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Fully degenerate poly-Bernoulli polynomials with a q parameter

Dae San Kim[a], Tae Kyun Kim[b], Toufik Mansour[c], Jong-Jin Seo[d]

[a] *Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea*
[b] *Department of Mathematics, Tianjin Polytechnic University, Tianjin, China and Department of Mathematics, Kwangwoon University, Seoul, Republic of Korea*
[c] *Department of Mathematics, University of Haifa, 3498838 Haifa, Israel*
[d] *Department of Applied Mathematics, Pukyong National University, Busan, Republic of Korea*

Abstract. In this paper, we consider the fully degenerate poly-Bernoulli polynomials with a q parameter. We present several properties, explicit formulas and recurrence relations for these polynomials by using the technique of umbral calculus.

1. Introduction

The goals of this paper are to use umbral calculus to obtain several new and interesting identities of fully degenerate poly-Bernoulli polynomials with a q parameter. The use of umbral calculus technique has been very attractive in numerous problems of mathematics and applied mathematics (for example, see [3, 6, 16, 19, 20]).

Throughout this paper, we assume that $\lambda, q \in \mathbb{C}$ with $\lambda, q \neq 0$ and $k \in \mathbb{Z}$. The *poly-Bernoulli polynomials* with a q parameter $B_{n,q}^{(k)}(x)$ are defined by (see [5])

$$\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}e^{xt} = \sum_{n\geq 0} B_{n,q}^{(k)}(x)\frac{t^n}{n!}. \tag{1}$$

In fact, they were defined by $B_{n,q}^{(k)}(-x)$ instead of $B_{n,q}^{(k)}(x)$ in [5]. Here $Li_k(x) = \sum_{n\geq 1}\frac{x^n}{n^k}$ is the kth *polylogarithm function* and $Li_1(x) = -\log(1-x)$.

In recent years, various kinds of degenerate versions of the familiar polynomials like Bernoulli polynomials, Euler polynomials and their variants regained some interest of many researchers. For instance, in [13] a degenerate version of poly-Cauchy polynomials with a q parameter were investigated by using umbral calculus (see [15]).

Here in the same vein the *fully degenerate poly-Bernoulli polynomials* with a q parameter $\beta_{n,q}^{(k)}(\lambda, x)$ are introduced as a degenerate version of the poly-Bernoulli polynomials with a q parameter $B_{n,q}^{(k)}(x)$. They are

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Email addresses: dskim@sogang.ac.kr (Dae San Kim), kimtk2015@gmail.com (Tae Kyun Kim), tmansour@univ.haifa.ac.il (Toufik Mansour), seo2011@pknu.ac.kr (Jong-Jin Seo)

defined by the generating function

$$\frac{qLi_k\left(\frac{1-(1+\lambda t)^{-\frac{q}{\lambda}}}{q}\right)}{1-(1+\lambda t)^{-\frac{q}{\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n\geq 0}\beta_{n,q}^{(k)}(\lambda,x)\frac{t^n}{n!}. \tag{2}$$

For $q = 1$, $\beta_n^{(k)}(\lambda,x) = \beta_{n,1}^{(k)}(\lambda,x)$ are called the fully degenerate poly-Bernoulli polynomials which are studied in [12]. On the other hand, we see that $\lim_{\lambda\to 0}\beta_{n,q}^{(k)}(\lambda,x) = B_{n,q}^{(k)}(x)$. For $x = 0$, $\beta_{n,q}^{(k)}(\lambda,0)$ are called the *fully degenerate poly-Bernoulli numbers* with a q parameter. Hence, our polynomials $\beta_{n,q}^{(k)}(\lambda,x)$ give a unified language to several families of polynomials, and several well known results (see [12–14]).

Now, from (2) it is immediate to see that the fully degenerate poly-Bernoulli polynomials with a q parameter are given by Sheffer sequence (for Sheffer sequence and umbral calculus, we refer the reader to [17, 18]) as

$$\beta_{n,q}^{(k)}(\lambda,x) \sim \left(\frac{1-e^{-qt}}{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}, \frac{e^{\lambda t}-1}{\lambda}\right). \tag{3}$$

Recently, several authors have studied special polynomials which are related to degenerate and umbral calculus(see [1-19]). In next section, we derive some properties of the fully degenerate poly-Bernoulli polynomials with a q parameter (for the case $q = 1$, see [12] and references therein).

