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On Variations of Quasi-Cauchy Sequences in Cone Metric Spaces

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Abstract. A sequence (x_n) of points in a topological vector space valued cone metric space (X, ρ) is called p-quasi-Cauchy if for each $c \in \overset{\circ}{K}$ there exists an $n_0 \in \mathbb{N}$ such that $\rho(x_{n+p}, x_n) - c \in \overset{\circ}{K}$ for $n \geq n_0$, where K is a proper, closed and convex pointed cone in a topological vector space Y with $\overset{\circ}{K} \neq \emptyset$. We investigate p-ward continuity in topological vector space valued cone metric spaces. It turns out that p-ward continuity coincides with uniform continuity not only on a totally bounded subset but also on a connected subset of X.

1. Introduction

A choice of a suitable definition of distance between images naturally leads to an environment in which many possible metrics can be considered simultaneously and cone metric spaces lend themselves to this requirement. One specific instance of this is in the analysis of the structural similarity (SSIM) index of images (see [4, 5, 28]). SSIM is used to improve the measuring of visual distortion between images (see [30]). In both of these contexts the difference between two images is calculated using multiple criteria, which leads in a natural way to consider vector-valued distances. In 1934, Kurepa ([29]) introduced an abstract metric space, in which the metric takes values in an ordered vector space. The metric spaces with vector valued are studied under various names ([31, 33, 38]). Huang and Zhang in 2007 called such spaces as cone metric spaces ([24]). Beg, Abbas, and Nazir [3], Beg, Azam, and Arshad [2] replaced the set of an ordered Banach space by a locally convex Hausdorff topological vector space in the definition of a cone metric and a generalized cone metric space. The connection between topological vector space valued cone metric spaces and standard metric spaces and the respective fixed point results were considered by several authors (see [22, 23, 25, 26, 36]).

Using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([12]), quasi-slowly oscillating continuity, Δ-quasi-slowly oscillating continuity ([13, 14, 20]), ward continuity, ([8]), δ-ward continuity, ([9]), p-ward continuity ([16]), statistical ward continuity, lacunary statistical ward continuity, ([10, 11]). Investigation of some of these kinds of continuities lead some authors to find certain characterizations of uniform continuity of a real function in terms of sequences in the above manner ([37, Theorem 8], [10, Theorem 6], [6, Theorem 1], [7, Theorem 3.8]).

The aim of this paper is to investigate p-ward continuity in topological vector space valued cone metric spaces.

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2. Preliminaries

Let Y be a topological vector space (TVS for short) with its zero vector θ. A nonempty subset K of Y is called a *convex cone* if $K+K \subseteq K$ and $\lambda K \subseteq K$ for $\lambda \geq 0$. A convex cone K is said to be *pointed* if $K \cap (-K) = \{\theta\}$. Now, we first recall the concept of a topological vector space valued cone metric space. Let X be a nonempty set. A vector-valued function $\rho : X \times X \to Y$ satisfying the following conditions is called a *topological vector space valued cone metric* (*TVS-cone metric* for short) on X, and (X, ρ) is said to be a *topological vector space valued cone metric space* (TVS-cone metric space for short):
(CM1) $\rho(x, y) \in K$ for all $x, y \in X$ and $\rho(x, y) = \theta$ if and only if $x = y$;
(CM2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
(CM3) $\rho(x, z) - \rho(x, y) - \rho(y, z) \in K$ for all $x, y, z \in X$.
On the other hand, one can define a partial ordering $\preceq$ with respect to K by $x \preceq y \Leftrightarrow y - x \in K$. Now using the notation $\preceq$ that the statement $\theta \precsim \rho(x, y)$ for all $x, y \in X$ is equivalent to the statement $\rho(x, y) \in K$ for all $x, y \in X$, and (CM3) is equivalent to the statement $\rho(x, z) \preceq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$. In what follows $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \overset{\circ}{K}$, where $\overset{\circ}{K}$ denotes the interior of K. There are topological vector space-valued cone metric spaces (TVS-cone metric spaces) which are not cone metric space (see [35, Example 2.1]). In the following, unless otherwise specified, we always suppose that Y is a locally convex Hausdorff TVS with its zero vector θ, K a proper, closed and convex pointed cone in Y with $\overset{\circ}{K} \neq \emptyset$, and $\preceq$ a partial ordering with respect to K. Throughout this paper, $\mathbb{N}$, and p will denote the set of positive integers, an element of $\mathbb{N}$, respectively. In the sequel, X will always stand for a TVS-cone metric space with a TVS-cone metric ρ. Recently in [19], it was proved that topology of a TVS-cone metric space coincides with a topology given by a metric defined in [22].

