Mannheim Partner Curves in Cartan-Vranceanu 3-space

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Abstract. In this paper, the Mannheim mate curves of the proper biharmonic curves in Cartan-Vranceanu 3-dimensional spaces $(M, ds^2_{l,m})$, with $l^2 \neq 4m$ and $m \neq 0$ are studied. We give the definition of the Mannheim mate of a proper biharmonic curve and give the explicit parametric equations of that Mannheim mate curve in Cartan-Vranceanu 3-dimensional space. Moreover, we show that the distance between corresponding points of the Mannheim pairs is constant in Cartan-Vranceanu 3-dimensional spaces.

1. Introduction

Harmonic maps $f : (M, g) \rightarrow (N, h)$ between a compact Riemannian manifold, (M, g), and a Riemannian manifold, (N, h) are the critical points of the energy functional

$$E(f) = \frac{1}{2} \int_M \left| df \right|^2 v_g,$$

by the definition given by J. Eells and J.H. Sampson in [7]. From the first variation formula it follows that f is harmonic if and only if its tension field

$$\tau(f) = trace \nabla df,$$

vanishes.

We can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2} \int_M \left| \tau(f) \right|^2 v_g,$$

and say that is biharmonic if it is a critical point of the bienergy (as suggested by the same authors in [7]). Jiang derived the first and the second variation formula for the bienergy in [10], showing that the Euler-Lagrange equation for E_2 is

$$\tau_2(f) = -J^f(\tau(f)) = -\Delta\tau(f) - trace R^N(df, \tau(f))df = 0,$$

where J^f denotes the Jacobi operator of f. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Note that the harmonic maps are also biharmonic. Therefore, we are interested in non-harmonic biharmonic maps, which are called proper biharmonic maps.

In this paper, we restrain our attention to curves $\gamma : I \rightarrow (N, h)$ parametrized by arc length, from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold. In this case, putting $T = \gamma\prime$, the tension field becomes $\tau(\gamma) = \nabla_T T$ and the biharmonic equation reduces to fourth order differential equation

$$\nabla_T^3 T + R(T, \nabla_T T)T = 0.$$

The homogeneous Riemannian spaces with a large isometry group play a special role among the 3-dimensional manifolds of non-constant sectional curvature. For these spaces, except for those with constant negative curvature, there is a nice local representation given by the following two-parameter family of Riemannian metrics which is called the Cartan-Vranceanu metric

$$ds_{l,m}^2 = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left(dz + \frac{l}{2}\frac{ydx - xdy}{[1 + m(x^2 + y^2)]}\right)^2, l, m \in \mathbb{R} \tag{1}$$

defined on 3-dimensional manifold M, where $M = \mathbb{R}^3$ if $m \geq 0$, and $M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < \frac{-1}{m}\right\}$ otherwise. The family of metrics given by equation (1) includes all 3-dimensional homogeneous metrics whose group of isometries has dimension 4 or 6, except for those of constant negative sectional curvature. Biharmonic curves on $(M, ds_{l,m}^2)$ have been already studied for particular values of ℓ and m. In particular, if $l = m = 0$, $(M, ds_{l,m}^2)$ is the Euclidean space and γ is biharmonic if and only if it is a line [6]; if $\ell^2 = 4m$ and $l \neq 0$, $(M, ds_{l,m}^2)$ is locally the 3-dimensional sphere and the proper biharmonic curves were classified in [1], where it was proved that they are helices; if $\ell = 0$ and $m < 0$, $(M, ds_{l,m}^2)$ is isometric to $H^2 \times \mathbb{R}$ with the product metric and it can be show that all biharmonic curves are geodesics; if $m = 0$ and $\ell \neq 0$, $(M, ds_{l,m}^2)$ is the Heisenberg space H_3 endowed with a left invariant metric and the explicit solutions of the biharmonic curves were obtained in [2] the generalized ones are obtained in [8]; if $\ell = 1$ a study of the biharmonic curves was given in [4]; if $\ell \neq 0$ and $m < 0$, $(M, ds_{l,m}^2)$ is locally $\widetilde{SL}(2, \mathbb{R})$; if $\ell \neq 0$ and $m > 0$, $(M, ds_{l,m}^2)$ is locally $SU(2)$; if $\ell = 0$, then $(M, ds_{l,m}^2)$ is the product of a surface S with constant Gaussian curvature $4m$ and the real line $\mathbb{R}$. In the case of $l^2 \neq 4m$ and $m \neq 0$,which contains the last three cases, explicit formulas for non-geodesic biharmonic curves of the 3- dimensional Cartan-Vranceanu space and the generalized ones are obtained in [3, 9].

