $uv \in E(G)$, then u is said to be a *neighbor* of v, and $N_G(v)$ is the set of neighbors of v. The *degree* $d(v)$ of a vertex v is $|N_G(v)|$, $\delta(G)$ is the minimum degree of G and $\Delta(G)$ is the maximum degree of G. A k-, k^+- or k^--vertex is a vertex of degree k, at least k, or at most k, respectively. A k-cycle is a cycle of length k. Two cycles are said to be *adjacent* (or *intersecting*) if they have at least one common edge (or vertex, respectively). Given a cycle C of length $k(k \geq 4)$ in G, an edge $xy \in E(G)\backslash E(C)$ is called a *chord* of C if $x, y \in V(C)$. Such a cycle C is also called a chordal-k-cycle.

If G is a planar graph, then we always assume that G has been embedded in the plane. Let G be a planar graph and $F(G)$ be the face set of G. For $f \in F(G)$, the *degree* of f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^--face is a face of degree k, at least k, or at most k, respectively. Let $n_i(v)$ denote the number of i-vertices of G adjacent to the vertex v, $f_i(v)$ the number of i-faces of G incident with v. All undefined notations and definitions follow that of Bondy and Murty [3].

A *linear forest* is a graph in which each component is a path. A map φ form $E(G)$ to $\{1, 2, \cdots, t\}$ is called a *t-linear coloring* if the induced subgraph of edges having the same color α is a linear forest for $1 \leq \alpha \leq t$. The linear arboricity $la(G)$ of a graph G defined by Harary [10] is the minimum number t for which G has a t-linear coloring. Akiyama et al.[1] conjectured that $la(G) = \lceil \frac{\Delta(G)+1}{2} \rceil$ for any simple regular graph G. The conjecture is equivalent to the following conjecture.

Conjecture A. *For any graph* G, $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

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The linear arboricity has been determined for complete bipartite graphs [1], complete regular multipartite graphs [20], Halin graphs [16], series-parallel graphs [18] and regular graphs with $\Delta = 3, 4$[2] and 5,6,8[9]. For planar graphs, more results are obtained. Conjecture A has already been proved to be true for all planar graphs (see [17] and [21]). Wu [17] proved that for a planar graph G with girth g and maximum degree Δ, $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$ if $\Delta(G) \geq 13$, or $\Delta(G) \geq 7$ and $g \geq 4$, or $\Delta(G) \geq 5$ and $g \geq 5$, $\Delta(G) \geq 3$ and $g \geq 6$. Recently, M. Cygan et al. [8] proved that if G is a planar graph with $\Delta \geq 9$, then $la(G) = \lceil \frac{\Delta}{2} \rceil$, and then they posed the following conjecture.

Conjecture B. *For any planar graph G of maximum degree $\Delta \geq 5$, $la(G) = \lceil \frac{\Delta}{2} \rceil$.*

There are more partial results to support the conjecture. The linear arboricity of a planar graph G is $\lceil \frac{\Delta}{2} \rceil$ if it satisfies one of the following conditions: (1) $\Delta(G) \geq 7$ and G contains no chordal i-cycles for some $i \in \{4, 5, 6, 7\}$ ([5, 6, 13]); (2) $\Delta \geq 7$ and for each vertex $v \in V(G)$, there exist two integers i_v, $j_v \in \{3, 4, 5, 6, 7, 8\}$ such that any two i_v, j_v-cycles incident with v are not adjacent ([7, 15]); (3) $\Delta \geq 5$ and G contains no 4-cycles ([22]); (4) $\Delta \geq 5$ and G has no intersecting 4-cycles and intersecting 5-cycles ([4]); (5) $\Delta \geq 5$ and G has no 5-, 6-cycles with chords ([5]); (6) $\Delta \geq 5$ and any 4-cycle is not adjacent to an i-cycle for any $i \in \{3, 4, 5\}$ or G has no intersecting 4-cycles and intersecting i-cycles for either $i = 3$ or $i = 6$ ([11]); (7) $\Delta \geq 5$ and any two 4-cycles are not adjacent, and any 3-cycle is not adjacent to a 5-cycle ([14]).

In the paper, we will prove that if G is a planar graph with $\Delta(G) \geq 7$ and any 5-cycle contains at most one chord, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. It generalizes some above results.

2. Main Result and its Proof

First, we give some more definitions. Given a t-linear coloring φ and $v \in V(G)$, we denote by $C^i_\varphi(v)$ the set of colors appear i times at v, where $i = 0, 1, 2$. Then $|C^0_\varphi(v)| + |C^1_\varphi(v)| + |C^2_\varphi(v)| = t$ and $d(v) = |C^1_\varphi(v)| + 2|C^2_\varphi(v)|$. For two adjacent edges uv and uw, we denote by $uv \rightleftharpoons uw$ to exchange the colors of uv and uw, by $uv \to c$ to color uv with a color c. If $i \in C^1_\varphi(v)$, we denote by (v, i) the edge colored with i. For two vertices u and v, we use $(u, i) \sim (v, i)$ to denote that there is a monochromatic path of color i between u and v. For a vertex v and an edge xy of G, $xy \sim (v, i)$ denote that there exists a monochromatic path of color i between x and v passing y. For two different edges x_1y_1 and x_2y_2 of G, we use $x_1y_1 \sim x_2y_2$ to denote more accurately that there is a monochromatic path from x_1 to y_2 passing through the edges x_1y_1 and x_2y_2 in G (that is, y_1 and x_2 are internal vertices in the path). We use $\nsim$ to denote that such monochromatic path does not exist.