3. Results

The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero, and more generally speaking, than that the distance between p-successive terms is tending to zero where by p-successive terms we mean x_{n+p} and x_n. Nevertheless, sequences which satisfy this weaker property are interesting in their own right.

Definition 3.1. A sequence $\mathbf{x} = (x_n)$ of points in X is called *p-quasi-Cauchy* if for each $c \in \overset{\circ}{K}$ there exists an $n_0 \in \mathbb{N}$ such that $\rho(x_{n+p}, x_n) - c \in \overset{\circ}{K}$ for $n \geq n_0$.

We note that for the special case, $p = 1$ we obtain the definition in [18, page 926]), and for the special case X is the real space we obtain [16, Definition 2.1]. A sequence $\boldsymbol{x}$ is quasi-Cauchy when $p = 1$. We will denote the set of all p-quasi-Cauchy sequences by $\Delta_p(X)$ for each $p \in \mathbb{N}$. We have

$$\rho(x_{n+p}, x_n) - \rho(x_{n+p}, x_{n+p-1}) - \rho(x_{n+p-1}, x_{n+p-2}) - \ldots - \rho(x_{n+2}, x_{n+1}) - \rho(x_{n+1}, x_n) \in K$$

for every $p \in \mathbb{N}$. Thus any quasi-Cauchy sequence is also p-quasi-Cauchy for any $p \in \mathbb{N}$, but the converse is not always true as it can be seen by considering the sequence (x_n) defined by $x_{2n-1} = x$ and $x_{2n} = y$ for different elements $x, y \in X$. Any slowly oscillating sequence is p-quasi-Cauchy for each $p \in \mathbb{N}$ ([18]). Cauchy sequences have the property that any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for p-quasi-Cauchy sequences.We see in the following example that for any $p \in \mathbb{N}$ there exists a p-quasi Cauchy sequence which is not a $p - 1$-quasi Cauchy sequence.

Example 3.2. Consider the sequence

$$(\xi_i) = (x_1, x_2, \ldots, x_p, x_1, x_2, \ldots x_p, \ldots, x_1, x_2, \ldots, x_p, \ldots)$$

where $x_1, x_2, \ldots, x_p$ are different elements of X. Then the sequence (ξ_i) is not $p - 1$-quasi Cauchy, not $p - 2$ quasi Cauchy, ..., not quasi Cauchy. It is not Cauchy as well. On the other hand, the subsequence

$$(\eta_i) = (x_1, x_2, \ldots, x_{p-1}, x_1, x_2, \ldots, x_{p-1}, \ldots, x_1, x_2, \ldots, x_{p-1}, \ldots)$$

of p-quasi-Cauchy sequence (ξ_i) is not p-quasi Cauchy.

Definition 3.3. A subset E of X is called *p-ward compact* if any sequence of points in E has a has p-quasi-Cauchy subsequence.

