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Refinements of Jensen's Inequality for Convex Functions on the Co-Ordinates in a Rectangle from the Plane

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Abstract. In this paper our aim is to give refinements of Jensen's type inequalities for the convex function defined on the co-ordinates of the bidimensional interval in the plane.

1. Introduction

A function $\phi : [a, b] \to \mathbb{R}$ is said to be convex if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \tag{1}$$

holds for all $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$. A function ϕ is said to be strictly convex if the inequality in (1) is strict whenever $x \neq y$ and $0 < \lambda < 1$.

Let $\phi : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$. If $x_i \in [a, b]$ and $p_i > 0$ such that $P_n = \sum_{i=1}^{n} p_i$ then

$$\phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi(x_i), \tag{2}$$

is well known in the literature as Jensen's inequality.

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the arithmetic-mean geometric-mean inequality, the Hölder and Minkowski inequalities, the Ky Fan inequality etc. can be obtained as particular cases of it.

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In [7], the authors have investigated the following refinement of (2):

$$\begin{aligned}\phi\left(\sum_{i=1}^{n} p_i x_i\right) \quad &\le \quad \min_{\mathcal{I}}\left[P_I\phi\left(\frac{\sum\limits_{i\in I} p_i x_i}{P_I}\right)+\sum_{i\in I_n\setminus I} p_i\phi(x_i)\right]\\ &\le \frac{1}{2^n-n-2}\left[\sum_{I\subset I_n} P_I\phi\left(\frac{\sum\limits_{i\in I} p_i x_i}{P_I}\right)+(2^{n-1}-n)\sum_{i=1}^{n} p_i\phi(x_i)\right]\\ &\le \max_{\mathcal{I}}\left[P_I\phi\left(\frac{\sum\limits_{i\in I} p_i x_i}{P_I}\right)+\sum_{i\in I_n\setminus I} p_i\phi(x_i)\right]\le\sum_{i=1}^{n} p_i\phi(x_i),\end{aligned}$$

where $\phi : C \to \mathbb{R}$ is a convex function defined on a convex set C, $x_i \in C$ and

$$\mathcal{I} = \{I \subset I_n,\ I \neq I_n = \{1,...,n\}\ \text{s.t.}\ |I| \geq 2\}\ ,\ i \in \{1,...,n\},\quad n \geq 3$$

and $P_I = \sum\limits_{i\in I} p_i$ together with $\sum\limits_{i=1}^{n} p_i = 1$.

In 2010 Dragomir obtained another refinement of Jensen's inequality (see [15]):

$$\phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le D(\phi,\mathbf{p},\mathbf{x},I) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi(x_i), \tag{3}$$

where

$$D(\phi,\mathbf{p},\mathbf{x},I) = P_I\phi\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) + P_{\bar{I}}\phi\left(\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i\right)$$

and

$$\emptyset \neq I \subset I_n = \{1,...,n\},\quad \bar{I} = I_n \setminus I \neq \emptyset,\quad i \in \{1,...,n\}$$

together with $P_I = \sum\limits_{i\in I} p_i$, $P_{\bar{I}} = \sum\limits_{i\in \bar{I}} p_i$ and $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{p} = (p_1, p_2, ..., p_n)$. Also in [6], the authors have proved a generalized refinement of (2) given as under:

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{j=1}^{k-1}\lambda_{j+1}x_{i+j}\right) \le \frac{\sum\limits_{i=1}^{n}\phi(x_i)}{n}, \tag{4}$$

where $\phi : [a,b] \to \mathbb{R}$ is a convex function, $\mathbf{x} := (x_1, ..., x_n) \in [a,b]^n$ such that $x_{i+n} = x_i$ and $\lambda := (\lambda_1, ..., \lambda_n)$ is a positive $n-$tuple together with $\sum\limits_{i=1}^{k} \lambda_i = 1$ for some k, $2 \le k \le n$. More recently in 2015, the authors have given further generalizations of the results presented in [2, 3].

In [14], the concept of convex functions defined on the co-ordinates of the bidimensional interval of the plane of two variables was introduced:

Definition 1.1. Let us consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $\phi : [a,b] \times [c,d] \to \mathbb{R}$ is called convex on the co-ordinates if the partial mappings $\phi_y : [a,b] \to \mathbb{R}$ defined as $\phi_y(t) := \phi(t,y)$ and $\phi_x : [c,d] \to \mathbb{R}$ defined as $\phi_x(s) := \phi(x,s)$, are convex for all $x \in [a,b]$, $y \in [c,d]$.

Remark 1.2. Note that every convex function $\phi : [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates, but the converse is not generally true [14].

The following Jensen's inequality for co-ordinate convex functions has been given in [4].