2. Explicit Expressions

In this section, we present several explicit formulas for the fully degenerate poly-Bernoulli polynomials with q parameter. To do so, we recall that the Stirling numbers $S_1(n,m)$ of the first kind are defined as

$$(x|\lambda)_n = \lambda^n(x/\lambda)_n = \sum_{\ell=0}^{n} S_1(n,\ell)\lambda^{n-\ell}x^\ell \sim (1,(e^{\lambda t}-1)/\lambda), \tag{4}$$

where $(x|\lambda)_n$ is defined by $(x|\lambda)_n = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, for $n \geq 1$, and $(x|\lambda)_0 = 1$. Note that the exponential generating function for the Stirling numbers of the first kind is given by

$$\frac{1}{j!}(\log(1+t))^j = \sum_{\ell\geq j} S_1(\ell,j)\frac{t^\ell}{\ell!}. \tag{5}$$

Also, we recall that the Stirling numbers $S_2(n,m)$ of the second kind are defined by

$$\frac{(e^t-1)^k}{k!} = \sum_{\ell\geq k} S_2(\ell,k)\frac{t^\ell}{\ell!}. \tag{6}$$

Theorem 2.1. *For all $n \geq 0$,*

$$\beta_{n,q}^{(k)}(\lambda,x) = -\sum_{r=0}^{n}\left(\sum_{\ell=r}^{n}\sum_{m=0}^{\ell-r}\frac{m!\binom{\ell}{r}}{(m+1)^k}S_1(n,\ell)S_2(\ell-r,m)\lambda^{n-\ell}(-q)^{\ell-r-m+1}\right)x^r.$$

Proof. By (3), we have

$$\frac{1-e^{-qt}}{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}\beta_{n,q}^{(k)}(\lambda,x) \sim \left(1, \frac{e^{\lambda t}-1}{\lambda}\right). \tag{7}$$

Thus, by (4), we obtain

$$\beta_{n,q}^{(k)}(\lambda,x) = \frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}(x|\lambda)_n = \sum_{\ell=0}^{n} S_1(n,\ell)\lambda^{n-\ell}\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}x^\ell$$

$$= \sum_{\ell=0}^{n}\sum_{m=0}^{\ell} S_1(n,\ell)\lambda^{n-\ell}\frac{(-1)^m}{(m+1)^k q^{m-1}}\left(e^{-qt}-1\right)^m x^\ell. \tag{8}$$

So, by using (6) and reordering the obtained expression, we have

$$\beta_{n,q}^{(k)}(\lambda,x) = \sum_{\ell=0}^{n}\sum_{m=0}^{\ell}\sum_{r=m}^{\ell} S_1(n,\ell)S_2(r,m)\lambda^{n-\ell}\frac{m!(-1)^{m+r}}{r!(m+1)^k q^{m-r-1}}t^r x^\ell$$

$$= -\sum_{r=0}^{n}\left(\sum_{\ell=r}^{n}\sum_{m=0}^{\ell-r}\frac{m!\binom{\ell}{r}}{(m+1)^k}S_1(n,\ell)S_2(\ell-r,m)\lambda^{n-\ell}(-q)^{\ell-r-m+1}\right)x^r, \tag{9}$$

as claimed. □

Theorem 2.2. *For all $n \geq 0$,*

$$\beta_{n,q}^{(k)}(\lambda,x) = \sum_{r=0}^{n}\left(\sum_{\ell=r}^{n}\sum_{m=0}^{\ell-r}\binom{\ell}{r}\lambda^{n-r-m}S_1(n,\ell)S_2(\ell-r,m)\beta_{m,q}^{(k)}(\lambda,0)\right)x^r.$$