Since any slowly oscillating sequence is also a quasi-Cauchy sequence so is a p-quasi-Cauchy sequence, we see that any slowly oscillating compact subset of X is p-ward compact for any $p \in \mathbb{N}$. We see that any finite subset of X is p-ward compact, the union of finite number of p-ward compact subsets of X is p-ward compact and the intersection of any family of p-ward compact subsets of X is p-ward compact. Furthermore any subset of a p-ward compact set of X is p-ward compact. Any totally bounded subset of X is p-ward compact. Before giving an equivalence of total boundedness and p-ward compactness we give the definition of total boundedness in TV cone metric spaces. A subset E of X is said to be a c-net in X if $X = \bigcup_{z\in A} B(z,c)$ where $B(z,c) = \{x \in X : \rho(z,x) - c \in \overset{\circ}{K}\}$ for a fixed element c of $\overset{\circ}{K}$. (X,ρ) is called totally bounded if it has a finite c-net in X for each $c \in \overset{\circ}{K}$. A subspace (E,ρ_E) of (X,ρ) is said to be *totally bounded* if it is totally bounded as a TVS-cone metric space in its own right (see [36] for the definition in Banach space valued cone metric spaces). A subset E of a TVS-cone metric space X is said to be totally bounded if it is totally bounded as a TVS-cone metric subspace.

Theorem 3.4. *A subset E of X is totally bounded if and only if it is p-ward compact.*

Proof. To prove that total boundedness implies p-ward compactness, take any sequence (x_n) of points in E. Let c be any fixed element of $\overset{\circ}{K}$. Since E can be covered by a finite number of subsets of X of diameter less than c, one of these sets, which we denote by E_1, must contain x_n for infinitely many values of n. We may choose a positive integer n_1 such that $x_{n_1} \in E_1$. Since E_1 is totally bounded, it can be covered by a finite number of subsets of E_1 satisfying $\delta(E_1) - \frac{c}{2} \in \overset{\circ}{K}$. One of these subsets of E_1, which we denote by E_2 with $\delta(E_2) - \frac{c}{3} \in \overset{\circ}{K}$, contains x_n for infinitely many n. Choose a positive integer n_2 such that $n_2 > n_1$ and $x_{n_2} \in E_2$. Since $E_2 \subset E_1$, it follows that $x_{n_2} \in E_1$ as well. Continuing in this way, we obtain for any positive integer k, and a subset E_k of E_{k-1} with $\delta(E_k) - \frac{c}{k+1} \in \overset{\circ}{K}$, and a term $x_{n_k} \in E_k$ of the sequence (x_n), where $n_k > n_{k-1}$. Since all x_{n_k}, $x_{n_{k+1}}$, $x_{n_{k+2}}$, ..., $x_{n_{k+j}}$, ... lie in E_k and $\delta(E_k) - \frac{c}{k+1} \in \overset{\circ}{K}$, it follows that (x_{n_k}) is a p-quasi-Cauchy subsequence of the sequence (x_n). To prove that p-ward compactness implies total boundedness, suppose that E is not totally bounded. Then there exists a $c \in \overset{\circ}{K}$ such that there does not exist a finite c-net. Take any $x_1 \in E$. Then $B_E(x_1,c) \neq E$, so there exists an $x_2 \in E$ such that $c - \rho(x_1,x_2) \notin \overset{\circ}{K}$, i.e. $x_2 \notin B_E(x_1,c)$, and $x_2 \in E$, where $B_E(x,c) = \{y \in E : \rho(x,y) - c \in \overset{\circ}{K}\}$. Then $B_E(x_1,c) \cup B_E(x_2,c) \neq E$. Let $x_3 \notin B_E(x_1,c) \cup B_E(x_2,c)$ i.e. $c - \rho(x_1,x_2) \notin \overset{\circ}{K}$, $c - \rho(x_1,x_3) \notin \overset{\circ}{K}$, and $c - \rho(x_2,x_3) \notin \overset{\circ}{K}$. Continuing the process in this manner, one can obtain a sequence (x_n) of points in E such that

$$i.e.\ c - \rho(x_i,x_n) \notin \overset{\circ}{K} \ \ (i = 1,2,\ldots,n-1) \text{ and } (n = 1,2,\ldots),\ n \neq i.$$

The sequence (x_n) constructed in this manner has no p-quasi-Cauchy subsequence. This completes the proof of the theorem. □

We note that [16, Theorem 2.3] is a special case of Theorem 3.4 for $X = \mathbb{R}$; [11, Theorem 3 (a)⇔ (b)] is a special case of Theorem 3.4 when X is a metric space and $p = 1$.