Theorem 1.3. ([4]) *Let $\phi : [a,b] \times [c,d] \to \mathbb{R}$ be a convex function on the co-ordinates on $[a,b] \times [c,d]$. If $\boldsymbol{x}$ is an n-tuple in $[a,b]$, $\boldsymbol{y}$ is an m-tuple in $[c,d]$, $\boldsymbol{p}$ is a non-negative n-tuple and $\boldsymbol{w}$ a non-negative m-tuple such that*

$$P_n = \sum_{i=1}^{n} p_i > 0 \text{ and } W_m = \sum_{j=1}^{m} w_j > 0, \text{ then}$$

$$\phi(\bar{x}, \bar{y}) \leq \frac{1}{2}\left\{\frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i, \bar{y}) + \frac{1}{W_m}\sum_{j=1}^{m} w_j \phi(\bar{x}, y_j)\right\} \leq \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j), \tag{5}$$

$$\text{where } \bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \quad \bar{y} = \frac{1}{W_m}\sum_{j=1}^{m} w_j y_j.$$

For other refinements and generalizations of Jensen's inequality and their applications see [1–6, 8–13, 17–23] and some of the references given in them.

In this article, we have generalized the results given in [6], [7] and [15] from convex functions defined on the subset of $\mathbb{R}$ to convex functions defined on the co-ordinates on the bidimensional interval of the plane by constructing some new functionals depending on the function ϕ and indexing sets, separating the discrete domain of it. Furthermore the result given in [6] is extended to co-ordinate convex functions.

2. Main Results

Terminologies and notations: Let $\phi : [a,b] \times [c,d] \to \mathbb{R}$ be convex on the co-ordinates on $[a,b] \times [c,d]$. If $x_i \in [a,b]$, $y_j \in [c,d]$, and $p_i, w_j > 0, i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$ such that $n, m \geq 3$ with $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$, and let $\Omega_1 = \{I^k : I^k \subset I_n = \{1,...,n\}, |I^k| \geq 2, I^k \neq I_n\}$ and $\Omega_2 = \{J^l : J^l \subset J_m = \{1,...,m\}, |J^l| \geq 2, J^l \neq J_m\}$, we assume $\bar{I}^k := \{1,2,\ldots,n\} \setminus I^k$ and $\bar{J}^l := \{1,2,\ldots,m\} \setminus J^l$. Define $P_{I^k} = \sum_{i \in I^k} p_i$ and $P_{\bar{I}^k} = \sum_{i \in \bar{I}^k} p_i$ and $W_{J^l} = \sum_{j \in J^l} w_j$, $W_{\bar{J}^l} = \sum_{j \in \bar{J}^l} w_j$. For the function ϕ and the n, m-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in [a,b]^n$, $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in [c,d]^m$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_m)$, we define the following functionals:

$$\begin{aligned}
F(\phi, \mathbf{p}, \mathbf{x}, I^k) &= \frac{P_{I^k}}{P_n}\phi\left(\frac{1}{P_{I^k}}\sum_{i \in I^k} p_i x_i, \bar{y}\right) + \frac{1}{P_n}\sum_{i \in \bar{I}^k} p_i \phi(x_i, \bar{y}), \\
F(\phi, \mathbf{w}, \mathbf{y}, J^l) &= \frac{W_{J^l}}{W_m}\phi\left(\bar{x}, \frac{1}{W_{J^l}}\sum_{j \in J^l} w_j y_j\right) + \frac{1}{W_m}\sum_{j \in \bar{J}^l} w_j \phi\left(\bar{x}, y_j\right), && (6) \\
D_1(I^k, J^l) &= F\left(\phi, \mathbf{w}, \mathbf{y}, J^l\right) + F\left(\phi, \mathbf{p}, \mathbf{x}, I^k\right) && (7) \\
D_2(I^k, J^l) &= \frac{1}{P_n}\sum_{i=1}^{n} p_i F\left(\phi, \mathbf{w}, \mathbf{y}, J^l, x_i\right) + \frac{1}{W_m}\sum_{j=1}^{m} w_j F\left(\phi, \mathbf{p}, \mathbf{x}, I^k, y_j\right) && (8)
\end{aligned}$$

where

$$
\begin{aligned}
F(\phi, \mathbf{p}, \mathbf{x}, I^k, y_j) &= \frac{P_{I^k}}{P_n}\phi\left(\frac{1}{P_{I^k}}\sum_{i\in I^k} p_i x_i, y_j\right) + \frac{1}{P_n}\sum_{i\in \bar{I}^k} p_i\phi\left(x_i, y_j\right),\\
F(\phi, \mathbf{w}, \mathbf{y}, J^l, x_i) &= \frac{W_{J^l}}{W_m}\phi\left(x_i, \frac{1}{W_{J^l}}\sum_{j\in J^l} w_j y_j\right) + \frac{1}{W_m}\sum_{j\in \bar{J}^l} w_j\phi\left(x_i, y_j\right),\\
\bar{x} &= \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \quad \bar{y} = \frac{1}{W_m}\sum_{j=1}^{m} w_j y_j.
\end{aligned}
$$

Remark 2.1. It is obvious that $|\Omega_1| = 2^n - n - 2$, $|\Omega_2| = 2^m - m - 2$, that is, $k = 1, ..., 2^n - n - 2$ and $l = 1, ..., 2^m - m - 2$ and throughout the paper we will denote $2^n - n - 2$ by N and $2^m - m - 2$ by M.

