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Third and Higher Order Convolution Identities for Cauchy Numbers

Takao Komatsu[a], **Yilmaz Simsek**[b]

[a] *School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China*
[b] *Department of Mathematics, Faculty of Science University of Akdeniz TR-07058 Antalya, Turkey*

Abstract. The n-th Cauchy number c_n ($n \ge 0$) are defined by the generating function $x/\ln(1+x) = \sum_{n=0}^{\infty} c_n x^n/n!$. In this paper, we deal with formulae of the type

$$\sum_{\substack{l_1+\cdots+l_m=\mu\\ l_1,\dots,l_m\ge 0}} \frac{\mu!}{l_1!\cdots l_m!}(c_{l_1}+\cdots+c_{l_m})^n = a_0 c_{n+\mu}+\cdots+a_{m-1}c_{n+\mu-m+1}\,,$$

where the a_i are suitable rational numbers, the c_i are Cauchy numbers and

$$(c_{l_1}+\cdots+c_{l_m})^n := \sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 0}} \frac{n!}{k_1!\cdots k_m!} c_{k_1+l_1}\cdots c_{k_m+l_m}\,.$$

In particular, we give explicit formulae for $m = 3$ and $m = 4$.

1. Introduction

The Cauchy numbers c_n ($n \ge 0$) are defined by

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx$$

and the (exponential) generating function of c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \quad (|x|<1)$$

([4, 9]). $b_n = c_n/n!$ are sometimes called the Bernoulli numbers of the second kind. The first few initial values are

$$c_0 = 1,\ c_1 = \frac{1}{2},\ c_2 = -\frac{1}{6},\ c_3 = \frac{1}{4},\ c_4 = -\frac{19}{30},\ c_5 = \frac{9}{4},\ c_6 = -\frac{863}{84},\ c_7 = \frac{1375}{24}.$$

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Email addresses: komatsu@whu.edu.cn (Takao Komatsu), ysimsek@akdeniz.edu.tr (Yilmaz Simsek)

In [7], an explicit expression of $(c_l + c_m)^n$ for $l, m, n \ge 0$ was determined, where with the classical umbral calculus notation (see, e.g., [10]), $(c_l + c_m)^n$ is defined by

$$(c_l + c_m)^n := \sum_{j=0}^{n} \binom{n}{j} c_{l+j} c_{m+n-j}\,.$$

As some special cases, we gave explicit formulae:

$$(c_0 + c_1)^n = -\frac{1}{2}(n+1)(n-1)c_n - \frac{1}{2} n c_{n+1}\,, \tag{1}$$

$$(c_0 + c_2)^n = \frac{n!}{6} \sum_{k=0}^{n} \frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6} n(2n+1)c_{n+1} - \frac{1}{3} n c_{n+2}\,, \tag{2}$$

$$(c_1 + c_1)^n = -\frac{n!}{6} \sum_{k=0}^{n} \frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6} n(n+5)c_{n+1} - \frac{1}{6}(n+3)c_{n+2}\,. \tag{3}$$

Some similar expressions were obtained for Cauchy numbers of the second kind ([8]).

The analogous concept for the Bernoulli numbers B_n, defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi)\,,$$

has been extensively studied by many authors, including Agoh and Dilcher ([1, 2, 5] and references there). Define

$$(B_l + B_m)^n := \sum_{j=0}^{n} \binom{n}{j} B_{l+j} B_{m+n-j}\,.$$

Then Euler's famous formula can be written as

$$(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n \quad (n \ge 1)\,. \tag{4}$$

The corresponding formula for the Cauchy numbers c_n was written as

$$(c_0 + c_0)^n = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \ge 0) \tag{5}$$

(see [12]).