Corollary 3.5. *A subset of X is slowly oscillating compact if and only if it is p-ward compact for a $p \in \mathbb{N}$ (see* [12, 18]).

Corollary 3.6. *A subset of X is slowly oscillating compact if and only if it is totally bounded (see* [11]).

Corollary 3.7. *A subset of X is ward compact if and only if it is p-ward compact for every $p \in \mathbb{N}$* ([8]).

Theorem 3.8. *If X is p-ward compact, then it is separable.*

Proof. Let X be p-ward compact. X is totally bounded by Theorem 3.4. Take any $c_0 \in \overset{\circ}{K}$. Total boundedness of X implies that X has a finite $\frac{c_0}{n}$-net for each $n \in \mathbb{N}$. Now denote this $\frac{c_0}{n}$-net by $A_n = \{a_1, a_2, \ldots, a_{k_n}\}$. Write $A = \bigcup_{n=1}^{\infty} A_n$. It is clear that A is countable since each A_n is countable. Now we are going to show that $\overline{A} = X$. Let x be any element in X and c be any element in $\overset{\circ}{K}$. Then we can find an $n_0 \in \mathbb{N}$ such that $c - \frac{c_0}{n_0} \in \overset{\circ}{K}$. Then $X = \bigcup_{i=1}^{k_n} B(a_i, \frac{c_0}{n_0})$. There exists an a_j such that $x \in B(a_j, \frac{c_0}{n_0})$. Hence $B(x, c) \bigcap A \neq \emptyset$, so we have $x \in \overline{A}$. Hence $\overline{A} = X$. The proof is therefore complete. □

Since every compact TVS cone metric space is p-ward compact, we obtain that any compact TVS cone metric space is separable.

Now we introduce the definition of p-ward continuity in a TVS-cone metric space in the following.

Definition 3.9. A function defined on a subset E of X is called *p-ward continuous* if it preserves p-quasi-Cauchy sequences, i.e. $(f(x_n))$ is a p-quasi-Cauchy sequence whenever (x_n) is a p-quasi-Cauchy sequence of points in E.

We note that one can obtain [8, page 227, Definition 2], for the real case with the special case $p = 1$, one can also obtain [6, page 328], [11, Definition 1] for the metric space setting with the special case $p = 1$, and one can also obtain [16, Definition 3.1] for the real case. On the other hand, p-ward continuity cannot be obtained by any G-sequential continuity in the manner of [15] and [32].

In connection with p-quasi-Cauchy sequences, slowly oscillating sequences, and convergent sequences the problem arises to investigate the following types of continuity of a function on $\mathbb{R}$.

($\Delta_p(X)$) $(x_n) \in \Delta_p(X) \Rightarrow (f(x_n)) \in \Delta_p(X)$,

($\Delta_p(X)c$) $(x_n) \in \Delta_p(X) \Rightarrow (f(x_n)) \in c$,

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$,

(d) $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta_p(X)$,

(e) $(x_n) \in w \Rightarrow (f(x_n)) \in \Delta_p(X)$,

where w denotes the set of slowly oscillating sequences. We see that $(\Delta_p(X))$ is p-ward continuity of f, and (c) states the ordinary continuity of f. It is easy to see that $(\Delta_p(X)c)$ implies $(\Delta_p(X))$, and $(\Delta_p(X))$ does not imply $(\Delta_p(X)c)$; and $(\Delta_p(X))$ implies (d), and (d) does not imply $(\Delta_p(X))$; and $(\Delta_p(X))$ implies (e), and (e) does not imply $(\Delta_p(X))$; $(\Delta_p(X)c)$ implies (c) and (c) does not imply $(\Delta_p(X)c)$; and (c) is equivalent to (d).
Now we give the implication $(\Delta_p(X))$ implies $(\Delta_1(X))$, i.e. any p-ward continuous function is 1-ward continuous, i.e. ward continuous.