The following lemma will be proved helpful in the further elaboration of the next refinement:

Lemma 2.2. *Let $\phi : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a function defined on Δ. If $x_i \in [a, b]$, $y_j \in [c, d]$, and $p_i, w_j > 0$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, n, $m \geq 3$, with $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$, then we have*

$$
\begin{aligned}
\frac{\sum_{l=1}^{M}\sum_{k=1}^{N} D_1(I^k, J^l)}{MN} &= \frac{1}{N}\left[\sum_{k=1}^{N}\frac{P_{I^k}}{P_n}\phi\Big(\frac{\sum_{i\in I^k} p_i x_i}{P_{I^k}}, \bar{y}\Big) + \frac{(2^{n-1} - n)}{P_n}\sum_{i=1}^{n} p_i\phi(x_i, \bar{y})\right]\\
&\quad + \frac{1}{M}\left[\sum_{l=1}^{M}\frac{W_{J^l}}{W_m}\phi\Big(\bar{x}, \frac{\sum_{j\in J^l} w_j y_j}{W_{J^l}}\Big) + \frac{(2^{m-1} - m)}{W_m}\sum_{j=1}^{m} w_j\phi(\bar{x}, y_j)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\sum_{l=1}^{M}\sum_{k=1}^{N} D_2(I^k, J^l)}{MN} &= \frac{1}{W_m}.\frac{1}{N}\left[\sum_{k=1}^{N}\sum_{j=1}^{m}\frac{P_{I^k}}{P_n}w_j\phi\Big(\frac{\sum_{i\in I^k} p_i x_i}{P_{I^k}}, y_j\Big) + \frac{(2^{n-1} - n)}{P_n}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j\phi(x_i, y_j)\right]\\
&\quad + \frac{1}{P_n}.\frac{1}{M}\left[\sum_{l=1}^{M}\sum_{i=1}^{n}\frac{W_{J^l}}{W_m}p_i\phi\Big(x_i, \frac{\sum_{j\in J^l} w_j y_j}{W_{J^l}}\Big) + \frac{(2^{m-1} - m)}{W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j\phi(x_i, y_j)\right].
\end{aligned}
$$

Proof. Since, from (6) we know that

$$
D_1(I^k, J^l) = F\left(\phi, \mathbf{w}, \mathbf{y}, J^l\right) + F\left(\phi, \mathbf{p}, \mathbf{x}, I^k\right).
$$

Therefore, we have

$$\begin{aligned}
\frac{\sum_{l=1}^{M}\sum_{k=1}^{N} D_1(I^k, J^l)}{MN} &= \frac{1}{MN}\left[\sum_{l=1}^{M}\sum_{k=1}^{N}\left\{F(\phi, \mathbf{p}, \mathbf{x}, I^k) + F(\phi, \mathbf{w}, \mathbf{y}, J^l)\right\}\right] \\
&= \frac{1}{MN}\Bigg[\sum_{l=1}^{M}\sum_{k=1}^{N}\left\{\frac{P_{I^k}}{P_n}\phi\left(\frac{1}{P_{I^k}}\sum_{i\in I^k} p_i x_i, \bar{y}\right) + \frac{1}{P_n}\sum_{i\in \bar{I}^k} p_i\phi\left(x_i, \bar{y}\right)\right. \\
&+ \left.\frac{W_{J^l}}{W_m}\phi\left(\bar{x}, \frac{1}{W_{J^l}}\sum_{j\in J^l} w_j y_j\right) + \frac{1}{W_m}\sum_{j\in \bar{J}^l} w_j\phi\left(\bar{x}, y_j\right)\right\}\Bigg] \\
&= \frac{1}{N}\left[\sum_{k=1}^{N}\frac{P_{I^k}}{P_n}\phi\Big(\frac{\sum_{i\in I^k} p_i x_i}{P_{I^k}}, \bar{y}\Big) + \frac{(2^{n-1}-n)}{P_n}\sum_{i=1}^{n} p_i\phi(x_i, \bar{y})\right] \\
&+ \frac{1}{M}\left[\sum_{l=1}^{M}\frac{W_{J^l}}{W_m}\phi\Big(\bar{x}, \frac{\sum_{j\in J^l} w_j y_j}{W_{J^l}}\Big) + \frac{(2^{m-1}-m)}{W_m}\sum_{j=1}^{m} w_j\phi(\bar{x}, y_j)\right].
\end{aligned}$$