In [2] the higher order recurrences for Bernoulli numbers,

$$(B_{l_1} + \cdots + B_{l_m})^n := \sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m \ge 0}} \frac{n!}{k_1! \cdots k_m!} B_{k_1+l_1} \cdots B_{k_m+l_m}\,,$$

were discussed. However, explicit formulae for the third and the fourth order are not obtained, but some special cases can be derived. For example,

$$(B_0 + B_0 + B_0)^n = \frac{(n-1)(n-2)}{2} B_n + \frac{3n(n-2)}{2} B_{n-1} + n(n-1)B_{n-2}\,,$$

$$(B_0 + B_0 + B_1)^n = \frac{n(n-1)}{6} B_{n+1} + \frac{(n-1)(n+1)}{2} B_n + \frac{n(n+1)}{3} B_{n-1}\,,$$

$$(B_0 + B_1 + B_1)^n = \frac{n(n+3)}{24} B_{n+2} + \frac{n(n-8)}{12} B_{n+1} - \frac{n^2 - 19n - 6}{24} B_n - \frac{n(n-2)}{12} B_{n-1}\,,$$

$$(B_0 + B_0 + B_2)^n = \frac{n(n-1)}{12} B_{n+2} + \frac{n(n-1)}{3} B_{n+1} + \frac{(5n-2)(n-1)}{12} B_n + \frac{n(n-2)}{6} B_{n-1}\,.$$

In this paper, we consider formulae of the type

$$\sum_{\substack{l_1+\cdots+l_m=\mu\\ l_1,\dots,l_m\ge 0}}\frac{\mu!}{l_1!\cdots l_m!}(c_{l_1}+\cdots+c_{l_m})^n=a_0c_{n+\mu}+\cdots+a_{m-1}c_{n+\mu-m+1}\,,$$

where the a_i are suitable rational numbers, the c_i are Cauchy numbers and

$$(c_{l_1}+\cdots+c_{l_m})^n:=\sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 0}}\frac{n!}{k_1!\cdots k_m!}c_{k_1+l_1}\cdots c_{k_m+l_m}\,. \tag{6}$$

In particular, we deal with the cases for $m=3$ and $m=4$. For example, we have

$$\begin{aligned}&(c_0+c_0+c_0)^n\\&=\frac{(n-1)(n-2)}{2}c_n+\frac{n(n-2)(2n-5)}{2}c_{n-1}+\frac{n(n-1)(n-3)^2}{2}c_{n-2}\,,\\&(c_0+c_0+c_1)^n\\&=\frac{n(n-1)}{6}c_{n+1}+\frac{(n+1)(n-1)(2n-3)}{6}c_n+\frac{n(n+1)(n-2)^2}{6}c_{n-1}\end{aligned}$$

and

$$\begin{aligned}&(c_0+c_0+c_0+c_0)^n\\&=-\frac{(n-1)(n-2)(n-3)}{6}c_n-\frac{n(n-2)(n-3)^2}{2}c_{n-1}\\&\quad-\frac{n(n-1)(n-3)(3n^2-21n+37)}{6}c_{n-2}-\frac{n(n-1)(n-2)(n-4)^3}{6}c_{n-3}\,,\\&(c_0+c_0+c_0+c_1)^n\\&=-\frac{n(n-1)(n-2)}{24}c_{n+1}-\frac{(n+1)(n-1)(n-2)^2}{8}c_n\\&\quad-\frac{(n+1)n(n-2)(3n^2-15n+19)}{24}c_{n-1}-\frac{(n+1)n(n-1)(n-3)^3}{24}c_{n-2}\,.\end{aligned}$$

2. Prelimilaries

Let $b_0,b_1,b_2,\dots$ be any sequence of complex numbers with $b_0\ne 0$. Consider the polynomial sequence $b_0(t),b_1(t),b_2(t),\dots$, defined by

$$b_n(t)=\sum_{i=0}^n\binom{n}{i}b_it^{n-i}=n![x^n]e^{tx}\left(\sum_{i=0}^\infty b_i\frac{x^i}{i!}\right)\,.$$