Proof. By (8), we have

$$\beta_{n,q}^{(k)}(\lambda,x) = \sum_{\ell=0}^{n} S_1(n,\ell)\lambda^{n-\ell}\left.\frac{qLi_k\left(\frac{1-(1+\lambda s)^{-\frac{q}{\lambda}}}{q}\right)}{1-(1+\lambda s)^{-\frac{q}{\lambda}}}\right|_{s=\frac{e^{\lambda t}-1}{\lambda}} x^\ell = \sum_{\ell=0}^{n}\sum_{m=0}^{\ell} S_1(n,\ell)\lambda^{n-\ell}\beta_{m,q}^{(k)}(\lambda,0)\frac{(e^{\lambda t}-1)^m}{m!\lambda^m}x^\ell.$$

Thus, by (6), we obtain

$$\beta_{n,q}^{(k)}(\lambda,x) = \sum_{\ell=0}^{n}\sum_{m=0}^{\ell}\sum_{r=m}^{\ell} S_1(n,\ell)S_2(r,m)\lambda^{n-\ell}\beta_{m,q}^{(k)}(\lambda,0)\lambda^{r-m}\binom{\ell}{r}x^{\ell-r},$$

which, by reordering the sums, completes the proof. □

Theorem 2.3. *For all $n \geq 1$,*

$$\beta_{n,q}^{(k)}(\lambda,x) = -\sum_{r=0}^{n}\left(\sum_{\ell=0}^{n-r}\sum_{m=0}^{n-\ell-r}\binom{n-1}{\ell}\binom{n-\ell}{r}\frac{m!\lambda^\ell(-q)^{n-\ell-r-m+1}}{(m+1)^k}B_\ell^{(n)}S_2(n-\ell-r,m)\right)x^r,$$

where $B_\ell^{(n)}$ is the Bernoulli number of order n given by $(\frac{t}{e^t-1})^n = \sum_{\ell\geq 0} B_\ell^{(n)}\frac{t^\ell}{\ell!}$.

Proof. By applying the transfer formula to $x^n \sim (1,t)$ and (7), for $n \geq 1$ we have

$$\frac{1-e^{-qt}}{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}\beta_{n,q}^{(k)}(\lambda,x) = x\frac{\lambda^n t^n}{(e^{\lambda t}-1)^n}x^{-1}x^n = x\frac{\lambda^n t^n}{(e^{\lambda t}-1)^n}x^{n-1},$$

which implies

$$\frac{1-e^{-qt}}{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}\beta_{n,q}^{(k)}(\lambda,x) = x\sum_{\ell=0}^{n-1}B_\ell^{(n)}\frac{\lambda^\ell}{\ell!}t^\ell x^{n-1} = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^\ell B_\ell^{(n)}x^{n-\ell}.$$

Therefore,

$$\beta_{n,q}^{(k)}(\lambda,x)=\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}\lambda^{\ell}B_{\ell}^{(n)}\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}x^{n-\ell}, \tag{10}$$

which, by using (9), leads to

$$\begin{aligned}\beta_{n,q}^{(k)}(\lambda,x)&=\sum_{\ell=0}^{n-1}\sum_{m=0}^{n-\ell}\sum_{r=m}^{n-\ell}\binom{n-1}{\ell}\binom{n-\ell}{r}\frac{(-1)^m m!\lambda^{\ell}}{(m+1)^k q^{m-1}}B_{\ell}^{(n)}S_2(r,m)(-q)^r x^{n-\ell-r}\\&=\sum_{\ell=0}^{n-1}\sum_{m=0}^{n-\ell}\sum_{r=0}^{n-\ell-m}\binom{n-1}{\ell}\binom{n-\ell}{r}\frac{(-1)^m m!\lambda^{\ell}}{(m+1)^k q^{m-1}}B_{\ell}^{(n)}S_2(n-\ell-r,m)(-q)^{n-\ell-r}x^r\\&=-\sum_{r=0}^{n}\left(\sum_{\ell=0}^{n-r}\sum_{m=0}^{n-\ell-r}\binom{n-1}{\ell}\binom{n-\ell}{r}\frac{m!\lambda^{\ell}(-q)^{n-\ell-r-m+1}}{(m+1)^k}B_{\ell}^{(n)}S_2(n-\ell-r,m)\right)x^r,\end{aligned}$$

as required. □

Theorem 2.4. *For all $n\geq 1$,*

$$\beta_{n,q}^{(k)}(\lambda,x)=\sum_{r=0}^{n}\left(\sum_{\ell=0}^{n-r}\sum_{m=0}^{n-\ell-r}\binom{n-1}{\ell}\binom{n-\ell}{r}\lambda^{n-r-m}B_{\ell}^{(n)}\beta_{m,q}^{(k)}(\lambda,0)S_2(n-\ell-r,m)\right)x^r,$$

where $B_{\ell}^{(n)}$ is the Bernoulli number of order n.