Now we begin to give the main result of the paper and its proof.

Theorem 2.1. *Let G be a planar graph with $\Delta(G) \geq 7$. If any 5-cycle contains at most one chord, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.*

Proof. Since all planar graphs G with $\Delta(G) \geq 9$ have been proved in [8] to be $\lceil \frac{\Delta(G)}{2} \rceil$-linear colorable, it suffices to prove the following result.

(A) *Any planar graph G of maximum degree at most 8 has an 4-linear coloring using colors 1, 2, 3, 4 if G contains no 5-cycles with two chords.*

Let $G = (V, E)$ be a minimal counterexample to (A). First, we show some known claims for G.

Claim 2.2. *Let $uv \in E(G)$ and $G - uv$ has an 4-linear coloring φ. Let $C_\varphi(u, v) = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$. Then*

(1) $|C_\varphi(u, v)| = 4$;

(2) *If there is a color i such that $i \in C^1_\varphi(u) \cap C^1_\varphi(v)$ then $(u, i) \sim (v, i)$.*

Proof. (1) Suppose that $|C_\varphi(u, v)| < 4$, We may extend φ to an 4-linear coloring of G by setting $\varphi(uv) \in \{1, 2, 3, 4\} \setminus C_\varphi(u, v)$, a contradiction.

(2) If $(u, i) \nsim (v, i)$, we may extend φ to an 4-linear coloring of G by setting $\varphi(u, v) = i$, a contradiction. □

By Claim 2.2, we have
(a) $\delta(G) \geq 2$,
(b) for any edge $uv \in E(G)$, $d_G(u) + d_G(v) \geq 10$,
(c) any two 4^--vertices are not adjacent,
(c) any 3-face is incident with three 5^+-vertices, or at least two 6^+-vertices, and
(d) any 7^--vertex has no neighbors of degree 2.

Claim 2.3. [13] *If a 7-vertex u is adjacent to a 3-vertex v such that uv is incident with a 3-cycle, then all neighbors of u except v are 4^+-vertices.*

Claim 2.4. [22] *Every vertex is adjacent to at most two 2-vertices. Moreover, suppose that a vertex v is adjacent to two 2-vertices x, y. Let x', y' be the other neighbors of x, y, respectively. Then $x'v, y'v \notin E(G)$.*

Claim 2.5. [5, 11] *If a vertex u is adjacent to two 2-vertices v, w and incident with a 3-face $uxyu$, then $d(x) \geq 4$ and $d(y) \geq 4$.*

Claim 2.6. [5, 13] *If a vertex u is adjacent to a 2-vertex v and incident with two adjacent 3-cycles $uwxu$, $uwyu$, then $d(w) \geq 4$ and $\max\{d(y), d(x)\} \geq 4$.*

Claim 2.7. [8] *If there are two adjacent 3-face $uvwu$ and $uvxu$ such that $d(w) = 2$, then $d(x) \geq 4$.*

By Claim 2.7, we have the following corollary.

Corollary 2.8. *If a 3-face $uxvu$ is adjacent to a 4-face $uxvyu$ such that $d(x) = 2$, then $d(y) \geq 4$.*

Claim 2.9. [13] *If G has a 3-face $uvwu$ such that $d(u) + d(v) = 10$, then $d(w) = 8$.*

Claim 2.10. *G has no configurations depicted in Figure 1.*

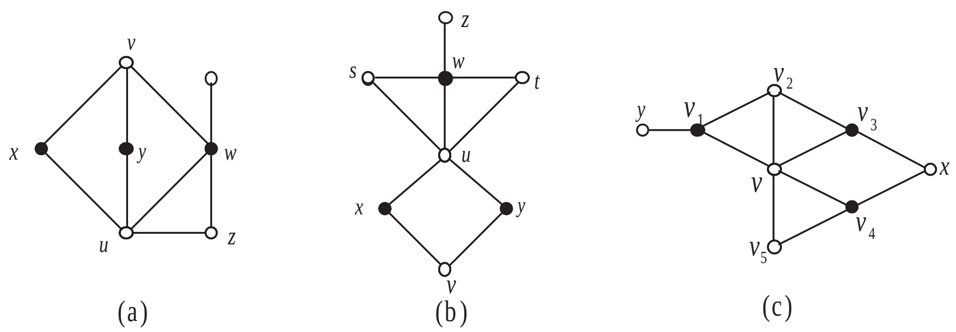

Figure 1.