An increasing interest of the theory of curves makes a development of special curves, which can be characterized by the relationship between Frenet vectors of the curves, to be examined. Liu and Wang [11, 14] gave a new definition of the curves known as Mannheim curves. According to the definition given by Liu and Wang [11], the principal normal vector field of γ is linearly dependent with the binormal vector field of γ^*. Then γ is called a Mannheim curve and γ^* a Mannheim mate curve of γ. The pair (γ, γ^*) is said to be a Mannheim pair. Mannheim mate of a biharmonic curve in the Heisenberg group $Heis^3$, which is a special case of Cartan-Vranceanu metrics, are studied in [12, 13]. Choi and his colleagues characterized Mannheim curves and their mate curves in 3-dimensional space forms, making a generalization of the results obtained by Liu and Wang in [5].

In this paper, the Mannheim mate curves of the proper biharmonic curves in Cartan-Vranceanu 3-dimensional spaces $(M, ds_{l,m}^2)$, with $l^2 \neq 4m$ and $m \neq 0$ are studied. We give the definition of Mannheim mate of a curve and give the explicit parametric equations of Mannheim mate of biharmonic curves given by [3] in Cartan-Vranceanu 3-dimensional space. Moreover, we show that the distance between corresponding points of the Mannheim pair is constant in Cartan-Vranceanu 3-dimensional spaces.

2. Biharmonic Curves in Cartan-Vranceanu 3-space

In this section, the Riemannian structure of Cartan-Vranceanu 3-space and the parametric equations of all proper biharmonic curves of $(M, ds_{l,m}^2)$ are given. These curves are important. We give the Mannheim

mate of these biharmonic curves in Section 3.

The Cartan-Vranceanu metric given by (1) can be written as:

$$ds^2_{l,m} = \sum_{i=1}^{3} \omega^i \otimes \omega^i,$$

where, putting $F = 1 + m(x^2 + y^2)$,

$$\omega^1 = \frac{dx}{F}, \ \omega^2 = \frac{dy}{F}, \ \omega^3 = dz + \frac{l}{2} \frac{ydx - xdy}{F}, \tag{2}$$

and the orthonormal base of dual vector fields to the 1-forms given by equation (2) is

$$E_1 = F\frac{\partial}{\partial x} - \frac{ly}{2}\frac{\partial}{\partial z}, \ E_2 = F\frac{\partial}{\partial y} + \frac{lx}{2}\frac{\partial}{\partial z}, \ E_3 = \frac{\partial}{\partial z}. \tag{3}$$

The Levi-Civita connection with respect to the orthonormal basis given by equation (3) is given by,

$$\nabla_{E_1}E_1 \ = \ 2myE_2, \ \nabla_{E_1}E_2 = -2myE_1 + \frac{l}{2}E_3,$$

$$\nabla_{E_2}E_2 \ = \ 2mxE_1, \ \nabla_{E_2}E_1 = -2mxE_2 - \frac{l}{2}E_3,$$

$$\nabla_{E_3}E_3 \ = \ 0, \qquad \nabla_{E_1}E_3 = \nabla_{E_3}E_1 = -\frac{l}{2}E_2,$$

$$\nabla_{E_2}E_3 \ = \ \nabla_{E_3}E_2 = \frac{l}{2}E_1.$$

Also, one obtains the following bracket relations:

$$[E_1, E_2] \ = \ (-2my) E_1 + (2mx) E_2 + (l) E_3, \ [E_1, E_3] = 0, \ [E_2, E_3] = 0,$$

$$[E_1, E_1] \ = \ 0, [E_2, E_2] = 0, [E_3, E_3] = 0.$$

We shall adopt the following notation and sign convention. The curvature operator is given by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]}Z,$$

while the Riemann-Christoffel tensor field and the Ricci tensor field are given by

$$R(X, Y, Z, W) = ds^2_{l,m}((X, Y)Z, W), \ \rho(X, Y) = trace(Z \to R(X, Z)Y),$$

where X, Y, Z, W are smooth vector fields on $(M, ds^2_{l,m})$. The nonvanishing components of the Riemann-Christoffel and the Ricci tensor fields are

$$R_{1212} = 4m - \frac{3}{4}l^2, \ R_{1313} = \frac{l^2}{4}, \ R_{2323} = \frac{l^2}{4},$$

and

$$\rho_{11} = \rho_{22} = 4m - \frac{l^2}{2}, \ \rho_{33} = \frac{l^2}{2}.$$

Let $\gamma : I \to (M, ds^2_{l,m})$ be a differentiable curve parametrized by arc length and let $\{T = T_iE_i, N = N_iE_i, B = B_iE_i\}$ be the orthonormal frame field tangent to M along γ decomposed with respect to the orthonormal basis given by equation (3). Then we have the following Frenet equations:

$$\nabla_T T \ = \ \kappa N$$
$$\nabla_T N \ = \ -\kappa T + \tau B$$
$$\nabla_T B \ = \ -\tau N \tag{4}$$

where κ and τ are curvatures of γ.