Proof. We proceed by using the proof of Theorem 2.3 as follows. By (10), we have

$$\beta_{n,q}^{(k)}(\lambda,x)=\sum_{\ell=0}^{n}\sum_{m=0}^{n-\ell}\sum_{r=0}^{n-\ell-m}\binom{n-1}{\ell}\lambda^{\ell-m}B_{\ell}^{(n)}\beta_{m,q}^{(k)}(\lambda,0)S_2(n-\ell-r,m)\lambda^{n-\ell-r}\binom{n-\ell}{r}x^r,$$

which, by changing the order of the summations, completes the proof. □

To proceed further, we observe the following. Note that $Li_k(x)=\int_0^x\frac{Li_{k-1}(x)}{x}dx$ with $Li_1(x)=-\log(1-x)$. Thus, by induction on $k\geq 2$,

$$Li_k(x)=\int_0^x\int_0^{x_1}\cdots\int_0^{x_{k-2}}\frac{Li_1(x_{k-1})}{x_1x_2\cdots x_{k-1}}dx_{k-1}\cdots dx_2dx_1.$$

By setting $x=\frac{1-e^{-qt}}{q}$, we obtain

$$\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}=\frac{q}{1-e^{-qt}}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-2}}\frac{q^{k-1}e^{-q(t_1+\cdots+t_{k-1})}Li_1\left(\frac{1-e^{-qt_{k-1}}}{q}\right)}{(1-e^{-qt_1})\cdots(1-e^{-qt_{k-1}})}dt_{k-1}\cdots dt_2dt_1.$$

By induction on k together with the fact that

$$\frac{qLi_1\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}=\sum_{j\geq0}B_{j,q}^{(1)}\frac{t^j}{j!}=\sum_{j\geq0}B_{j,q}\frac{t^j}{j!},$$

we obtain

$$\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}=\sum_{j_1,\cdots,j_k\geq0}t^{j_1+\cdots+j_k}\frac{B_{j_1,q}(-q)}{j_1!(j_1+1)}\frac{B_{j_k}(1)q^{j_k}}{j_k!}\prod_{i=2}^{k-1}\frac{B_{j_i}q^{j_i}}{j_i!(j_1+\cdots+j_i+1)}. \tag{11}$$

where $B_{j_1,q}(-q)=B_{j_1,q}^{(1)}(-q)$ (see (1)) and $B_n(x)$ are the ordinary Bernoulli polynomials.

Theorem 2.5. *Let $k \geq 2$. Then*

$$\beta_{n,q}^{(k)}(\lambda, x) = \sum_{j_1+\cdots+j_k \leq n} \frac{B_{j_1,q}(-q)}{j_1!(j_1+1)} \frac{B_{j_k}(1)q^{j_k}}{j_k!} \prod_{i=2}^{k-1} \frac{B_{j_i}q^{j_i}}{j_i!(j_1+\cdots+j_i+1)} \alpha_{j_1+\cdots+j_k},$$

where

$$\alpha_{j_1+\cdots+j_k} = \frac{(j_1+\cdots+j_k)!}{\lambda^{j_1+\cdots+j_k}} \sum_{\ell=j_1+\cdots+j_k}^{n} \binom{n}{\ell} S_1(\ell, j_1+\cdots+j_k)\lambda^{\ell}(x|\lambda)_{n-\ell}.$$

Proof. By (3) (with help of umbral calculus, see [17, 18]), we obtain

$$\beta_{n,q}^{(k)}(\lambda, y) = \left\langle \frac{qLi_k\left(\frac{1-(1+\lambda t)^{-\frac{q}{\lambda}}}{q}\right)}{1-(1+\lambda t)^{-\frac{q}{\lambda}}}(1+\lambda t)^{y/\lambda} \mid x^n \right\rangle.$$