Proof. Suppose that G has a configuration as depicted in Figure 1(a). By the minimality of G, $G' = G - uy$ has a 4-linear coloring φ. Without loss of generality, assume $\varphi(vy) = 1$. Then $1 \in C^1_\varphi(u)$ and $(u, 1) \sim vy$ by Claim 2.2. If $\varphi(xu) \neq 1$, then $\varphi(xv) \neq 1$, and then $xu \to 1$ and $uy \to \varphi(xu)$. Otherwise, we must have $\varphi(xv) = \varphi(xu) = 1$, and $\varphi(uw) \neq 1$, $\varphi(uz) \neq 1$, $\varphi(vw) \neq 1$. If $1 \in C^0_\varphi(w) \cup C^1_\varphi(w)$, then $wu \to 1$ and $uy \to \varphi(wu)$. Otherwise $1 \in C^2_\varphi(w)$, that is, $\varphi(wz) = 1$. We recolor wz and xu with $\varphi(uz)$, and then $uz \to 1$, $vy \rightleftharpoons vw$, and $uy \to 1$. Hence we can obtain a 4-linear coloring of G, a contradiction.

Suppose that G has a configuration as depicted in Figure 1(b). By the minimality of G, $G' = G - uy$ has a 4-linear coloring φ. Without loss of generality, assume $\varphi(vy) = 1$. By the same argument as above, we have $\varphi(xv) = \varphi(xu) = 1$ and $1 \in C^2_\varphi(w)$. Suppose that $\varphi(wt) = \varphi(ws) = 1$. If $\varphi(ut) = \varphi(zw)$ and $ut \sim zw$, then $us \rightleftharpoons ws$, $uy \to \varphi(us)$. Otherwise, $ut \rightleftharpoons wt$, $uy \to \varphi(ut)$. Suppose that $\varphi(wt) = 1$ and $\varphi(ws) \neq 1$ (It is similar to settle the case $\varphi(wt) \neq 1$ and $\varphi(ws) = 1$). Then $\varphi(wz) = 1$. First, $wu \to 1, ut \to 1$, $wt \to \varphi(ut)$. Then, if

$\varphi(ut) = \varphi(sw)$ and $ut \sim sw$, then $us \rightleftharpoons ws$. Finally, $ux \to \varphi(wu)$ and $uy \to \varphi(ut)$. Hence, we can obtain a 4-linear coloring of G, a contradiction.

Suppose that G has a configuration as depicted in Figure 1(c). By the minimality of G, there exists a 4-linear coloring ϕ of $G - vv_3$ with colors $1, 2, 3, 4$. We also show how to extend ϕ to G and obtain a contradiction with the minimality. The only non-colored edge is vv_3. Let $C^1_\phi(v) = \{a\}$.

Case 1. $\phi(v_2v_3) = \phi(v_3x)$. Without loss of generality, assume that $\phi(v_2v_3) = 1$.

Then $a = 1$ for otherwise we can color vv_3 with a directly. If $(v, 1) \nsim v_2v_3$, then $\phi(vv_2) \neq 1$, $v_2v \rightleftharpoons v_2v_3$ and $vv_3 \to \phi(vv_2)$, a contradiction. So

$$(v, 1) \sim v_2v_3. \qquad (*)$$

Subcase 1.1. $\phi(vv_1) = 1$.

Then $vv_1 \sim v_2v_3$ by (*). If $\phi(v_1y) = 1$, then $v_2v \rightleftharpoons v_2v_3$, $vv_1 \to \phi(vv_2)$ and $vv_3 \to 1$. Otherwise, $\phi(v_1v_2) = 1$. If $\phi(v_1v_2) = 1$ and $vv_2 \sim yv_1$, then $v_2v \rightleftharpoons v_2v_3$, $vv_1 \to \phi(vv_2)$ and $vv_3 \to 1$. Otherwise, $v_2v_1 \rightleftharpoons v_2v$ and $vv_3 \to \phi(vv_2)$.

Subcase 1.2. $\phi(vv_2) = 1$.

Then $\phi(v_1y) = 1$ and $v_1y \sim v_2v$ for otherwise we can recolor v_1v with 1 and color vv_3 with $\phi(vv_1)$. If $1 \in C^0_\phi(v_4) \cup C^1_\phi(v_4)$, then $vv_4 \to 1$ and $vv_3 \to \phi(vv_4)$. Otherwise, $\phi(v_4v_4) = \phi(v_4x) = 1$. Thus $vv_5 \rightleftharpoons v_5v_4$ and $vv_3 \to \phi(vv_5)$.

Subcase 1.3. $1 \notin \{\phi(vv_1), \phi(vv_2)\}$.

If $\phi(v_1v_2) = 1$, then $(v, 1) \sim v_1v_2$ by (*) and then $v_2v_1 \rightleftharpoons v_2v$ and $vv_3 \to \phi(vv_2)$. Otherwise $\phi(v_1v_2) = b \neq 1$. By the same argument, we have $1 \in C^2_\phi(v_4)$, $\phi(v_1y) = 1$ and $(v, 1) \sim v_1y$. It follows that $\phi(v_4v_5) = 1$ and $\phi(vv_5) \neq 1$. First, $vv_5 \rightleftharpoons v_5v_4$ and $vv_3 \to \phi(vv_5)$. Then if $\phi(vv_5) = \phi(xv_4)$ and $vv_5 \sim xv_4$, then $xv_4 \rightleftharpoons xv_3$.