Theorem 3.10. *If f is p-ward continuous on a subset E of X, then it is ward continuous on E.*

Proof. Since the sequence $(x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_n, x_n, \ldots, x_n, \ldots)$ is p-quasi-Cauchy whenever (x_n) is quasi-Cauchy, we have that
$(f(x_1), f(x_1), \ldots, f(x_1), f(x_2), f(x_2), \ldots, f(x_2), \ldots, f(x_n), f(x_n), \ldots, f(x_n), \ldots)$
is p-quasi-Cauchy whenever (x_n) is quasi-Cauchy where the same term repeats p-times. Therefore for every $c \in \overset{\circ}{K}$ there exits an $n_0 \in \mathbb{N}$ such that $c - \rho(f(x_{n+1}), f(x_n) \in \overset{\circ}{K}$ for $n \geq n_0$. This completes the proof of the theorem. □

Corollary 3.11. *If f is p-ward continuous on a subset E of X, then it is sequentially continuous on E in the ordinary case.*

Proof. The proof can be obtained by using a technique analogous to that of Theorem 3.10, so is omitted. □

We note that [16, Corollary 3.3] is a special case of Corollary 3.11 for $X = \mathbb{R}$, [8, Theorem 1] is a special case of Corollary 3.11 for $X = \mathbb{R}$ with $p = 1$.

Now we prove that any uniformly continuous function is p-ward continuous for any $p \in \mathbb{N}$.

Theorem 3.12. *If a function f is uniformly continuous on a subset E of X, then it is p-ward continuous on E for any $p \in \mathbb{N}$.*

Proof. To prove that f is p-ward continuous take any p-quasi-Cauchy sequence (x_n) of points in E. Let $c \in \overset{\circ}{K}$. Uniform continuity of f implies that there exists a $c_0 \in \overset{\circ}{K}$, depending on c, such that $\rho(f(x), f(y)) - c \in \overset{\circ}{K}$ whenever $\rho(x, y) - c_0 \in \overset{\circ}{K}$. For this c_0, there exists an $N = N(c_0) = N_1(c)$ such that $\Delta_p x_n - c_0 \in \overset{\circ}{K}$, whenever $n > N$ where $\Delta_p x_n = \rho(x_{n+p}, x_n)$. Hence $\Delta_p f(x_n) - c \in \overset{\circ}{K}$ if $n > N$, where $\Delta_p f(x_n) = \rho(f(x_{n+p}), f(x_n))$. It follows from this that $(f(x_n))$ is p-quasi-Cauchy. This completes the proof of the theorem. □

Theorem 3.13. *p-ward continuous image of any p-ward compact subset of $\mathbb{R}$ is p-ward compact.*

Proof. The proof can be obtained straightforwardly so is omitted. □

In the following theorem we see that p-ward continuity coincides with uniform continuity on totally bounded subsets of a TVS-cone metric space.