Thus, by (11), we have

$$\beta_{n,q}^{(k)}(\lambda, y) = \sum_{j_1+\cdots+j_k \leq n} \frac{B_{j_1,q}(-q)}{j_1!(j_1+1)} \frac{B_{j_k}(1)q^{j_k}}{j_k!} \prod_{i=2}^{k-1} \frac{B_{j_i}q^{j_i}}{j_i!(j_1+\cdots+j_i+1)} \alpha_{j_1+\cdots+j_k},$$

where $\alpha_{j_1+\cdots+j_k} = \left\langle \frac{\log^{j_1+\cdots+j_k}(1+\lambda t)(1+\lambda t)^{y/\lambda}}{\lambda^{j_1+\cdots+j_k}} \mid x^n \right\rangle$. By (5), we obtain that $\alpha_{j_1+\cdots+j_k}$ is given by

$$\begin{aligned}
&\frac{(j_1+\cdots+j_k)!}{\lambda^{j_1+\cdots+j_k}} \left\langle \sum_{\ell=j_1+\cdots+j_k}^{n} S_1(\ell, j_1+\cdots+j_k)\frac{\lambda^{\ell}t^{\ell}}{\ell!}(1+\lambda t)^{y/\lambda} \mid x^n \right\rangle \\
&= \frac{(j_1+\cdots+j_k)!}{\lambda^{j_1+\cdots+j_k}} \sum_{\ell=j_1+\cdots+j_k}^{n} \sum_{j\geq 0} \binom{j+\ell}{\ell} S_1(\ell, j_1+\cdots+j_k)\lambda^{\ell}(y|\lambda)_j \left\langle \frac{t^{j+\ell}}{(j+\ell)!} \mid x^n \right\rangle \\
&= \frac{(j_1+\cdots+j_k)!}{\lambda^{j_1+\cdots+j_k}} \sum_{\ell=j_1+\cdots+j_k}^{n} \binom{n}{\ell} S_1(\ell, j_1+\cdots+j_k)\lambda^{\ell}(y|\lambda)_{n-\ell},
\end{aligned}$$

which completes the proof. □

Note that the above theorem holds for $k \geq 2$. In the case $k = 1$, we can use similar technique to obtain $\beta_{n,q}^{(1)}(\lambda, x) = \sum_{j=0}^{n}\sum_{\ell=j}^{n}\binom{n}{\ell}\lambda^{\ell-j}\beta_{j,q}S_1(\ell, j)(x|\lambda)_{n-\ell}$, where we leave the proof to the interested reader.

3. Recurrences

Note that, by (3) and the fact that $(x|\lambda)_n \sim (1, \frac{e^{\lambda t}-1}{\lambda})$, we obtain the following Sheffer identities: $\beta_{n,q}^{(k)}(\lambda, x+y) = \sum_{j=0}^{n}\binom{n}{j}\beta_{j,q}^{(k)}(\lambda, x)(y|\lambda)_{n-j}$. Moreover, in the next results, we present several recurrences for the fully degenerate poly-Bernoulli polynomials with a q parameter.

Theorem 3.1. *For all $n \geq 1$, $\beta_{n,q}^{(k)}(\lambda, x+\lambda) = \beta_{n,q}^{(k)}(\lambda, x) + n\lambda\beta_{n-1,q}^{(k)}(\lambda, x)$.*

Proof. Using the fact that $f(t)S_n(x) = nS_{n-1}(x)$ for all $S_n(x) \sim (g(t), f(t))$ (see [17, 18]) in our case, see (3), we obtain $\frac{1}{\lambda}(e^{\lambda t}-1)\beta_{n,q}^{(k)}(\lambda, x) = n\beta_{n-1,q}^{(k)}(\lambda, x)$, which implies $\beta_{n,q}^{(k)}(\lambda, x+\lambda) - \beta_{n,q}^{(k)}(\lambda, x) = n\lambda\beta_{n-1,q}^{(k)}(\lambda, x)$, as claimed. □