Case 2. $\phi(v_2v_3) \neq \phi(v_3x)$. Without loss of generality, assume that $\varphi(v_2v_3) = 1$ and $\varphi(v_3x) = 2$.

Then $a \in \{1, 2\}$ and $(v, a) \sim (v_3, a)$, for otherwise we directly color vv_3 with a.

Subcase 2.1. $a = 1$.

Then $(v, 1) \sim v_2v_3$.

Subcase 2.1.1. $\phi(vv_1) = 1$, that is, $(v, 1) \sim vv_1$.

Subcase 2.1.1.1. $1 \in C^0_\phi(v_4) \cup C^1_\phi(v_4)$.

If $\phi(vv_4) = 2$ and $v_4v \sim xv_3$, then $\phi(xv_4) \neq 2$ and then $vv_4 \to 1$, $vv_3 \to \phi(vv_4)$ and $xv_3 \rightleftharpoons xv_4$. Otherwise, $vv_4 \to 1$ and $vv_3 \to \phi(vv_4)$.

Subcase 2.1.1.2. $1 \in C^2_\phi(v_4)$. Then $\phi(xv_4) = \phi(v_4v_5) = 1$.

Suppose that $\phi(vv_2) = c \neq 2$. If $\phi(v_1y) = c$ and $v_2v \sim yv_1$, then $\phi(v_1v_2) = 1$ and we do $vv_2 \sim v_1v_2$ and $vv_3 \to c$. Otherwise, $vv_2 \sim vv_1$, $v_2v_3 \to c$ and $vv_3 \to 1$.

Suppose that $\phi(vv_2) = \phi(v_1v_2) = 2$. If $v_2v \nsim xv_3$, then $v_2v \rightleftharpoons v_2v_3$, $vv_1 \to 2$ and $vv_3 \to 1$. Otherwise, $\phi(vv_4) \notin \{1, 2\}$ and then $v_2v \rightleftharpoons v_2v_3$, $vv_1 \to \phi(vv_4)$, $vv_4 \to 2$ and $vv_3 \to 1$.

Suppose that $\phi(vv_2) = 2$ and $\phi(v_1v_2) = c \neq 2$. If $c > 2$, then $\phi(v_1y) = 1$, and $vv_2 \rightleftharpoons vv_1$, $v_1v_2 \to 2$, $vv_3 \to c$ and $vv_3 \to 1$. Otherwise, $\phi(v_1v_2) = 1$. If $\phi(vv_5) = 2$ and $vv_2 \sim xv_3$, then $v_5v_4 \rightleftharpoons v_5v$, $vv_3 \to \phi(vv_4)$ and $vv_4 \to 2$. Otherwise, $v_5v_4 \rightleftharpoons v_5v$ and $vv_3 \to \phi(vv_5)$.

Subcase 2.1.2. $\phi(vv_2) = 1$.

Then $\phi(v_1v_2) \neq 1$. If $\phi(vv_1) \neq 2$, or $\phi(vv_1) = 2$ but $vv_1 \nsim xv_3$, then $vv_1 \to 1$ and $vv_3 \to \phi(vv_1)$. Otherwise, if $1 \in C^2_\phi(v_4)$, then $vv_1 \to 1$, $vv_4 \to 2$ and $vv_3 \to \phi(vv_4)$. Otherwise, $vv_4 \to 1$ and $vv_3 \to vv_4$.

Subcase 2.1.3. $1 \notin \{\phi(vv_1), \phi(vv_2)\}$.

Suppose that $\phi(v_1v_2) \neq 1$. If $\phi(vv_1) = 2$ and $v_1v \sim xv_3$, then $\phi(v_1v_2) > 2$ and $v_1v_2 \rightleftharpoons v_2v_3$, $vv_3 \to 1$. Otherwise, $vv_1 \to 1$ and $vv_3 \to \phi(vv_1)$.

Suppose that $\phi(v_1v_2) = 1$. Since $(v, 1) \sim v_2v_3$, $\phi(v_1y) = 1$. If $\phi(vv_2) \neq 2$, then $vv_2 \sim v_1v_2$ and $vv_3 \to \phi(vv_2)$. If $\phi(vv_2) = 2$ and $\phi(vv_1) \neq 2$, then $vv_2 \sim v_1v_2$, $vv_1 \to \phi(vv_2)$ and $vv_3 \to \phi(vv_1)$. Suppose that $\phi(vv_1) = \phi(vv_2) = 2$. We also have $vv_2 \sim xv_3$ for otherwise $vv_2 \sim v_1v_2$ and $vv_3 \to \phi(vv_2)$. Thus, if $1 \in C^0_\phi(v_4) \cup C^1_\phi(v_4)$, then $vv_4 \to 1$ and $vv_3 \to \phi(vv_4)$. Otherwise, if $\phi(vv_4) = \phi(v_4v_5) = 1$, then $\phi(v_4x) > 2$ and $vv_2 \sim v_1v_2$, $vv_3 \to \phi(vv_2)$ and $xv_3 \rightleftharpoons xv_4$. If $\phi(vv_4) = \phi(xv_4) = 1$, then $vv_2 \sim v_2v_3$, $vv_3 \to \phi(vv_2)$ and $xv_3 \rightleftharpoons xv_4$. If $\phi(v_5v_4) = \phi(xv_4) = 1$, then $vv_2 \sim v_1v_2$, $vv_3 \to \phi(vv_4)$ and $vv_4 \to 2$.