Here it is obvious that

$$\sum_{k=1}^{N}\sum_{i\in \bar{I}^k} p_i\phi(x_i, y_j) = (2^{n-1} - (n-1) - 1)\sum_{i=1}^{n} p_i\phi(x_i, y_j),$$

since every $p_i\phi(x_i, y_j)$ appears as many times as there is a subset $I^k \subset I_n$, $|I^k| \geq 2$, and that doesn't contain the index i. Similarly we can prove the second part of the lemma. □

The following refinement of Theorem 1.3 holds:

Theorem 2.3. *Suppose that $\phi : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. If $x_i \in [a, b]$, $y_j \in [c, d]$, and $p_i, w_j > 0, i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, n, $m \geq 3$, with $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$, then for any $I^k \in \Omega_1$ and $J^l \in \Omega_2$ we have*

$$\begin{aligned}
\phi\left(\bar{x}, \bar{y}\right) &\leq \frac{1}{2}\min_{\substack{k=1,\ldots,N\\ l=1,\ldots,M}} D_1(I^k, J^l) \leq \frac{1}{2}\frac{\sum_{l=1}^{M}\sum_{k=1}^{N} D_1(I^k, J^l)}{MN} \leq \frac{1}{2}\max_{\substack{k=1,\ldots,N\\ l=1,\ldots,M}} D_1(I^k, J^l) \\
&\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i\phi\left(x_i, \bar{y}\right) + \frac{1}{W_m}\sum_{j=1}^{m} w_j\phi\left(\bar{x}, y_j\right)\right] \\
&\leq \frac{1}{2}\min_{\substack{k=1,\ldots,N\\ l=1,\ldots,M}} D_2(I^k, J^l) \leq \frac{1}{2}\frac{\sum_{l=1}^{M}\sum_{k=1}^{N} D_2(I^k, J^l)}{MN} \leq \frac{1}{2}\max_{\substack{k=1,\ldots,N\\ l=1,\ldots,M}} D_2(I^k, J^l) \\
&\leq \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j\phi(x_i, y_j), \qquad (9)
\end{aligned}$$

$$\textit{where}\quad \bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \quad \bar{y} = \frac{1}{W_m}\sum_{j=1}^{m} w_j y_j, 1 \leq k \leq N,\ 1 \leq l \leq M.$$

Proof. One-dimensional Jensen's inequality gives us

$$\phi\left(x_i,\bar{y}\right)\leq\frac{1}{W_m}\sum_{j=1}^{m}w_j\phi\left(x_i,y_j\right)\ \text{ and }\ \phi\left(\bar{x},y_j\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi\left(x_i,y_j\right).$$

By Jensen's inequality, we get

$$\begin{aligned}F(\phi,\mathbf{p},\mathbf{x},I^k,y_j) &= \frac{P_{I^k}}{P_n}\phi\left(\frac{1}{P_{I^k}}\sum_{i\in I^k}p_ix_i,y_j\right)+\frac{1}{P_n}\sum_{i\in\bar{I}^k}p_i\phi\left(x_i,y_j\right)\\ &\leq \frac{P_{I^k}}{P_n}\frac{1}{P_{I^k}}\sum_{i\in I^k}p_i\phi(x_i,y_j)+\frac{1}{P_n}\sum_{i\in\bar{I}^k}p_i\phi\left(x_i,y_j\right)=\frac{1}{P_n}\sum_{i\in I^k\cup\bar{I}^k}p_i\phi(x_i,y_j)\end{aligned}$$

$$\Rightarrow F(\phi,\mathbf{p},\mathbf{x},I^k,y_j)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi(x_i,y_j). \tag{10}$$

As the function ϕ is convex on the first co-ordinate, so we have

$$\begin{aligned}F(\phi,\mathbf{p},\mathbf{x},I^k,y_j) &= \frac{P_{I^k}}{P_n}\phi\left(\frac{1}{P_{I^k}}\sum_{i\in I^k}p_ix_i,y_j\right)+\frac{1}{P_n}\sum_{i\in\bar{I}^k}p_i\phi\left(x_i,y_j\right)\\ &\geq \frac{P_{I^k}}{P_n}\phi\left(\frac{1}{P_{I^k}}\sum_{i\in I^k}p_ix_i,y_j\right)+\frac{P_{\bar{I}^k}}{P_n}\phi\left(\frac{1}{P_{\bar{I}^k}}\sum_{i\in\bar{I}^k}p_ix_i,y_j\right)\\ &\geq \phi\left(\frac{P_{I^k}}{P_n}\frac{1}{P_{I^k}}\sum_{i\in I^k}p_ix_i+\frac{P_{\bar{I}^k}}{P_n}\frac{1}{P_{\bar{I}^k}}\sum_{i\in\bar{I}^k}p_ix_i,y_j\right)=\phi\left(\frac{1}{P_n}\sum_{I^k\cup\bar{I}^k}p_ix_i,y_j\right)\end{aligned}$$

$$\Rightarrow F(\phi,\mathbf{p},\mathbf{x},I^k,y_j)\geq\phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i,y_j\right). \tag{11}$$