Theorem 3.2. *For all $n \geq 0$,*

$$\beta^{(k)}_{n+1,q}(\lambda, x) = x\beta^{(k)}_{n,q}(\lambda, x-\lambda) - \sum_{m=0}^{n}\sum_{\ell=0}^{m+1} \frac{\lambda^{n-m}q^{\ell}}{m+1}\binom{m+1}{\ell} S_1(n,m)(B^{(k)}_{m+1-\ell,q} - B^{(k-1)}_{m+1-\ell,q})B_\ell((x-\lambda)/q).$$

Proof. We proceed the proof by using the fact that $S_{n+1}(x) = (x - \frac{g'(t)}{g(t)})\frac{1}{f'(t)}S_n(x)$, for all $S_n(x) \sim (g(t), f(t))$ (see [17, 18]). By the above fact and (3), we have that

$$\beta^{(k)}_{n+1,q}(\lambda, x) = x\beta^{(k)}_{n,q}(\lambda, x-\lambda) - e^{-\lambda t}\frac{g'(t)}{g(t)}\beta^{(k)}_{n,q}(\lambda, x) \tag{12}$$

with $g(t) = \frac{1-e^{-qt}}{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}$. Note that $\frac{d}{dx}(Li_k(x)) = \frac{Li_{k-1}(x)}{x}$. So,

$$\frac{g'(t)}{g(t)} = \frac{qe^{-qt}}{1-e^{-qt}}\left(1 - \frac{Li_{k-1}\left(\frac{1-e^{-qt}}{q}\right)}{Li_k\left(\frac{1-e^{-qt}}{q}\right)}\right).$$

Thus, by (4) and (7), we have

$$\begin{aligned} e^{-\lambda t}\frac{g'(t)}{g(t)}\beta^{(k)}_{n,q}(\lambda, x) &= e^{-\lambda t}\frac{q}{e^{qt}-1}\left\{\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}} - \frac{qLi_{k-1}\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}\right\}\frac{1-e^{-qt}}{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}\beta^{(k)}_{n,q}(\lambda, x) \\ &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}e^{-\lambda t}\frac{qt}{e^{qt}-1}\frac{1}{t}\left\{\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}} - \frac{qLi_{k-1}\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}\right\}x^m \\ &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}e^{-\lambda t}\frac{qt}{e^{qt}-1}\left\{\frac{qLi_k\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}} - \frac{qLi_{k-1}\left(\frac{1-e^{-qt}}{q}\right)}{1-e^{-qt}}\right\}\frac{x^{m+1}}{m+1}, \end{aligned}$$

where we note that the expression in the curly bracket has order at least one. So,

$$e^{-\lambda t}\frac{g'(t)}{g(t)}\beta^{(k)}_{n,q}(\lambda, x) = \sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}\lambda^{n-m}e^{-\lambda t}\frac{qt}{e^{qt}-1}(B^{(k)}_{m+1,q}(x) - B^{(k-1)}_{m+1,q}(x)).$$

Note that by (1) we observe that $B^{(k)}_{n,q}(x) = \sum_{\ell=0}^{n}\binom{n}{\ell}B^{(k)}_{n-\ell,q}x^\ell$. Thus,

$$\begin{aligned} e^{-\lambda t}\frac{g'(t)}{g(t)}\beta^{(k)}_{n,q}(\lambda, x) &= \sum_{m=0}^{n}\sum_{\ell=0}^{m+1}\frac{S_1(n,m)}{m+1}\lambda^{n-m}\binom{m+1}{\ell}(B^{(k)}_{m+1-\ell,q} - B^{(k-1)}_{m+1-\ell,q})e^{-\lambda t}\frac{qt}{e^{qt}-1}x^\ell \\ &= \sum_{m=0}^{n}\sum_{\ell=0}^{m+1}\frac{S_1(n,m)}{m+1}\lambda^{n-m}\binom{m+1}{\ell}(B^{(k)}_{m+1-\ell,q} - B^{(k-1)}_{m+1-\ell,q})q^\ell B_\ell(\frac{x-\lambda}{q}). \end{aligned}$$

By substituting this expression into (12), we complete the proof. □

In next result, we express $\frac{d}{dx}\beta^{(k)}_{n,q}(\lambda, x)$ in terms of $\beta^{(k)}_{n,q}(\lambda, x)$.