We have $\deg b_n(t)=n$ (because of $b_0\ne 0$) and $b_n=b_n(0)$ for all n. Any polynomial $q(t)$ of degree n expands in a unique way as

$$q(t)=a_0b_n(t)+a_1b_{n-1}(t)+\cdots+a_nb_0(t)\,,$$

where a_i is a suitable complex number. Now, choose

$$q(t)=\sum_{\substack{l_1+\cdots+l_m=\mu\\ l_1,\dots,l_m\ge 0}}\frac{\mu!}{l_1!\cdots l_m!}\Big(b_{l_1}(t)+\cdots+b_{l_m}(t)\Big)^n\,,$$

and set $t = 0$ to obtain

$$\sum_{\substack{l_1+\cdots+l_m=\mu\\ l_1,\dots,l_m\ge 0}} \frac{\mu!}{l_1!\cdots l_m!}(b_{l_1}+\cdots+b_{l_m})^n = a_0 b_{\mu+n} + a_1 b_{\mu+n-1} + \cdots + a_{\mu+n} b_0 \,.$$

The umbral notation used here comes from heuristic techniques largely used at the end of the nineteenth century within invariant theory (see for instance [6]). Many mathematicians have attempted to give rigorous foundation to these techniques (see for instance [3]). Among them, in the 1970s, Gian-Carlo Rota and his collaborators (see for instantce [11]) founded the modern umbral calculus by means of linear operators acting on a ring of polynomials (that's the umbral calculus of [10]). To be precise, following Roman's notation, in place of the symbolic representation of Cauchy numbers (written $c^n = c_n$) one defines a linear functional $C : \mathbf{Q[t]} \to \mathbf{Q}$ satisfying

$$\langle C, t^n \rangle = c_n \quad \text{for all } n\,.$$

Now, we can linearly extend the domain of C from $\mathbf{Q[t]}$ to $\mathbf{Q[t_1, t_2, \dots, t_m]}$ by assuming

$$\left\langle C, t_1^{n_1}\cdots t_m^{n_m}\right\rangle = \left\langle C, t_1^{n_1}\right\rangle \cdots \langle C, t_m^{n_m}\rangle \,.$$

Finally, one obtains

$$\left\langle C, t_1^{n_1}\cdots t_m^{n_m}(t_1+\cdots+t_m)^n\right\rangle = \sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 0}} \frac{n!}{k_1!\cdots k_m!} c_{k_1+l_1}\cdots c_{k_m+l_m} \,.$$

By using the compact notation introduced by Rota and Taylor ([?]), we write $p \simeq q$ to mean $\langle C, p\rangle = q$, and obtain

$$t_1^{n_1}\cdots t_m^{n_m}(t_1+\cdots+t_m)^n \simeq \sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 0}} \frac{n!}{k_1!\cdots k_m!} c_{k_1+l_1}\cdots c_{k_m+l_m} \,,$$

which is (6).

3. Basic results

$c(x) = x/\ln(1+x)$ satisfies the identity

$$c(x)^2 = (1+x)c(x) - (1+x)xc'(x)\,. \tag{7}$$

Since for $i, \nu \ge 0$ we have

$$x^i c^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} c_{n+\nu-i} \frac{x^n}{n!}\,, \tag{8}$$

the identity (7) immediately leads to the formula

$$\sum_{k=0}^{n} \binom{n}{k} c_k c_{n-k} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \ge 0)\,, \tag{9}$$

which is in fact identical with (5). Differentiating both sides of (7) and dividing them by 2, we obtain

$$c(x)c'(x) = -\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x)\,. \tag{10}$$

Proposition 1.

$$c(x)^3 = \frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x)\,. \tag{11}$$

Proof. By (7) and (10),

$$\begin{aligned} c(x)^3 &= (1+x)\Big((1+x)c(x) - (1+x)xc'(x)\Big) \\ &\quad - (1+x)x\left(-\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x)\right) \\ &= \frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x)\,. \end{aligned}$$