Subcase 2.2. $a = 2$.

Then $(v,2) \sim xv_3$. Suppose that $2 \in C^0_\phi(v_1) \cup C^1_\phi(v_1)$. Then $\phi(vv_1) \neq 2$ and we can recolor vv_1 with 2. If $\phi(vv_1) = 1$, then we come back to Subcase 2.1. Otherwise, $vv_3 \to \phi(vv_1)$. Suppose that $1 \in C^0_\phi(v_4) \cup C^1_\phi(v_4)$. Then $\phi(vv_4) \neq 2$ and we can recolor vv_4 with 2. If $\phi(vv_4) = 1$, then we go back to Subcase 2.1. Otherwise, $vv_3 \to \phi(vv_4)$. So in the following, we assume that $2 \in C^2_\phi(v_1) \cap C^2_\phi(v_4)$.

Subcase 2.2.1. $\phi(vv_4) = \phi(xv_4) = 2$.

Then $\phi(v_2v_1) = \phi(v_1y) = 2$. It follows that $v_2v_1 \sim v_2v_3$, $vv_1 \to 2$ and $vv_3 \to \phi(vv_1)$.

Subcase 2.2.2. $\phi(vv_4) = \phi(v_4v_5) = 2$.

Then $\phi(v_2v_1) = \phi(v_1y) = 2$. Suppose that $\phi(vv_5) = 1$. If $\phi(xv_4) = 1$ and $vv_5 \sim xv_4$, then $vv_5 \rightleftharpoons v_4v_5$, $xv_4 \rightleftharpoons xv_3$, $vv_4 \to 1$ and $vv_2 \to 2$. Otherwise, $vv_5 \rightleftharpoons v_4v_5$ and we go back to Subcase 2.1.

Suppose that $\phi(vv_5) = c > 2$. If $\phi(xv_4) = c$ and $vv_5 \sim xv_4$, then $vv_5 \rightleftharpoons v_4v_5$, $xv_4 \rightleftharpoons xv_3$, $vv_4 \to c$ and $vv_2 \to 2$. Otherwise, $vv_5 \rightleftharpoons v_4v_5$ and $vv_3 \to c$.

Subcase 2.2.3. $\phi(v_5v_4) = \phi(xv_4) = 2$.

Subcase 2.2.3.1. $\phi(vv_1) = \phi(v_1v_2) = 2$.

Suppose that $\phi(vv_2) = 1$. If $\phi(v_1y) = 1$ and $v_2v \sim yv_1$, then $v_2v_1 \sim v_2v$, $vv_4 \to 1$ and $vv_3 \to \phi(vv_4)$. Otherwise, $v_2v_1 \sim v_2v$ and $vv_3 \to 1$.

Suppose that $\phi(vv_2) = c > 2$. If $\phi(v_1y) = c$, then $v_2v_1 \to 1$, $vv_2 \to 2$, $v_2v_3 \to c$ and $vv_3 \to c$. Otherwise, $v_2v_1 \sim v_2v$ and $vv_3 \to c$.

Subcase 2.2.3.2. $\phi(vv_1) = \phi(v_1y) = 2$.

First, $vv_5 \rightleftharpoons v_4v_5$, $vv_1 \to \phi(vv_5)$ and $vv_3 \to 2$. Then, if $\phi(v_1v_2) = \phi(vv_5) \neq 1$, then $v_1v_2 \rightleftharpoons v_2v_3$.

Subcase 2.2.3.3. $\phi(v_1v_2) = \phi(yv_1) = 2$.

Suppose that $v_1v_2 \nsim xv_3$. If $\phi(vv_1) = 1$ and $v_1v \sim v_2v_3$, then $v_2v_1 \rightleftharpoons v_2v_3$ and $vv_1 \rightleftharpoons vv_4$. Otherwise, $v_2v_1 \rightleftharpoons v_2v_3$. Thus, we go back to Subcase 2.1.