Now, from (10) and (11), we have

$$\phi\left(\bar{x},y_j\right)\leq F\left(\phi,\mathbf{p},\mathbf{x},I^k,y_j\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi(x_i,y_j). \tag{12}$$

Similarly, we can write

$$\phi\left(x_i,\bar{y}\right)\leq F\left(\phi,\mathbf{w},\mathbf{y},J^l,x_i\right)\leq\frac{1}{W_m}\sum_{j=1}^{m}w_j\phi(x_i,y_j). \tag{13}$$

Multiplying (12) and (13) respectively by w_j and p_i and summing over i and j, we obtain

$$\frac{1}{W_m}\sum_{j=1}^{m}w_j\phi\left(\bar{x},y_j\right)\leq\frac{1}{W_m}\sum_{j=1}^{m}w_jF\left(\phi,\mathbf{p},\mathbf{x},I^k,y_j\right)\leq\frac{1}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(x_i,y_j), \tag{14}$$

and

$$\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi\left(x_i,\bar{y}\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_iF\left(\phi,\mathbf{w},\mathbf{y},J^l,x_i\right)\leq\frac{1}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(x_i,y_j). \tag{15}$$

Adding (14) and (15), one has the following

$$\begin{aligned}\frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi\left(x_i,\bar{y}\right)+\frac{1}{W_m}\sum_{j=1}^{m}w_j\phi\left(\bar{x},y_j\right)\right] &\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_iF\left(\phi,\mathbf{w},\mathbf{y},J^l,x_i\right)+\frac{1}{W_m}\sum_{j=1}^{m}w_jF\left(\phi,\mathbf{p},\mathbf{x},I^k,y_j\right)\right]\\ &\leq \frac{1}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(x_i,y_j). \qquad (16)\end{aligned}$$

Now, setting $x_i=\bar{x}$ and $y_j=\bar{y}$ in (12), (13) and adding we have

$$\phi\left(\bar{x},\bar{y}\right) \leq \frac{1}{2}\left[F\left(\phi,\mathbf{w},\mathbf{y},J^l\right)+F\left(\phi,\mathbf{p},\mathbf{x},I^k\right)\right]\leq\frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi\left(x_i,\bar{y}\right)+\frac{1}{W_m}\sum_{j=1}^{m}w_j\phi\left(\bar{x},y_j\right)\right].$$

Combining (16) and (17) we obtain

$$\begin{aligned}\phi\left(\bar{x},\bar{y}\right) &\leq \frac{1}{2}\left[F\left(\phi,\mathbf{w},\mathbf{y},J^l\right)+F\left(\phi,\mathbf{p},\mathbf{x},I^k\right)\right]\leq\frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi\left(x_i,\bar{y}\right)+\frac{1}{W_m}\sum_{j=1}^{m}w_j\phi\left(\bar{x},y_j\right)\right]\\ &\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_iF\left(\phi,\mathbf{w},\mathbf{y},J^l,x_i\right)+\frac{1}{W_m}\sum_{j=1}^{m}w_jF\left(\phi,\mathbf{p},\mathbf{x},I^k,y_j\right)\right]\leq\frac{1}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(x_i,y_j).\end{aligned}$$

The statement in the theorem follows by taking the min and max of $D_1(I^k,J^l)$ and $D_2(I^k,J^l)$ over the indices k and l with $1\leq k\leq N$, $1\leq l\leq M$ and together with Lemma 2.2 and using the fact that

$$\min_{\substack{k=1,\dots,N\\ l=1,\dots,M}}D_1(I^k,J^l)\leq\frac{\sum\limits_{l=1}^{M}\sum\limits_{k=1}^{N}D_1(I^k,J^l)}{MN}\leq\max_{\substack{k=1,\dots,N\\ l=1,\dots,M}}D_1(I^k,J^l) \qquad (17)$$

and

$$\min_{\substack{k=1,\dots,N\\ l=1,\dots,M}}D_2(I^k,J^l)\leq\frac{\sum\limits_{l=1}^{M}\sum\limits_{k=1}^{N}D_2(I^k,J^l)}{MN}\leq\max_{\substack{k=1,\dots,N\\ l=1,\dots,M}}D_2(I^k,J^l). \qquad (18)$$

