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Constant Curvature Ratios in $\mathbb{L}^6$

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Abstract. In this paper, we find a relation between Frenet formulas and harmonic curvatures, and also a relation between Frenet formulas and e-curvature functions of a curve of osculating order 6 in 6 dimensional Lorentzian space $\mathbb{L}^6$. Moreover, we give a relation between harmonic curvatures and ccr-curves of a curve in $\mathbb{L}^6$.

1. Introduction

Let $X = (x_1, x_2, x_3, x_4, x_5, x_6)$ and $Y = (y_1, y_2, y_3, y_4, y_5, y_6)$ be two non-zero vectors in 6-dimensional Lorentz Minkowski space $\mathbb{R}^6_1$. We briefly denoted $\mathbb{R}^6_1$ by $\mathbb{L}^6$. For $X, Y \in \mathbb{L}^6$

$$\langle X, Y \rangle = -x_1 y_1 + \sum_{i=2}^{6} x_i y_i$$

is called *Lorentzian inner product*. The couple $\left\{\mathbb{R}^6_1, \langle , \rangle\right\}$ is called *Lorentzian space* and denoted by $\mathbb{L}^6$. Then a vector X of $\mathbb{L}^6$ is called **i)** time-like if $\langle X, X \rangle < 0$, **ii)** space-like if $\langle X, X \rangle > 0$ or $X = 0$, **iii)** null (or light-like) vector if $\langle X, X \rangle = 0$, $X \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{L}^6$ can locally be space-like, time-like or null, if all of its velocity vectors $\alpha^{'}(s)$ are space-like, time-like or null, respectively. Also, recall that the norm of a vector X is given by $\|X\| = (|\langle X, X \rangle|)^{\frac{1}{2}}$. Therefore, X is a unit vector if $\langle X, X \rangle = \pm 1$. Next, two vectors X, Y in $\mathbb{L}^6$ are said to be orthogonal if $\langle X, Y \rangle = 0$. The velocity of the curve α is given by $\left\|\alpha^{'}\right\|$. Thus, a space-like or a time-like α is said to be parametrized by arclength function s if $\left\langle X^{'}, X^{'} \right\rangle = \pm 1$ [1].

2. Basic Definitions of $\mathbb{L}^6$

Definition 2.1. Let $\alpha : I \longrightarrow \mathbb{L}^6$ be a unit speed non-null curve in $\mathbb{L}^6$. The curve α is called the Frenet curve of osculating order 6 if its 6^{th} order derivatives $\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s), \cdots \alpha^{iv}(s), \alpha^{v}(s), \alpha^{vi}(s)$ are linearly independent and $\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s), \alpha^{iv}(s), \alpha^{v}(s), \alpha^{vi}(s), \alpha^{vii}(s)$ are no longer linearly independent for all $s \in I$. For each Frenet curve of order 6 one can associate an orthonormal $6-$frame $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ along

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α (such that $\alpha'(s) = V_1$) which is called the Frenet frame and k_i: $I \longrightarrow \mathbb{R}$, $1 \le i \le 5$ are called the Frenet curvatures, such that the Frenet formulas are defined in the usual way

$$\left\{ \begin{array}{c} \nabla_{V_1} V_1 = \varepsilon_2 k_1 V_2 \\ \nabla_{V_1} V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3 \\ \nabla_{V_1} V_3 = -\varepsilon_2 k_2 V_2 + \varepsilon_4 k_3 V_4 \\ \nabla_{V_1} V_4 = -\varepsilon_3 k_3 V_3 + \varepsilon_5 k_4 V_5 \\ \nabla_{V_1} V_5 = -\varepsilon_4 k_4 V_4 + \varepsilon_6 k_5 V_6 \\ \nabla_{V_1} V_6 = -\varepsilon_5 k_5 V_5, \end{array} \right\}$$

where $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$.