□

Theorem 1. *For $n \ge 2$ we have*

$$(c_0+c_0+c_0)^n = \frac{(n-1)(n-2)}{2}c_n + \frac{n(n-2)(2n-5)}{2}c_{n-1} + \frac{n(n-1)(n-3)^2}{2}c_{n-2}\,.$$

Remark. This result is analogous to

$$(B_0+B_0+B_0)^n = \frac{(n-1)(n-2)}{2}B_n + \frac{3n(n-2)}{2}B_{n-1} + n(n-1)B_{n-2}\,,$$

which was already mentioned above ([2, Corollary 3]).
Proof of Theorem 1. By using (8) for the identity in Proposition 1

$$\begin{aligned} &\frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x) \\ &= \sum_{n=0}^{\infty}\left(c_n + \frac{3}{2}nc_{n-1} + \frac{1}{2}n(n-1)c_{n-2}\right)\frac{x^n}{n!} \\ &\quad - \sum_{n=0}^{\infty}\left(nc_n + \frac{3}{2}n(n-1)c_{n-1} + \frac{1}{2}n(n-1)(n-2)c_{n-2}\right)\frac{x^n}{n!} \\ &\quad + \sum_{n=0}^{\infty}\left(\frac{1}{2}n(n-1)c_n + n(n-1)(n-2)c_{n-1} + \frac{1}{2}n(n-1)(n-2)(n-3)c_{n-2}\right)\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{(n-1)(n-2)}{2}c_n + \frac{n(n-2)(2n-5)}{2}c_{n-1} + \frac{n(n-1)(n-3)^2}{2}c_{n-2}\right)\frac{x^n}{n!}\,. \end{aligned}$$

□

4. Fundamental results

By differentiating both sides of (7) μ times with respect to x, we have

$$\begin{aligned} &\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}c^{(\kappa)}(x)c^{(\mu-\kappa)}(x) \\ &= -\mu(\mu-2)c^{(\mu-1)}(x) - \Big((2\mu-1)x + (\mu-1)\Big)c^{(\mu)}(x) - x(x+1)c^{(\mu+1)}(x)\,. \end{aligned} \tag{12}$$

Therefore, for $n, \mu \ge 0$, we obtain

$$\begin{aligned}&\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}(c_\kappa + c_{\mu-\kappa})^n\\&= -\Big(\mu(\mu-2) + (2\mu-1)n + n(n-1)\Big)c_{n+\mu-1} - \Big((\mu-1)+n\Big)c_{n+\mu}\\&= -m(m-2)c_{m-1} - (m-1)c_m\,,\end{aligned} \tag{13}$$

where $m = n + \mu$. Hence, if $\mu = 0$ in (13), then we have (5). If $\mu = 1$ in (13), then we have

$$2(c_0 + c_1)^n = -(n+1)(n-1)c_n - nc_{n+1}\,,$$

which matches (1). If $\mu = 2$ in (13), then we have

$$2(c_0 + c_2)^n + 2(c_1 + c_1)^n = -(n+2)nc_{n+1} - (n+1)c_{n+2}\,,$$

which is also obtained from (2) and (3). This idea can be extended to the higher-order convolution identities for Cauchy numbers.

The fundamental result of this paper is given by the following.

Theorem 2. *For $\mu, n \ge 0$, we have*

$$\begin{aligned}&\sum_{\substack{\kappa_1+\kappa_2+\kappa_3=\mu\\ \kappa_1,\kappa_2,\kappa_3\ge 0}}\frac{\mu!}{\kappa_1!\kappa_2!\kappa_3!}(c_{\kappa_1}+c_{\kappa_2}+c_{\kappa_3})^n\\&= \frac{(m-1)(m-2)}{2}c_m + \frac{m(m-2)(2m-5)}{2}c_{m-1} + \frac{m(m-1)(m-3)^2}{2}c_{m-2}\,,\end{aligned}$$

where $m = n + \mu$.