This completes the desired proof. □

Remark 2.4. For $I^k=\{\mu\}$, $\mu\in\{1,\dots,n\}$ and $J^l=\{\nu\}$, $\nu\in\{1,\dots,m\}$, the above functionals take the form given below

$$\begin{aligned}F(\phi,\mathbf{p},\mathbf{x},I^k,y_j) &= F(\phi,\mathbf{p},\mathbf{x},\{\mu\},y_j)=\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi(x_i,\,y_j),\\ F(\phi,\mathbf{w},\mathbf{y},J^l,x_i) &= F(\phi,\mathbf{w},\mathbf{y},\{\nu\},x_i)=\frac{1}{W_m}\sum_{j=1}^{m}w_j\phi(x_i,\,y_j),\\ D_1(\{\mu\},\{\nu\}) &= F\left(\phi,\mathbf{w},\mathbf{y},\{\nu\}\right)+F\left(\phi,\mathbf{p},\mathbf{x},\{\mu\}\right),\\ &= \frac{1}{W_m}\sum_{j=1}^{m}w_j\phi(\bar{x},\,y_j)+\frac{1}{P_n}\sum_{i=1}^{n}p_i\phi(x_i,\,\bar{y})\\ D_2(\{\mu\},\{\nu\}) &= \frac{2}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(x_i,y_j)\end{aligned}$$

and the refinement given in Theorem 2.3 shrinks to the result given in Theorem 1.3.

In the next theorem, subsets of equivalent cardinality are observed.

Theorem 2.5. *Suppose that $\phi : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. If $x_i \in [a,b]$, $y_j \in [c,d]$, and $p_i, w_j > 0, i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$, n, $m \geq 3$, with $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$, then for any $I^k \in \Omega_1$ and $J^l \in \Omega_2$ such that $|I^k| = s \geq 2$ and $|J^l| = r \geq 2$ we have*

$$\begin{aligned}\phi(\bar{x},\bar{y}) &\leq \frac{1}{2}\min_{\substack{|I^k|=s\\|J^l|=r}} D_1(I^k,J^l) \leq \frac{1}{2}\frac{\sum_{l=1}^{\binom{m}{r}}\sum_{k=1}^{\binom{n}{s}} D_1(I^k,J^l)}{\binom{n}{s}\binom{m}{r}} \leq \frac{1}{2}\max_{\substack{|I^k|=s\\|J^l|=r}} D_1(I^k,J^l) \leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i\phi(x_i,\bar{y}) + \frac{1}{W_m}\sum_{j=1}^{m} w_j\phi(\bar{x},y_j)\right]\\ &\leq \frac{1}{2}\min_{\substack{|I^k|=s\\|J^l|=r}} D_2(I^k,J^l) \leq \frac{1}{2}\frac{\sum_{l=1}^{\binom{m}{r}}\sum_{k=1}^{\binom{n}{s}} D_2(I^k,J^l)}{\binom{n}{s}\binom{m}{r}} \leq \frac{1}{2}\max_{\substack{|I^k|=s\\|J^l|=r}} D_2(I^k,J^l) \leq \frac{1}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_iw_j\phi(x_i,y_j),\end{aligned}$$

where $\bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_ix_i, \quad \bar{y} = \frac{1}{W_m}\sum_{j=1}^{m} w_jy_j.$

Proof. The statement in the theorem follows by taking the min and max of the functionals given in (1) and (2), after choosing every subset $I^k \in \Omega_1$ and $J^l \in \Omega_2$, such that $|I^k| = s$ and $|J_l| = r$, with $2 \leq s < n$ and $2 \leq r < m$. We use the facts mentioned in (17) and (18), where $\binom{n}{s}$ and $\binom{m}{r}$ represent the number of subsets $I^k \subset I_n$ and $J^l \subset J_m$, $|I^k| = s$, $|J^l| = r$. Note that

$$\sum_{I^k\subset I_n, |I^k|=s}\ \sum_{i\in \bar{I}^k} p_i\phi(x_i,y_j) = \left[\binom{n}{s} - \binom{n-1}{s-1}\right]\sum_{i=1}^{n} p_i\phi(x_i,y_j),$$

since every $p_i\phi(x_i,y_j)$ in the double sum appears as many times as there are subsets $I^k \subset I_n$, $|I^k| = s \geq 2$ such that $i \notin I^k$. The subsets $I^k \subset I_n$, with $|I^k| = s$ and $i \in I^k$ is constructed by adding $s-1$ elements from the $n-1$ available once. Algebraically,

$$\left[\binom{n}{s} - \binom{n-1}{s-1}\right]\sum_{i=1}^{n} p_i\phi(x_i,y_j) = \binom{n-1}{s}\sum_{i=1}^{n} p_i\phi(x_i,y_j).$$

Similar arguments can be given for the subsets $J^l \subset J_m$, with $|J^l| = r \geq 2$ and one has

$$\sum_{J^l\subset J_m, |J^l|=r}\ \sum_{j\in \bar{J}^l} w_j\phi(x_i,y_j) = \left[\binom{m}{r} - \binom{m-1}{r-1}\right]\sum_{j=1}^{m} w_j\phi(x_i,y_j).$$

□

Every partition of $I_n = \{1,...,n\}$ and $J_m = \{1,...,m\}$ gives the statement obtained in the next theorem.