Lemma 2.1. *[3] Let $\gamma : I \to (M, ds^2_{l,m})$ a non-geodesic curve parametrized by arc length. If $N_3 = 0$, then*

$$T = \sin\alpha_0 \cos\beta(s)E_1 + \sin\alpha_0 \sin\beta(s)E_2 + \cos\alpha_0 E_3, \tag{5}$$

where $\alpha_0 \in (0, \pi)$.

In the following theorem the explicit parametric equations of proper biharmonic curves in $(M, ds^2_{l,m})$ is given. This theorem is important to obtain Mannheim mate of the proper biharmonic curve in Cartan-Vranceanu 3-dimensional spaces. In our main Theorem (3.4), we use these biharmonic curves.

Theorem 2.2. *[3] Let $(M, ds^2_{l,m})$ be the Cartan-Vranceanu space with $l^2 \neq 4m$ and $m \neq 0$. Assume that $\delta = l^2 + (16m - 5l^2)\sin^2\alpha_0 \geq 0$, $\alpha_0 \in (0, \pi)$, and denote by $2\omega_{1,2} = -l\cos\alpha_0 \pm \sqrt{\delta}$. Then, the parametric equations of all proper biharmonic curves of $(M, ds^2_{l,m})$ are of the following three types:*

Type I

$$
\begin{aligned}
x(s) &= b\sin\alpha_0 \sin\beta(s) + c; \ b, c \in \mathbb{R}, b > 0,\\
y(s) &= -b\sin\alpha_0 \cos\beta(s) + d; \ d \in \mathbb{R},\\
z(s) &= \frac{l}{4m}\beta(s) + \frac{1}{4m}\left[\left(4m - l^2\right)\cos\alpha_0 - l\omega_{1,2}\right]s,
\end{aligned}
\tag{6}
$$

where β is a non-constant solution of the following ODE:

$$\beta' + 2md\sin\alpha_0 \cos\beta(s) - 2mc\sin\alpha_0 \sin\beta(s) = l\cos\alpha_0 + 2mb\sin^2\alpha_0 + \omega_{1,2},$$

and the constants satisfy

$$c^2 + d^2 = \frac{b}{m}\left\{(l\cos\alpha_0 + \omega_{1,2} - \frac{1}{b}) + mb\sin^2\alpha_0\right\}.$$

Type II If $\beta = \beta_0 = cnst$ and $\cos\beta_0 \sin\beta_0 \neq 0$, the parametric equations are

$$
\begin{aligned}
x(s) &= x(s),\\
y(s) &= x(s)\tan\beta_0 + a,\\
z(s) &= \frac{1}{4m}\left[\left(4m - l^2\right)\cos\alpha_0 - l\omega_{1,2}\right]s + b, \ b \in \mathbb{R},
\end{aligned}
$$

where

$$a = \frac{\omega_{1,2} + l\cos\alpha_0}{2m\sin\alpha_0 \cos\beta_0}$$

and $x(s)$ is a solution of the following ODE:

$$x' = (1 + m[x^2 + (x\tan\beta_0 + a)^2])\sin\alpha_0 \cos\beta_0.$$

Type III If $\cos\beta_0 \sin\beta_0 = 0$, up to interchange of x with y, $\cos\beta_0 = 0$ and the parametric equations are

$$
\begin{aligned}
x(s) &= x_0 = \pm\frac{\omega_{1,2} + l\cos\alpha_0}{2m\sin\alpha_0},\\
y(s) &= y(s),\\
z(s) &= \frac{1}{4m}\left[\left(4m - l^2\right)\cos\alpha_0 - l\omega_{1,2}\right]s + b, \ b \in \mathbb{R},
\end{aligned}
$$

where $y(s)$ is a solution of the following ODE:

$$y' = \pm(1 + m[x_0^2 + y^2])\sin\alpha_0.$$

3. Mannheim Mate of Biharmonic Curves in Cartan-Vranceanu 3-Space

In this section, we define Mannheim partner curves in Cartan-Vranceanu 3-dimensional spaces. We give the explicit parametric equations of Mannheim mate of the biharmonic curves given by [3] and show that the distance between corresponding points of the Mannheim pairs is constant in Cartan-Vranceanu 3-dimensional spaces.