Remark. If we put $\mu = 0$, we have the identity in Theorem 1. If we put $\mu = 1$, we have

$$\begin{aligned}&(c_0 + c_0 + c_1)^n\\&= \frac{n(n-1)}{6}c_{n+1} + \frac{(n+1)(n-1)(2n-3)}{6}c_n + \frac{n(n+1)(n-2)^2}{6}c_{n-1}\,.\end{aligned}$$

If we put $\mu = 2$, we have

$$\begin{aligned}&(c_0 + c_0 + c_2)^n + 2(c_0 + c_1 + c_1)^n\\&= \frac{n(n+1)}{6}c_{n+2} + \frac{n(n+2)(2n-1)}{6}c_{n+1} + \frac{(n+1)(n+2)(n-1)^2}{6}c_n\,.\end{aligned}$$

If we put $\mu = 3$, we have

$$\begin{aligned}&(c_0 + c_0 + c_3)^n + 6(c_0 + c_1 + c_2)^n + 2(c_1 + c_1 + c_1)^n\\&= \frac{(n+1)(n+2)}{6}c_{n+3} + \frac{(n+1)(n+3)(2n+1)}{6}c_{n+2} + \frac{n^2(n+2)(n+3)}{6}c_{n+1}\,.\end{aligned}$$

The proof of Theorem 2 is based upon a relation about the function $c(x)$.

Proposition 2. *For $\mu \geq 0$, we have*

$$\begin{aligned}
&\sum_{\substack{\kappa_1+\kappa_2+\kappa_3=\mu\\ \kappa_1,\kappa_2,\kappa_3\geq 0}} \frac{\mu!}{\kappa_1!\kappa_2!\kappa_3!} c^{(\kappa_1)}(x)c^{(\kappa_2)}(x)c^{(\kappa_3)}(x)\\
&= \frac{1}{2}x^2(x+1)^2 c^{(\mu+2)}(x)\\
&\quad + \frac{1}{2}x(x+1)\Big((4\mu-1)x+(2\mu-2)\Big)c^{(\mu+1)}(x)\\
&\quad + \frac{1}{2}\Big((6\mu^2-9\mu+1)x^2+3(2\mu^2-4\mu+1)x+(\mu-1)(\mu-2)\Big)c^{(\mu)}(x)\\
&\quad + \frac{\mu}{2}\Big((4\mu^2-15\mu+13)x+(2\mu-5)(\mu-2)\Big)c^{(\mu-1)}(x)\\
&\quad + \frac{1}{2}\mu(\mu-1)(\mu-3)^2 c^{(\mu-2)}(x)\,.
\end{aligned}$$

Proof. By differentiating both sides of (11) μ times with respect to x, we have the desired result. The left-hand side is due to the General Leibniz's rule. The right-hand side can be proved by induction. □