Suppose that $v_1v_2 \sim xv_3$, that is, there is a monochromatic path $v \cdots yv_1v_2 \cdots v_5v_4x\, v_3$. It follows that $2 \notin \{\phi(vv_1), \phi(vv_2), \phi(vv_4), \phi(vv_5)\}$. If $\phi(vv_1) = \phi(vv_2) = 1$, then $v_2v_1 \rightleftharpoons v_2v$, $vv_4 \to 1$ and $vv_3 \to \phi(vv_4)$. Otherwise, $v_2v_1 \rightleftharpoons v_2v$ and $vv_3 \to \phi(vv_2)$. □

Claim 2.11. *If a planar graph G contains no* 5*-cycles with two chords and* $\delta(G) > 2$*, then the following results hold.*

(*a*) *Every* 4^+*-vertex* v *is incident with at most* $\lfloor \frac{2d(v)}{3} \rfloor$ 3*-faces;*

(*b*) *If a vertex* v *is incident with three continuous faces* f_1,f_2 *and* f_3 *such that* $d(f_1) = 3$, $d(f_2) = 4$ *and* f_1,f_2 *have a common* 2*-vertex, then* $d(f_3) \geq 4$*;*

(*c*) *If a vertex* v *is incident with four continuous faces* f_1,f_2 ,f_3 *and* f_4 *such that* $d(f_1) = d(f_3) = 3$, $d(f_2) = 4$ *and a* 2*-vertex is incident with* f_2 *and* f_3*, then* $d(f_4) \geq 4$*;*

(*d*) *If a face is adjacent to two nonadjacent* 3*-face, then the face must be a* 4^+*-face.*

The proof of the claim is obvious, we omit here. By the Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0. \tag{1}$$

We define ch to be the initial charge. Let $ch(v) = 2d(v) - 6$ for each $v \in V(G)$ and $ch(f) = d(f) - 6$ for each $f \in F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12 < 0. \tag{2}$$

In the following, we will show that $ch'(x) \geq 0$ for $x \in V(G) \cup F(G)$, a contradiction to (2), completing the proof.

For a face $f = (v_1, v_2, \cdots, v_t)$ of G, we use $(d(v_1), d(v_2), \cdots, d(v_t)) \to (c_1, c_2, \cdots, c_t)$ to denote that vertex v_i sends f the amount of charge c_i for any $i \in \{1, 2, \cdots, t\}$. Now, let us introduce the needed discharging rules as follows.

R1. Every 8^+-vertex sends 1 to each of its adjacent 2-vertices.

R2. Let f be a 3-face. Then

$$(3,7^+,7^+)\rightarrow(0,\frac{3}{2},\frac{3}{2}),$$
$$(4,6^+,7^+)\rightarrow(\frac{1}{2},\frac{5}{4},\frac{5}{4}),$$
$$(5^+,5^+,5^+)\rightarrow(1,1,1).$$

R3. Let f be a 4-face. Then

$$(3^-,7^+,3^-,7^+)\rightarrow(0,1,0,1),$$
$$(3^-,7^+,4^+,7^+)\rightarrow(0,\frac{3}{4},\frac{1}{2},\frac{3}{4}),$$
$$(4^+,4^+,4^+,4^+)\rightarrow(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}).$$

R4. Let f be a 5-face. Then

$$(3^-,7^+,7^+,3^-,7^+)\rightarrow(0,\frac{1}{3},\frac{1}{3},0,\frac{1}{3}),$$
$$(3^-,7^+,4^+,4^+,7^+)\rightarrow(0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}),$$
$$(4^+,4^+,4^+,4^+,4^+)\rightarrow(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}).$$

Now we begin to check $ch'(x)\geq 0$ for all $x\in V(G)\cup F(G)$. Let $f\in F$. If $d(f)\geq 6$, then $ch'(f)=d(f)-6\geq 0$. If $d(f)=5$, then $ch'(f)=ch(f)+\max\{\frac{1}{3}\times 3,\frac{1}{4}\times 4,\frac{1}{4}\times 5\}=0$ by R4. If $d(f)=4$, then $ch'(f)=ch(f)+\max\{1\times 2,\frac{1}{2}+\frac{3}{4}\times 2,\frac{1}{2}\times 4\}=0$. If $d(f)=3$, then $ch'(f)=ch(f)+\max\{\frac{3}{2}\times 2,\frac{1}{2}+\frac{5}{4}\times 2,1\times 3\}=0$.

Let $v\in V$. If $d(v)=2$, then $ch'(v)=ch(v)+2=0$ by R1. If $d(v)=3$, then $ch'(v)=ch(v)=0$ by R2-R4. If $d(v)=4$, then it sends every incident face at most $\frac{1}{2}$. So $ch'(v)=ch(v)-\frac{1}{2}\times f_3(v)-\frac{1}{2}\times(4-f_3(v))=0$ by R2-R4. If $d(v)=5$, then $f_3(v)\leq 3$ by Claim 2.11. So $ch'(v)\geq ch(v)-1\times f_3(v)-\frac{1}{2}\times(5-f_3(v))=\frac{3-f_3(v)}{2}\geq 0$. If $d(v)=6$, then $f_3(v)\leq 4$ and $ch'(v)\geq ch(v)-\frac{5}{4}\times f_3(v)-\frac{1}{2}\times(6-f_3(v))=\frac{12-3f_3(v)}{4}\geq 0$. Suppose $d(v)=7$. By Claim 2.11, $f_3(v)\leq 4$. If v has a 3-neighbor u such that uv is incident with a 3-cycle (note that uv may be incident with two 3-faces), then all neighbors of v except u are 4^+-vertices, and it follows that $ch'(v)\geq ch(v)-(\frac{3}{2}\times 2+\frac{5}{4}\times(f_3(v)-2)+\frac{3}{4}\times(7-f_3(v)))=\frac{9-2f_3(v)}{4}>0$. Otherwise, $ch'(v)=ch(v)-\frac{5}{4}\times f_3(v)-1\times(7-f_3(v))=\frac{4-f_3(v)}{4}\geq 0$.