Definition 3.1. *Let unit speed curve $\gamma : I \to (M, ds_{l,m}^2)$ and the curve $\gamma^* : I \to (M, ds_{l,m}^2)$ be given. If there exists a corresponding relationship between γ and γ^* such that, at the corresponding points of the curves, the principal normal of γ coincides with the binormal lines of γ^*, then γ is called a Mannehim curve, γ^* is called Mannheim mate curve of γ and (γ, γ^*) is called Mannheim pair.*

From the above definition, we obtain

$$\gamma^*(s) = \gamma(s) + \lambda(s)B(s), \ \forall s \in I, \tag{7}$$

where $\lambda : I \to \mathbb{R}$.

Let the Frenet frames of the curves γ and γ^* be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively.

Theorem 3.2. *Let $(M, ds_{l,m}^2)$ be the Cartan-Vranceanu space with $l^2 \neq 4m$ and $m \neq 0$. Let $\gamma : I \to (M, ds_{l,m}^2)$ be a unit curve and γ^* its Mannheim mate on $(M, ds_{l,m}^2)$. Then, the distance between corresponding points of the Mannheim pairs (γ, γ^*) is constant in $(M, ds_{l,m}^2)$.*

Proof. By taking the derivative of equation (7) and using the third Frenet equation (4), we obtain

$$T^*(s) = T(s) + \lambda'(s)B(s) - \lambda(s)\tau(s)N(s).$$

Since N^* and B are linearly dependent, $ds_{l,m}^2(T^*, B) = 0$, we have $\lambda'(s) = 0$ and in this case λ_0 is a nonzero constant. On the other hand, from the distance functions between two points, we have

$$d(\gamma^*(s), \gamma(s)) = \left\| \gamma(s) - \gamma^*(s) \right\|$$

$$= \left| \sqrt{ds_{l,m}^2(\lambda_0 B, \lambda_0 B)} \right|$$

$$= |\lambda_0| \left| \sqrt{ds_{l,m}^2(B, B)} \right|$$

$$= |\lambda_0|. \tag{8}$$

Namely, $ds_{l,m}^2(\gamma^*(s), \gamma(s)) = cnst.$ $\square$

Corollary 3.3. *Let unit speed curve $\gamma : I \to (M, ds_{l,m}^2)$ and its Mannheim mate $\gamma^* : I \to (M, ds_{l,m}^2)$ be given. For $\forall s \in I$, the curves γ and γ^* can be written as the following equation:*

$$\gamma^*(s) = \gamma(s) + \lambda_0 B(s), \tag{9}$$

where $\lambda_0 \in \mathbb{R}$.

Proof. Bu using Definition (3.1) and equation (8), we get equation (9). $\square$

In the following theorem, we obtain the explicit parametric equations of the Mannheim mate of the proper biharmonic curve in Cartan-Vranceanu 3-dimensional spaces.

Theorem 3.4. *Let $(M, ds^2_{l,m})$ be the Cartan-Vranceanu space with $l^2 \neq 4m$ and $m \neq 0$. Assume that $\delta = l^2 + (16m - 5l^2)\sin^2 \alpha_0 \geq 0$, $\alpha_0 \in (0, \pi)$, and denote by $2\omega_{1,2} = -l\cos \alpha_0 \pm \sqrt{\delta}$. Let $\gamma : I \to (M, ds^2_{l,m})$ be a unit curve and γ^* its Mannheim mate on $(M, ds^2_{l,m})$. Then, the parametric equations of γ^* are of the following three types:*

Type I

$$
\begin{aligned}
\gamma^*(s) \ = \ & \left(\frac{b \sin \alpha_0 \sin \beta(s) + c}{b\beta'} - \lambda_0 \cos \alpha_0 \cos \beta(s) \right) E_1 \\
& + \left(\frac{-b \sin \alpha_0 \cos \beta(s) + d}{b\beta'} - \lambda_0 \cos \alpha_0 \sin \beta(s) \right) E_2 \\
& + \left(\frac{l}{4m}\beta(s) + \frac{1}{4m}\left[\left(4m - l^2\right)\cos \alpha_0 - l w_{1,2} \right]s + \lambda_0 \sin \alpha_0 \right) E_3,
\end{aligned}
\tag{10}
$$

β is a non-constant solution of the following ODE:

$$\beta' + 2md \sin \alpha_0 \cos \beta(s) - 2mc \sin \alpha_0 \sin \beta(s) = l \cos \alpha_0 + 2mb \sin^2 \alpha_0 + w_{1,2}$$

and the constants satisfy

$$c^2 + d^2 = \frac{b}{m} \left\{ (l \cos \alpha_0 + w_{1,2} - \frac{1}{b}) + mb \sin^2 \alpha_0 \right\}.$$

Moreover $b, \lambda_0 \in \mathbb{R}$, $b > 0$.