Theorem 3.14. *Any p-ward continuous function defined on a totally bounded subset E of X is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous on E so that there exist a $c_0 \in \overset{\circ}{K}$ and sequences (x_n) and (y_n) of points in E such that $\rho(x_n, y_n) - \frac{c_0}{n} \in \overset{\circ}{K}$ and $c_0 - \rho(f(x_n), f(y_n)) \notin \overset{\circ}{K}$ for all $n \in \mathbb{N}$. On the other hand, by Theorem 3.4, E is p-ward compact, therefore there is a p-quasi-Cauchy subsequence of (x_{n_k}) of (x_n). There is also a p-quasi-Cauchy subsequence of $(y_{n_{k_j}})$ of (y_{n_k}) since E is p-ward compact. It is clear that the corresponding sequence $(x_{n_{k_j}})$ is also p-quasi-Cauchy, since $(y_{n_{k_j}})$ is p-quasi-Cauchy and

$$\rho(x_{n_{k_j}}, x_{n_{k_{j+p}}}) - \rho(x_{n_{k_j}}, y_{n_{k_j}}) - \rho(y_{n_{k_j}}, y_{n_{k_{j+p}}}) - \rho(y_{n_{k_{j+p}}}, x_{n_{k_{j+p}}}) \in K.$$

It is easy to construct a sequence $\mathbf{z} = (z_j)$ which is p-quasi-Cauchy while $f(\mathbf{z}) = (f(z_j))$ is not p-quasi-Cauchy. This contradiction completes the proof of the theorem. □

Corollary 3.15. *The set of p-ward continuous functions on a totally bounded subset E of X is equivalent to the set of uniformly continuous functions on E.*

Corollary 3.16. *The set of slowly oscillating continuous functions on a totally bounded subset E of X is equivalent to the set of p-ward continuous functions on E (see [18]).*

When the domain is restricted to a totally bounded subset of X, p-ward continuity coincides with not only ward continuity, but also uniform continuity.

Lemma 3.17. *If (ξ_n, η_n) is a sequence of ordered pairs of points in a connected subset E of X such that $\lim_{n\to\infty} \rho(\xi_n, \eta_n) = 0$, then there exists a p-quasi-Cauchy sequence (x_n) with the property that for any positive integer i there exists a positive integer j such that $(\xi_i, \eta_i) = (x_{j-p}, x_j)$.*

Theorem 3.18. *If f is p-ward continuous on a connected subset E of X for a positive integer p, then it is uniformly continuous on E.*

Proof. Let p be any positive integer. To prove that p-ward continuity of f on E implies uniform continuity of f on E suppose that f is not uniformly continuous on E so that there exists a $c \in \overset{\circ}{K}$ such that for any $\delta \in \overset{\circ}{K}$, there exist $x, y \in E$ with $\rho(x, y) - \delta \in \overset{\circ}{K}$ but $c - \rho(f(x), f(y)) \notin \overset{\circ}{K}$. Hence for each $n \in \mathbb{N}$, there exist x_n and y_n in E such that $\rho(x_n, y_n) - \frac{c}{n} \in \overset{\circ}{K}$, and $c - \rho(f(x_n), f(y_n)) \notin \overset{\circ}{K}$. By Lemma 3.17, one can construct a p-quasi-Cauchy sequence (t_n) which has a subsequence $(z_n) = (t_{k_n})$ such that $\lim_{n\to\infty} \rho(z_{n+p}, z_n) = 0$, but $c - \rho(f(z_{n+p}), f(z_n)) \notin \overset{\circ}{K}$. Therefore the sequence $(f(z_n))$ is not p-quasi-Cauchy. Thus this contradiction yields that p-ward continuity implies uniform continuity. This completes the proof of the theorem. □

Combining Theorem 3.18 and Theorem 3.12 we have the following:

Corollary 3.19. *A function f is uniformly continuous on a connected subset E if and only if it is p-ward continuous for any $p \in \mathbb{N}$.*

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case p-ward continuity, i.e. uniform limit of a sequence of p-ward continuous functions is p-ward continuous.