Definition 2.2. Let $M \subset \mathbb{L}^6$, $\alpha : I \longrightarrow \mathbb{L}^6$ be a curve in $\mathbb{L}^6$and k_i , $1 \le i \le 5$, be the Frenet curvatures of α. Then for the unit tangent vector $V_1 = \alpha'(s)$ over M, the i^{th} e-curvature function m_i, $1 \le i \le 6$ is defined by

$$m_i = \left\{ \begin{array}{ll} 0 & , \quad i = 1 \\ \dfrac{\varepsilon_1 \varepsilon_2}{k_1} & , \quad i = 2 \\ \left[\dfrac{d}{dt}(m_{i-1}) + \varepsilon_{i-2} m_{i-2} k_{i-2} \right] \dfrac{\varepsilon_i}{k_{i-1}} & , \quad 2 < i \le 6, \end{array} \right\}$$

where $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$.

Definition 2.3. A non-null curve $\alpha : I \longrightarrow \mathbb{L}^6$ is called a $W-curve$ (or helix) of *rank* 6, if α is a Frenet curve of osculating order 6 and the Frenet curvatures k_i, $1 \le i \le 5$ are non-zero constants.

3. A General Helix of Rank 4

Definition 3.1. Let α be a non-null curve of osculating order 6. The harmonic functions

$$H_j : I \longrightarrow \mathbb{R} \quad , \quad 0 \le j \le 4,$$

defined by

$$H_0 = 0 \, , H_1 = \frac{k_1}{k_2},$$

$$H_j = \left\{ \nabla_{v_1} \left(H_{(j-1)} \right) + \varepsilon_{(j-2)} H_{(j-2)} k_j \right\} \frac{\varepsilon_j}{k_{(j+1)}}, \; 2 \le j \le 4$$

are called the harmonic curvatures of α where k_i, for $1 \le i \le 5$, are Frenet curvatures of α which are not necessarily constant.

Definition 3.2. Let α be a non-null of osculating order 6. Then α is called a general helix of *rank* 4 if

$$\sum_{i=1}^{4} H_i^2 = c,$$

holds, where $c \neq 0$ is a real constant.

We have the following result.

Proposition 3.3. *If α is a general helix of rank 4 then*

$$H_1^2 + H_2^2 + H_3^2 + H_4^2 = c.$$

Proof. By the use of above definition we obtain the proof. □

Proposition 3.4. *Let α be a curve in $\mathbb{L}^6$ of osculating order 6. Then*

$$\nabla_{V_1} V_1 = \varepsilon_2 k_2 H_1 V_2,$$

$$\nabla_{V_1} V_2 = -\varepsilon_1 k_2 H_1 V_1 + \varepsilon_3 \frac{k_1}{H_1} V_3,$$

$$\nabla_{V_1} V_3 = -\varepsilon_2 \frac{k_1}{H_1} V_2 + \varepsilon_4 \varepsilon_2 \frac{H_1^{'}}{H_2} V_4,$$

$$\nabla_{V_1} V_4 = -\varepsilon_3 \varepsilon_2 \frac{H_1^{'}}{H_2} V_3 + \varepsilon_5 \varepsilon_3 \left(\frac{H_2 H_2^{'} + \varepsilon_1 \varepsilon_2 H_1 H_1^{'}}{H_2 H_3} \right) V_5,$$

$$\nabla_{V_1} V_5 = -\varepsilon_4 \varepsilon_3 \left(\frac{H_2 H_2^{'} + \varepsilon_1 \varepsilon_2 H_1 H_1^{'}}{H_2 H_3} \right) V_4 + \varepsilon_4 \varepsilon_6 \left(\frac{H_3 H_3^{'} + \varepsilon_2 \varepsilon_3 H_2 H_2^{'} + \varepsilon_1 \varepsilon_3 H_1 H_1^{'}}{H_3 H_4} \right) V_6,$$

$$\nabla_{V_1} V_6 = -\varepsilon_5 \varepsilon_4 \left(\frac{H_3 H_3^{'} + \varepsilon_2 \varepsilon_3 H_2 H_2^{'} + \varepsilon_1 \varepsilon_3 H_1 H_1^{'}}{H_3 H_4} \right) V_5,$$

where H_i, for $1 \leq i \leq 4$, are harmonic curvatures of α.