Type II If $\beta = \beta_0 = cnst$ and $\cos \beta_0 \sin \beta_0 \neq 0$, the parametric equations are

$$
\begin{aligned}
\gamma^*(s) \ = \ & \left(\frac{x(s)}{1 + m\left(\frac{x^2(s)}{\cos^2 \beta_0} + 2ax(s)\tan \beta_0 + a^2 \right)} - \lambda_0 \cos \alpha_0 \cos \beta_0 \right) E_1 \\
& + \left(\frac{x(s)\tan \beta_0 + a}{1 + m\left(\frac{x^2(s)}{\cos^2 \beta_0} + 2ax(s)\tan \beta_0 + a^2 \right)} - \lambda_0 \cos \alpha_0 \sin \beta_0 \right) E_2 \\
& + \left(\frac{1}{4m}\left[\left(4m - l^2\right)\cos \alpha_0 - l w_{1,2} \right]s + b + \lambda_0 \sin \alpha_0 \right) E_3,
\end{aligned}
$$

where $b, \lambda_0 \in \mathbb{R}$,

$$a = \frac{\omega_{1,2} + l \cos \alpha_0}{2m \sin \beta_0 \cos \beta_0}$$

and $x(s)$ is a solution of the following ODE:

$$x' = (1 + m[x^2 + (x \tan \beta_0 + a)^2]) \sin \alpha_0 \cos \beta_0.$$

Type III If $\cos \beta_0 \sin \beta_0 = 0$, up to interchange of x with y, $\cos \beta_0 = 0$ and the parametric equations are

$$
\begin{aligned}
\gamma^*(s) \ = \ & \left(\frac{x_0}{1 + m((x_0)^2 + y(s)^2)} \right) E_1 \\
& + \left(\frac{y(s)}{1 + m((x_0)^2 + y(s)^2)} - \lambda_0 \cos \alpha_0 \sin \beta_0 \right) E_2 \\
& + \left(\frac{1}{4m}\left[\left(4m - l^2\right)\cos \alpha_0 - l w_{1,2} \right]s + b + \lambda_0 \sin \alpha_0 \right) E_3,
\end{aligned}
$$

where $b, \lambda_0 \in \mathbb{R}$ and $y(s)$ is a solution of the following ODE:

$$y' = \pm(1 + m[x_0^2 + y^2]) \sin \alpha_0.$$

Proof. The covariant derivative of the vector field T given by (5) is

$$
\begin{aligned}
\nabla_T T &= [-\beta' \sin \alpha_0 \sin \beta - 2my \sin^2 \alpha_0 \cos \beta \sin \beta \\
&\quad +2mx \sin^2 \alpha_0 \sin^2 \beta + l \cos \alpha_0 \sin \alpha_0 \sin \beta]E_1 \\
&\quad +[\beta' \sin \alpha_0 \cos \beta + 2my \sin^2 \alpha_0 \cos^2 \beta \\
&\quad -2mx \sin^2 \alpha_0 \cos \beta \sin \beta - l \cos \alpha_0 \sin \alpha_0 \cos \beta]E_2 \\
&= \kappa N,
\end{aligned}
$$

where

$$
\kappa = \left| \beta' + 2my \sin \alpha_0 \cos \beta - 2mx \sin \alpha_0 \sin \beta - l \cos \alpha_0 \right| \sin \alpha_0.
$$

We assume that

$$
\omega = \beta' + 2my \sin \alpha_0 \cos \beta - 2mx \sin \alpha_0 \sin \beta - l \cos \alpha_0 > 0.
$$

Then we have

$$
\kappa = \omega \sin \alpha_0,
$$

and

$$
N = -\sin \beta(s)E_1 + \cos \beta(s)E_2.
$$

Noting that $T \times N = B$, we have

$$
B = -\cos \alpha_0 \cos \beta(s)E_1 - \cos \alpha_0 \sin \beta(s)E_2 + \sin \alpha_0 E_3. \tag{11}
$$

If we replace (6) and (11) in terms of the orthonormal basis given by (3) into (9), we get equation (10). The case of Type II and Type III can be derived in a similar way. □

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