Theorem 2.6. *Suppose that $\phi : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. If $x_i \in [a,b]$, $y_j \in [c,d]$, and $p_i, w_j > 0, i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$, n, $m \geq 4$, with $P_n = \sum_{i=1}^{n} p_i$ and $W_m = \sum_{j=1}^{m} w_j$. For every integers*

q, r, such that $4 \le 2q \le n$ and $4 \le 2r \le m$, there are partitions $I_1 \cup I_2 \cup ... \cup I_q = I_n$, $J_1 \cup J_2 \cup ... \cup J_r = J_m$ with $2 \le |I_\mu| < n$, $2 \le |J_\nu| < m$ for $\mu = 1, 2, ..., q$, $\nu = 1, 2, ..., r$. Then we have

$$\begin{aligned}
\phi(\bar{x}, \bar{y}) &\le \frac{1}{2} \min_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_1(I_\mu, J_\nu) \le \frac{1}{2} \frac{\sum_{\nu=1}^{r}\sum_{\mu=1}^{q} D_1(I_\mu, J_\nu)}{(qr)} \le \frac{1}{2} \max_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_1(I_\mu, J_\nu) \\
&\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i, \bar{y}) + \frac{1}{W_m}\sum_{j=1}^{m} w_j \phi\left(\bar{x}, y_j\right)\right] \\
&\le \frac{1}{2} \min_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_2(I_\mu, J_\nu) \le \frac{1}{2} \frac{\sum_{\nu=1}^{r}\sum_{\mu=1}^{q} D_2(I_\mu, J_\nu)}{(qr)} \le \frac{1}{2} \max_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_2(I_\mu, J_\nu) \le \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j),
\end{aligned}$$

where $\bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$, $\quad \bar{y} = \frac{1}{W_m}\sum_{j=1}^{m} w_j y_j$, $1 \le \mu \le q$, $1 \le \nu \le r$.

Proof. Every subset $I_\mu \subset I_n$ and $J_\nu \subset J_m$ induced theirs complement $\bar{I}_\mu$ and $\bar{J}_\nu$ and (9) is valid with the substitutions: $I^k \to I_\mu$, $J^l \to J_\nu$. For $D_1(I_\mu, J_\nu)$ and $D_2(I_\mu, J_\nu)$ we take the min and max over $\mu = 1, ..., n$ and $\nu = 1, ..., m$ and using the facts that

$$\min_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_1(I_\mu, J_\nu) \le \frac{\sum_{\nu=1}^{r}\sum_{\mu=1}^{q} D_1(I_\mu, J_\nu)}{(qr)} \le \max_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_1(I_\mu, J_\nu) \text{ and } \min_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_2(I_\mu, J_\nu) \le \frac{\sum_{\nu=1}^{r}\sum_{\mu=1}^{q} D_2(I_\mu, J_\nu)}{(qr)} \le \max_{\substack{\mu=1,...,q\\ \nu=1,...,r}} D_2(I_\mu, J_\nu).$$

Note: $\sum_{\mu=1}^{q}\sum_{i \in \bar{I}_\mu} p_i \phi(x_i, y_j) = (q-1)\sum_{i=1}^{n} p_i \phi(x_i, y_j)$. □

Theorem 2.3 ensures the next improvements of Jensen's difference.

Corollary 2.7. *Under the conditions of Theorem* 2.3*, we obtain:*

$$\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \phi(\bar{x}, \bar{y}) \ge \max_{I^k, J^l}\left[\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \frac{1}{2} D_1(I^k, J^l)\right] \ge 0. \tag{19}$$

Proof. Subtracting $\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j)$ from every side of (9), we obtain that for every choice of $I^k \subset I_n = \{1, ..., n\}$ and $J^l \subset J_m = \{1, ..., m\}$, there is a statement:

$$\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \phi(\bar{x}, \bar{y}) \ge \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \frac{1}{2} D_1(I^k, J^l) \ge 0. \tag{20}$$

Taking the max of the right hand side in (20) for $I^k \subset I_n$, $|I^k| \ge 2$ and $J^l \subset J_m$, $|J^l| \ge 2$, the proof is making through. □

Corollary 2.8. *Under the conditions of Theorem* 2.3*, we obtain:*

$$\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \phi(\bar{x}, \bar{y}) \ge \max_{I^k, J^l}\left[\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(x_i, y_j) - \frac{1}{2} D_2(I^k, J^l)\right] \ge 0. \tag{21}$$

Proof. The proof is similar to that of corollary 2.7 only use $D_2(I^k, J^l)$ instead of $D_1(I^k, J^l)$. $\square$

Now we give another refinement of the Jensen's inequality for the convex function defined on the co-ordinates of the bidimensional interval in the plane:

Theorem 2.9. *Let $\phi : [a, b] \times [c, d] \to \mathbb{R}$ be a co-ordinate convex function on $[a, b] \times [c, d]$. If $x_i \in [a, b]$, $y_j \in [c, d]$ such that $x_{i+n} = x_i$, $y_{j+m} = y_j$ and $p_i, w_j > 0, i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, with $\sum_{i=1}^{k} p_i = 1$ and $\sum_{j=1}^{l} w_j = 1$, for some k and l, $2 \le k \le n$ and $2 \le l \le m$, then we have*

$$\begin{aligned}
\phi(\bar{x}, \bar{y}) &\le \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r},\ \bar{y}\right) + \frac{1}{m}\sum_{j=1}^{m}\phi\left(\bar{x},\ \sum_{t=0}^{l-1}w_{t+1}y_{j+t}\right)\right] \le \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(x_i,\ \bar{y}) + \frac{1}{m}\sum_{j=1}^{m}\phi(\bar{x},\ y_j)\right] \\
&\le \frac{1}{2}\left[\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r},\ y_j\right) + \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\phi\left(x_i,\ \sum_{t=0}^{l-1}w_{t+1}y_{j+t}\right)\right] \le \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\phi(x_i,\ y_j), \\
\textit{where}\quad \bar{x} &= \frac{1}{n}\sum_{i=1}^{n}x_i, \quad \bar{y} = \frac{1}{m}\sum_{j=1}^{m}y_j.
\end{aligned}$$

Proof. Since $\phi_{y_j} : [a, b] \to \mathbb{R}$ is convex, so by Jensen's inequality, we have

$$\begin{aligned}
\phi_{y_j}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) = \phi_{y_j}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\sum_{r=1}^{k}p_r\right) &= \phi_{y_j}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{r=0}^{k-1}p_{r+1}x_{i+r}\right) \\
&\le \frac{1}{n}\sum_{i=1}^{n}\phi_{y_j}\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r}\right) \le \frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r}, y_j\right),
\end{aligned}$$

therefore,

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i, y_j\right) \le \frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r}, y_j\right). \tag{22}$$

On the other hand, since $\phi_{y_j} : [a, b] \to \mathbb{R}$ is convex, so again by Jensen's inequality and simple calculations one can get

$$\frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r},\ y_j\right) \le \frac{1}{n}\sum_{i=1}^{n}\phi\left(x_i,\ y_j\right) \tag{23}$$

the combination of (22) and (23) yields

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i,\ y_j\right) \le \frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r},\ y_j\right) \le \frac{1}{n}\sum_{i=1}^{n}\phi\left(x_i,\ y_j\right). \tag{24}$$

Similarly, the convexity of $\phi_{x_i} : [c, d] \to \mathbb{R}$ implies the following

$$\phi\left(x_i,\ \frac{1}{m}\sum_{j=1}^{m}y_i\right) \le \frac{1}{m}\sum_{j=1}^{m}\phi\left(x_i,\ \sum_{t=0}^{l-1}w_{t+1}y_{j+t}\right) \le \frac{1}{m}\sum_{j=1}^{m}\phi\left(x_i,\ y_j\right). \tag{25}$$

Multiplying (24) and (25) by $\frac{1}{m}$ and $\frac{1}{n}$ respectively and summing over j, i and then adding the obtained results, one has the following

$$\begin{aligned}\frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(x_i,\,\bar{y})+\frac{1}{m}\sum_{j=1}^{m}\phi(\bar{x},\,y_j)\right] &\leq \frac{1}{2}\left[\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r},\,y_j\right)+\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\phi\left(x_i,\,\sum_{t=0}^{l-1}w_{t+1}y_{j+t}\right)\right]\\ &\leq \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\phi(x_i,\,y_j). \qquad (26)\end{aligned}$$

Furthermore by setting $x_i \to \bar{x}$ and $y_j \to \bar{y}$ in (24) and (25) respectively we get

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i,\,\bar{y}\right)\leq\frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r},\,\bar{y}\right)\leq\frac{1}{n}\sum_{i=1}^{n}\phi\left(x_i,\,\bar{y}\right) \qquad (27)$$

and

$$\phi\left(\bar{x},\,\frac{1}{m}\sum_{j=1}^{m}y_i\right)\leq\frac{1}{m}\sum_{j=1}^{m}\phi\left(\bar{x},\,\sum_{t=0}^{l-1}w_{t+1}y_{j+t}\right)\leq\frac{1}{m}\sum_{j=1}^{m}\phi\left(\bar{x},\,y_j\right). \qquad (28)$$

Now adding them, we obtain

$$\phi(\bar{x},\bar{y}) \leq \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{r=0}^{k-1}p_{r+1}x_{i+r},\,\bar{y}\right)+\frac{1}{m}\sum_{j=1}^{m}\phi\left(\bar{x},\,\sum_{t=0}^{l-1}w_{t+1}y_{j+t}\right)\right]\leq\frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(x_i,\,\bar{y})+\frac{1}{m}\sum_{j=1}^{m}\phi(\bar{x},\,y_j)\right].$$

Hence, we have the desired result. □

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