Proof. By using the Frenet formulas and definitions of the harmonic curvatures, we get the result. □

Proposition 3.5. ([7]) *a)Let α be a time-like curve. Then*

$$k_r = \frac{\varepsilon_{(r-2)} \left(\sum\limits_{i=1}^{r-2} H_i^2 \right)^{'}}{2 H_{(r-2)} H_{(r-1)}}, 2 < r \leq 4,$$

where $(H_i)^{'}$ stands for differentiation with respect to parameter t.

b)Let α be a time-like curve. Then

$$k_r = \frac{\varepsilon_{(r-2)} \left(\sum\limits_{i=2}^{r} m_i^2 \right)^{'}}{2 m_r m_{(r+1)}}, 2 \leq r < 6,$$

where m_i , for $2 \leq i \leq 6$, are the i^{th} e-curvature functions of α .

4. ccr-curves in $\mathbf{L}^6$

Definition 4.1. A curve $\alpha : I \longrightarrow \mathbb{L}^6$ is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients $\varepsilon_i \left(\frac{k_{i+1}}{k_i} \right)$ are constant. Here; k_i, k_{i+1} are Frenet curvatures of α and $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1, (1 \leq i \leq 4)$.

Proposition 4.2. *a) For i=1, the ccr-curve is* $\frac{\varepsilon_1}{H_1}$.

b) For i=2, the ccr-curve is $\frac{H_1 H_1^{'}}{H_2 k_1}$.

c) For i=3, the ccr-curve is $\frac{\varepsilon_2 H_2 H_2^{'} + \varepsilon_1 H_1 H_1^{'}}{H_1^{'} H_3}$.

d) For i=4, the ccr-curve is $\frac{\varepsilon_3 H_2 H_3 H_3^{'} + \varepsilon_2 H_2^2 H_2^{'} + \varepsilon_1 H_2 H_1 H_1^{'}}{H_2 H_2^{'} H_4 + \varepsilon_1 \varepsilon_2 H_1 H_1^{'} H_4}$.

Here, for $1 \leq i \leq 4$, H_i are harmonic curvatures of α.

Proof. The proof can easily be seen from the definitions of harmonic curvature and ccr-curve. □

Proposition 4.3. *a) If the vector V_1 is time-like then the ccr-curve is $\frac{-1}{H_1}$, where $\varepsilon_1 = \langle V_1, V_1 \rangle = -1$.*

b) If the vector V_1 is space-like then the ccr-curve is $\frac{1}{H_1}$, where $\varepsilon_1 = \langle V_1, V_1 \rangle = 1$.

c) If the vector V_2 is time-like then the ccr-curve is $\frac{-H_2 H_2' + H_1 H_1'}{H_1' H_3}$, where $\varepsilon_2 = \langle V_1, V_1 \rangle = -1, \varepsilon_1 = \langle V_1, V_1 \rangle = 1$.

d) If the vector V_2 is space-like then the ccr-curve is $\frac{H_2 H_2' - H_1 H_1'}{H_1' H_3}$, where $\varepsilon_2 = \langle V_1, V_1 \rangle = 1, \varepsilon_1 = \langle V_1, V_1 \rangle = -1$.

Theorem 4.4. *α is a ccr-curve in $\mathbb{L}^6 \Leftrightarrow \sum_{i=1}^{4} \varepsilon_i H_i^2 = constant$.*

Proof. By using the definitions of a general helix of *rank* 4 and ccr-curve, the proof of theorem follows. □

Theorem 4.5. *i) Let $\alpha : I \longrightarrow \mathbb{L}^6$ be a non-null curve, $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ be a Frenet frame and k_1, k_2, k_3, k_4, k_5 ($k_6 = 0$) be curvature functions. If $k_1 = 1$ and k_2, k_3, k_4, k_5 are constants, then*

$$\nabla_{V_1}^6 V_1 - \left(1 + 2\varepsilon_1\varepsilon_3 \frac{k_1^2}{H_1^2} + \frac{k_1^4}{H_1^4}\right) \nabla_{V_1}^2 V_1 = 0,$$

where H_1 is the harmonic curvature of α.

ii)

$$\nabla_{V_1}^6 V_1 - \left(1 + 2\varepsilon_1\varepsilon_3 \frac{\left(m_2'\right)^2}{(m_3)^2} + \frac{\left(m_2'\right)^4}{(m_3)^4}\right) \nabla_{V_1}^2 V_1 = 0,$$

where m_2 and m_3 are 2^{nd} and 3^{rd} e-curvature functions of α .