Suppose $d(v)=8$. Then $f_3(v)\leq 5$. Let $v_1,v_2,...,v_8$ be neighbors of v in a clockwise order, and denote by $f_1,f_2,...,f_8$ be faces incident with v such that v_i is incident with f_i, f_{i+1}, $i=1,2,...,7$ and v_8 is incident with f_8 and f_1. By Claim 2.4, we consider the following three cases.

Case 1. $n_2(v)=2$.

Without loss of generality, assume that v_1 and v_i are 2-vertices($2\leq i\leq 5$). By Claim 2.4, f_1,f_2,f_i,f_{i+1} are 4^+-faces. Note that if some f_j is a 3-face, then all vertices incident with f_j are 4^+-vertices by Claim 2.5, and it follows that v sends at most $\frac{5}{4}$ to f_j. If f_j is a 3-face and f_{j+1} is a 4-face, then f_{j+1} is incident with at least three 4^+-vertices, and it follows that it receives at most $\frac{3}{4}$ from v.

Subcase 1.1. $i=2$.

Then $f_3(v)\leq 4$ since G contains no 5-cycles with two chords. If $f_3(v)<4$, then $ch'(v)\geq ch(v)-2-\frac{5}{4}\times f_3(v)-\frac{3}{4}\times f_3(v)-1\times(8-2f_3(v))=0$ by R2-R4. Otherwise, we must have that f_4,f_5,f_7,f_8 are 3-faces and f_6 is a 4^+-face. If $d(f_2)=4$, then f_3(or f_1) is 5^+-face or f_4(or f_8) is a $(5^+,5^+,5^+)$-face by Claim 2.10, respectively, and it follows that $ch'(v)\geq ch(v)-2-\max\{1\times 2+\frac{5}{4}\times 2+1+\frac{3}{4}\times 3,\frac{5}{4}\times 3+1+\frac{3}{4}\times 2+\frac{1}{3},\frac{5}{4}\times 4+1+\frac{3}{4}+\frac{1}{3}\times 2\}=\frac{1}{4}>0$. Otherwise, $d(f_2)\geq 5$, and we have $ch'(v)\geq ch(v)-2-\frac{5}{4}\times 4-\frac{3}{4}\times 3-\frac{1}{3}>0$.

Subcase 1.2. $i=3$.

Then $f_3(v) \le 3$. So $ch'(v) \ge ch(v) - 2 - \frac{5}{4} \times f_3(v) - \frac{3}{4} \times f_3(v) - 1 \times (8 - 2f_3(v)) = 0$ by R2, R3 and R4.

Subcase 1.3. $i = 4$.

Then $f_3(v) \le 3$. So $ch'(v) \ge ch(v) - 2 - \frac{5}{4} \times f_3(v) - \frac{3}{4} \times f_3(v) - 1 \times (8 - 2f_3(v)) \ge 0$ by R2, R3 and R4.

Subcase 1.4. $i = 5$.

Then $f_3(v) \le 4$. So $ch'(v) \ge ch(v) - 2 - \frac{5}{4} \times f_3(v) - \frac{3}{4} \times f_3(v) - 1 \times (8 - 2f_3(v)) \ge 0$ by R2, R3 and R4.

Case 2. $n_2(v) = 1$. Without loss of generality, assume that v_8 is the 2-vertex.

Suppose that there is an integer $i(2 \le i \le 6)$ such that f_i and f_{i+1} are 3-faces, then f_i or f_{i+1} is incident with three 4^+-vertices by Claim 2.6, and f_i or f_j receive at most $\frac{5}{4}$ from v, and accordingly, f_{i-1} or f_{j+1} is a 4^+-face incident with at least three 4^+-vertices and receive at most $\frac{3}{4}$ from v.

Subcase 2.1. f_1 and f_8 are 4^+-faces.

By the hypothesis of the theorem, $f_3(v) \le 4$. If $f_3(v) \le 2$, then $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 2 - 1 \times 6 = 0$. Suppose that $f_3(v) = 3$. Let f_i, f_j, f_k be three 3-faces, where $1 < i < j < k < 8$. If $i + 1 < j < k - 1$, then there are at least three 4^+-faces each of which is incident with at least three 4^+-vertices, and it follows that $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{3}{4} \times 3 - 1 \times 2 > 0$. Otherwise, there is a 3-face received $\frac{5}{4}$ from v and a 4^+-face received $\frac{3}{4}$ from v, so $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 2 - \frac{5}{4} - \frac{3}{4} - 1 \times 4 = 0$.