Theorem 3.3. *For all $n \geq 1$, $\frac{d}{dx}\beta^{(k)}_{n,q}(\lambda, x) = n!\sum_{\ell=0}^{n-1}\frac{(-\lambda)^{n-\ell-1}}{(n-\ell)\ell!}\beta^{(k)}_{\ell,q}(\lambda, x)$.*

Proof. In the case of (3), we obtain $\langle \bar{f}(t)|x^{n-\ell}\rangle = \sum_{j\geq 1}(-1)^{j-1}\langle\frac{t^j}{j}|x^{n-\ell}\rangle = (-\lambda)^{n-\ell-1}(n-\ell-1)!$. Thus, by using the fact that $\frac{d}{dx}S_n(x) = \sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle\bar{f}(t)|x^{n-\ell}\rangle S_\ell(x)$, for all $S_n(x) \sim (g(t), f(t))$ (see [17, 18]), we complete the proof. □

References

[1] Araci, S. and Acikgoz, M., A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials, *Adv. Stud. Contemp. Math.* **22(3)** (2012) 399-406.
[2] Bayad A. and Kim, T., Identities involving values of Bernstein, q-Bernoulli, and q-Euler polynomials, *Russ. J. Math. Phys.* **18(2)** (2011) 133–143.
[3] Di Bucchianico, A. and Loeb, D., A selected survey of umbral calculus, *Electron. J. Combin.* **2** (2000) #DS3.
[4] Carlitz, L., Degenerate Stirling, Bernoulli and Eulerian numbers, *Utilitas Math.* **15** (1979) 51–88.
[5] Cenkci, M. and Komatsu, T., Poly-Bernoulli numbers and polynomials with a q parameter, *J. Number Theory* **152** (2015), 38-54.
[6] Dattoli, G., Levi, D. and Winternitz, P., Heisenberg algebra, umbral calculus and orthogonal polynomials, *J. Math. Phys.* **49** (2008), no. 5, 053509.
[7] Dere, R. and Simsek, Y., Applications of umbral algebra to some special polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)* **22**(2012), no. 3, 433–438.
[8] Ding, D. and Yang, J., Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials, *Adv. Stud. Contemp. Math.* **20(1)** (2010) 7–21.
[9] Ernst, T., Examples of a q-umbral calculus, . *Adv. Stud. Contemp. Math. (Kyungshang)* **16** (2008), no. 1, 1–22.
[10] Kim, D. and Kim, T., A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials, *Russ. J. Math. Phys.* **22** (2015), no. 1, 26–33.
[11] Kim, D.S. and Kim, T., Higher-order Bernoulli and poly-Bernoulli mixed type polynomials, *Georgian Math. J.* **22** (2015), 265–272
[12] Kim, D.S. and Kim, T., Fully degenerate poly-Bernoulli numbers and polynomials, preprint.
[13] Kim, D.S., Kim, T., Dolgy, D.V. and Mansour, T., Degenerate poly-Cauchy polynomials with a q parameter, preprint.
[14] Kim, D.S., Kim, T., Kwon, H.I. and Mansour, T., Degenerate poly-Bernoulli polynomials with umbral calculus viepoint, *J. Inequal. Appl.* **2015** (2015).
[15] Komatsu, T., Poly-Cauchy numbers with a q parameter, *Ramanujan J.* **31** (2013), 353-371.
[16] Kwasniewski, A. K., On ψ-umbral extensions of Stirling numbers and Dobinski-like formulas, *Adv. Stud. Contemp. Math. (Kyungshang)* **12**(2006), no. 1, 73–100.
[17] Roman, S., More on the umbral calculus, with emphasis on the q-umbral calculus, *J. Math. Anal. Appl.* **107** (1985) 222–254.
[18] Roman, S., The umbral calculus, Dover Publ. Inc. New York, 2005.
[19] Qi, F. and Chapman, R.J., Two closed forms for the Bernoulli polynomials, *J. Number Theory* **159** (2016) 89–100.
[20] Wilson, B.G. and Rogers, F.G., Umbral calculus and the theory of multispecies nonideal gases, *Phys. A* **139** (1986) 359–386.