Theorem 3.20. *If (f_n) is a sequence of p-ward continuous functions defined on a subset E of X, and (f_n) is uniformly convergent to a function f, then f is p-ward continuous on E.*

Proof. Let $c \in \overset{\circ}{K}$, $x \in X$. Then there exists a positive integer N such that $\rho(f_n(x), f(x)) - \frac{c}{3} \in \overset{\circ}{K}$ for all $x \in E$ whenever $n \geq N$. Let (x_n) be a p-quasi-Cauchy sequence of points in E. Since f_N is p-ward continuous, there exists a positive integer N_1, depending on c, and greater than N, such that $\rho(f_N(x_{n+p}), f_N(x_n)) - \frac{c}{3} \in \overset{\circ}{K}$ for $n \geq N_1$. Now for $n \geq N_1$, we have

$$\rho(f(x_{n+p}), f(x_n)) - \rho(f(x_{n+p}), f_N(x_{n+p})) + \rho(f_N(x_{n+p}), f_N(x_n)) - \rho(f_N(x_n), f(x_n)) \in \overset{\circ}{K}$$

and so $\rho(f(x_{n+p}), f(x_n)) - \frac{c}{3} - \frac{c}{3} - \frac{c}{3} \in \overset{\circ}{K}$, therefore $\rho(f(x_{n+p}), f(x_n)) - c \in \overset{\circ}{K}$. This completes the proof of the theorem. □

Theorem 3.21. *The set of all p-ward continuous functions defined on a subset E of X is a closed subset of the set of all continuous functions on E, i.e. $\overline{\Delta_p(X)FC(E)} = \Delta_p(X)FC(E)$, where $\Delta_p(X)FC(E)$ is the set of all p-ward continuous functions on E, $\overline{\Delta_p(X)FC(E)}$ denotes the set of all cluster points of $\Delta_p(X)FC(E)$.*

Proof. Let f be any element in the closure of $\Delta_p(X)FC(E)$. Then there exists a sequence of points in $\Delta_p(X)FC(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is p-ward continuous, take any p-quasi-Cauchy sequence (x_n) of points in E. Let $c \in \overset{\circ}{K}$. Since (f_k) converges to f, there exists an N such that $\rho(f(x), f_n(x)) - \frac{c}{3} \in \overset{\circ}{K}$ for all $x \in E$ and for all $n \geq N$. As f_N is p-ward continuous, there is an N_1, greater than N, such that $\rho(f_N(x_{n+p}), f_N(x_n)) - \frac{c}{3} \in \overset{\circ}{K}$ for all $n \geq N_1$. Hence

$$\rho(f(x_{n+p}), f(x_n)) - \rho(f(x_{n+p}), f_N(x_{n+p})) - \rho(f_N(x_{n+p}), f_N(x_n)) - \rho(f(x_n), f_N(x_n)) \in \overset{\circ}{K}$$

for all $n \geq N_1$, therefore $\rho(f(x_{n+p}), f(x_n)) - \frac{c}{3} - \frac{c}{3} - \frac{c}{3} \in \overset{\circ}{K}$, so $\rho(f(x_{n+p}), f(x_n)) - c \in \overset{\circ}{K}$.
This completes the proof of the theorem. □

Corollary 3.22. *The set of all p-ward continuous functions on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on E.*

Proof. The proof follows from the preceding theorem. □

4. Conclusion

We introduce and investigate the notion of a p-quasi Cauchy sequence in a topological vector space valued cone metric space. All results are new not only for a topological vector space valued cone metric space, but also new for a metric space. It turns out that p-ward compactness coincides with total boundedness, and p-ward continuity coincides with uniform continuity both on a totally bounded subset, and on a connected subset of X. We note that the results in this paper are also valid both in Banach space valued cone normed spaces ([34]) and in topological vector space valued cone normed spaces as any topological vector space valued cone normed space is a topological vector space valued cone metric space with the induced topological vector space valued cone metric $\rho(x, y) = |||x - y|||$. For a further study, we suggest to investigate p-quasi-Cauchy sequences of fuzzy points. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see [1, 17, 27] for the definitions and related concepts in fuzzy setting). We also suggest to investigate p-quasi-Cauchy sequences of double sequences (see [21] for the definitions and related concepts in the double case).

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