Proof. i) As $k_1 = 1$, we have

$$\begin{aligned} \nabla_{V_1} V_1 &= \varepsilon_2 V_2 \Longrightarrow \nabla_{V_1}^2 V_1 = \varepsilon_2 \nabla_{V_1} V_2 \Longrightarrow \nabla_{V_1}^3 V_1 = \varepsilon_2 \nabla_{V_1}^2 V_2 \Longrightarrow \nabla_{V_1}^4 V_1 = \varepsilon_2 \nabla_{V_1}^3 V_2 \\ &\Longrightarrow \nabla_{V_1}^5 V_1 = \varepsilon_2 \nabla_{V_1}^4 V_2 \Longrightarrow \nabla_{V_1}^6 V_1 = \varepsilon_2 \nabla_{V_1}^5 V_2. \end{aligned}$$

Since H_1 = constant, $H_1' = 0$, that is $k_3 = 0$. Thus we have

$$\nabla_{V_1}^2 V_1 = -\varepsilon_1\varepsilon_2 V_1 + \varepsilon_2\varepsilon_3 k_2 V_3,$$

$$\nabla_{V_1}^3 V_1 = -\varepsilon_1 V_2 - \varepsilon_3 k_2^2 V_2,$$

$$\nabla_{V_1}^4 V_1 = \left(-\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 k_2^2\right) \nabla_{V_1}^2 V_1,$$

and

$$\nabla_{V_1}^5 V_1 = \left(-\varepsilon_1 - \varepsilon_3 k_2^2\right) \nabla_{V_1}^2 V_2,$$

where

$$\nabla_{V_1}^2 V_2 = -\varepsilon_1\varepsilon_2 V_2 - \varepsilon_2\varepsilon_3 k_2^2 V_2,$$

$$\nabla_{V_1}^3 V_2 = \left(-\varepsilon_1 - \varepsilon_3 k_2^2\right) \nabla_{V_1}^2 V_1,$$

$$\nabla^4_{V_1} V_2 = \left(-\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 k_2^2\right)\nabla^2_{V_1} V_2,$$
$$\nabla^5_{V_1} V_2 = \left(\varepsilon_2 + 2\varepsilon_1\varepsilon_2\varepsilon_3 k_2^2 + \varepsilon_2 k_2^4\right)\nabla^2_{V_1} V_1$$

or since $k_2 = \dfrac{k_1}{H_1}$, we obtain

$$\nabla^6_{V_1} V_1 - \left(1 + 2\varepsilon_1\varepsilon_3 \frac{k_1^2}{H_1^2} + \frac{k_1^4}{H_1^4}\right)\nabla^2_{V_1} V_1 = 0.$$

ii) By using definitions of the m_2 and m_3 which are 2^{nd} and 3^{rd} e-curvature functions of α, we get the result. □

Corollary 4.6. *i) If the vector V_1 is time-like, then*

$$\nabla^6_{V_1} V_1 - \left(1 - 2\varepsilon_3 \frac{k_1^2}{H_1^2} + \frac{k_1^4}{H_1^4}\right)\nabla^2_{V_1} V_1 = 0.$$

ii) If the vector V_3 is time-like, then

$$\nabla^6_{V_1} V_1 - \left(1 - 2\varepsilon_1 \frac{k_1^2}{H_1^2} + \frac{k_1^4}{H_1^4}\right)\nabla^2_{V_1} V_1 = 0.$$

iii) If the vector V_1 is time-like, then

$$\nabla^6_{V_1} V_1 - \left(1 - 2\varepsilon_3 \frac{\left(m_2'\right)^2}{(m_3)^2} + \frac{\left(m_2'\right)^4}{(m_3)^4}\right)\nabla^2_{V_1} V_1 = 0.$$

iv) If the vector V_3 is time-like, then

$$\nabla^6_{V_1} V_1 - \left(1 - 2\varepsilon_1 \frac{\left(m_2'\right)^2}{(m_3)^2} + \frac{\left(m_2'\right)^4}{(m_3)^4}\right)\nabla^2_{V_1} V_1 = 0.$$

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