Suppose that $f_3(v) = 4$. Let f_i, f_j, f_k, f_l be four 3-faces, where $2 \le i < j < k < l \le 7$. If $i + 1 = j$ and $k + 1 = l$, then $ch'(v) \ge ch(v) - 1 - (\frac{3}{2} + \frac{5}{4}) \times 2 - \max\{1 \times 3 + \frac{1}{2}, \frac{3}{4} \times 2 + 1 \times 2\} = 0$. Otherwise, there is a pair of adjacent 3-faces in $\{f_i, f_j, f_k, f_l\}$ and there are at least three 4^+-faces incident with at least three 4^+-vertices, and it follows that $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{5}{4} - \frac{3}{4} \times 3 - 1 = 0$.

Subcase 2.2. f_1 or f_8 is a 3-face. Without loss of generality, assume that $d(f_1) = 3$.

Then $d(f_8) \ge 4$ and $f_3(v) \le 5$.

Subcase 2.2.1. $f_3(v) \le 2$.

Then $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 2 - 1 \times 6 = 0$.

Subcase 2.2.2. $f_3(v) = 3$.

Let f_1, f_i, f_j be three 3-faces, where $1 < i < j < 8$. If $i = 2$, that is, f_1 and f_2 are two adjacent 3-faces, then $d(v_2) \ge 4$ by Claim 2.7, and it follows that v sends at most $\frac{5}{4}$ to f_2, at most $\frac{3}{4}$ to f_3, and we have $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 2 - \frac{5}{4} - \frac{3}{4} - 1 \times 4 = 0$. Otherwise $ch'(v) \ge ch(v) - 1 - \max\{\frac{3}{2} \times 3 + \frac{3}{4} \times 3 + 2 \times 1, \frac{3}{2} \times 2 + \frac{5}{4} + \frac{3}{4} \times 2 + 1 \times 2\} > 0$.

Subcase 2.2.3. $f_3(v) = 4$.

Let f_1, f_i, f_j, f_k be three 3-faces, where $1 < i < j < k < 8$. Suppose that $i = 2$, that is, f_1 and f_2 are two adjacent 3-faces. Then $d(v_2) \ge 4$ by Claim 2.7, and it follows that v sends at most $\frac{5}{4}$ to f_2, at most $\frac{3}{4}$ to f_3. If f_j, f_k are not adjacent, then $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{5}{4} - \frac{3}{4} - \max\{\frac{1}{2} + 1 \times 2, \frac{3}{4} \times 2 + 1\} = 0$. Otherwise $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 2 - \frac{5}{4} \times 2 - \max\{\frac{3}{4} \times 2 + 2 \times 1, \frac{3}{4} \times 3 + 1\} = 0$.

Suppose that $i > 2$. If $i = 3, j = 5, k = 7$, then $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 4 - \max\{\frac{3}{4} \times 4, \frac{3}{4} \times 2 + \frac{1}{2} + 1, \frac{1}{2} + 1 \times 2\} = 0$. Otherwise, there are two adjacent 3-faces in $\{f_i, f_j, f_k\}$, and $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{5}{4} - \max\{\frac{3}{4} \times 3 + 1, \frac{3}{4} + \frac{1}{2} + 1 \times 2\} = 0$.

Subcase 2.2.4. $f_3(v) = 5$.

Then we must have $d(f_7) = 3$ and $d(f_8) \ge 5$. Suppose that $d(f_2) \ge 4$. Then f_3, f_4, f_6, f_7 are 3-faces. By Claim 2.6, $\max\{d(v_2), d(v_4)\} \ge 4$ and $\max\{d(v_5), d(v_7)\} \ge 4$. So $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{5}{4} \times 2 - \frac{1}{3} - \max\{\frac{3}{4} \times 2, \frac{1}{2} + 1\} > 0$.

Suppose that $d(f_2) = 3$, that is, f_1 and f_2 are two adjacent 3-faces. Then $d(v_2) \ge 4$ by Claim 2.7, and $d(f_3) \ge 4$. We also have $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 3 - \frac{5}{4} \times 2 - \frac{1}{3} - \max\{\frac{3}{4} \times 2, \frac{1}{2} + 1\} > 0$.

Case 3. $n_2(v) = 0$.

Then $f_3(v) \le 5$. If $f_3(v) \le 4$, then $ch'(v) \ge ch(v) - \frac{3}{2} \times 4 - 1 \times 4 = 0$. Otherwise, assume that f_1, f_2, f_4, f_5, f_7 are 3-faces. If there is a 5^+-face in $\{f_3, f_6, f_8\}$, then $ch'(v) \ge ch(v) - \frac{3}{2} \times 5 - \frac{1}{3} - 1 \times 2 > 0$. Otherwise, $d(f_3) = d(f_6) = d(f_8) = 4$. By Claim 2.10, there are at least two 4-faces in $\{f_3, f_6, f_8\}$ each of which is incident with at least three 4^+-vertices. So $ch'(v) \ge ch(v) - \frac{3}{2} \times 4 - \frac{5}{4} - \frac{3}{4} \times 2 - 1 > 0$.

Hence the proof is completed. □

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