Proof of Theorem 2. By using (8) for the identity in Proposition 2, we have

$$\begin{aligned}
&\frac{1}{2}x^2(x+1)^2 c^{(\mu+2)}(x)\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2}n(n-1)c_{n+\mu} + n(n-1)(n-2)c_{n+\mu-1} + \frac{1}{2}n(n-1)(n-2)(n-3)c_{n+\mu-2}\right)\frac{x^n}{n!}\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}x(x+1)\Big((4\mu-1)x+(2\mu-2)\Big)c^{(\mu+1)}(x)\\
&= \sum_{n=0}^{\infty}\left((\mu-1)nc_{n+\mu} + \frac{6\mu-3}{2}n(n-1)c_{n+\mu-1} + \frac{4\mu-1}{2}n(n-1)(n-2)c_{n+\mu-2}\right)\frac{x^n}{n!}\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\Big((6\mu^2-9\mu+1)x^2+3(2\mu^2-4\mu+1)x+(\mu-1)(\mu-2)\Big)c^{(\mu)}(x)\\
&= \sum_{n=0}^{\infty}\left(\frac{(\mu-1)(\mu-2)}{2}c_{n+\mu} + \frac{3(2\mu^2-4\mu+1)}{2}nc_{n+\mu-1} + \frac{6\mu^2-9\mu+1}{2}n(n-1)c_{n+\mu-2}\right)\frac{x^n}{n!}\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{\mu}{2}\Big((4\mu^2-15\mu+13)x+(2\mu-5)(\mu-2)\Big)c^{(\mu-1)}(x)\\
&= \sum_{n=0}^{\infty}\left(\frac{\mu(2\mu-5)(\mu-2)}{2}c_{n+\mu-1} + \frac{\mu(4\mu^2-15\mu+13)}{2}nc_{n+\mu-2}\right)\frac{x^n}{n!}
\end{aligned}$$

and

$$\frac{1}{2}\mu(\mu-1)(\mu-3)^2 c^{(\mu-2)}(x) = \sum_{n=0}^{\infty}\frac{\mu(\mu-1)(\mu-3)^2}{2}c_{n+\mu-2}\frac{x^n}{n!}\,.$$

Combining all the relations together, we obtain the desired result. □

5. The fourth power

In a similar manner, we have the following for the fourth power.

Theorem 3. *For $\mu, n \ge 0$, we have*

$$\sum_{\substack{\kappa_1+\kappa_2+\kappa_3+\kappa_4=\mu\\ \kappa_1,\kappa_2,\kappa_3,\kappa_4\ge 0}} \frac{\mu!}{\kappa_1!\kappa_2!\kappa_3!\kappa_4!}(c_{\kappa_1}+c_{\kappa_2}+c_{\kappa_3}+c_{\kappa_4})^n$$
$$= -\frac{(m-1)(m-2)(m-3)}{6}c_m - \frac{m(m-2)(m-3)^2}{2}c_{m-1}$$
$$- \frac{m(m-1)(m-3)\big(3m^2-21m+37\big)}{6}c_{m-2} - \frac{m(m-1)(m-2)(m-4)^3}{6}c_{m-3}\,,$$

where $m = n + \mu$.

Remark. If we put $\mu = 0$ in Theorem 3, we have

$$(c_0+c_0+c_0+c_0)^n$$
$$= -\frac{(n-1)(n-2)(n-3)}{6}c_n - \frac{n(n-2)(n-3)^2}{2}c_{n-1}$$
$$- \frac{n(n-1)(n-3)(3n^2-21n+37)}{6}c_{n-2} - \frac{n(n-1)(n-2)(n-4)^3}{6}c_{n-3}\,.$$

If we put $\mu = 1$ in Theorem 3, we have

$$(c_0+c_0+c_0+c_1)^n$$
$$= -\frac{n(n-1)(n-2)}{24}c_{n+1} - \frac{(n+1)(n-1)(n-2)^2}{8}c_n$$
$$- \frac{(n+1)n(n-2)(3n^2-15n+19)}{24}c_{n-1} - \frac{(n+1)n(n-1)(n-3)^3}{24}c_{n-2}\,.$$

If we put $\mu = 2$ in Theorem 3, we have

$$(c_0+c_0+c_0+c_2)^n + 3(c_0+c_0+c_1+c_1)^n$$
$$= -\frac{(n+1)n(n-1)}{24}c_{n+2} - \frac{(n+2)n(n-1)^2}{8}c_{n+1}$$
$$- \frac{(n+2)(n+1)(n-1)(3n^2-9n+7)}{24}c_n - \frac{(n+2)(n+1)n(n-2)^3}{24}c_{n-1}\,.$$

Conjecture 1. *$(c_0+c_0+\cdots+c_0)^n$ or $(\nu c)^n$ ν-th power sum may be computed by the same method.*

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