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Some Symmetry Identities for the Unified Apostol-Type Polynomials and Multiple Power Sums

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Abstract. The purpose of this paper is to introduce and investigate a new unification of unified family of Apostol-type polynomials and numbers.We obtain some symmetry identities between these polynomials and the generalized sum of integer powers.We give explicit relations for these polynomials and recurrence relations related to multiple power sums.

1. Introduction, Definitions and Notations

The generalized Apostol-Bernoulli polynomials $B_n^{\alpha}(x,\lambda)$ of order α in x are defined by Luo and Srivastava in [10, 11] through the generating relation

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty} B_n^{\alpha}(x,\lambda)\frac{t^n}{n!}, \quad \left(\left|t+\log\lambda\right|<2\pi,\ \ 1^{\alpha}:=1\right),$$

where α and λ are arbitrary real or complex parameters and $x\in\mathbb{R}$. The Apostol-Bernoulli polynomials and the Apostol-Bernoulli numbers can be obtained from the generalized Apostol-Bernoulli polynomials by

$$B_n(x,\lambda)=B_n^{1}(x,\lambda),\ B_n(\lambda)=B_n(0,\lambda) \quad n\in\mathbb{N},$$

respectively. The case $\lambda=1$ in the above relations gives the classical Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n. Recently for the arbitrary real or complex parameters α and λ and $x\in\mathbb{R}$. Luo in [9, 11] and Liu et al. [8] generalized the Apostol-Euler polynomials $E_n^{\alpha}(x,\lambda)$ by the generating relation

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{x.t}=\sum_{n=0}^{\infty} E_n^{\alpha}(x,\lambda), \quad \left(\left|t+\log\lambda\right|<\pi,\, 1^{\alpha}:=1\right).$$

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The Apostol-Euler polynomials and the Apostol-Euler numbers are given by

$$E_n(x,\lambda) = E_n^1(x,\lambda),\, E_n(\lambda) = E_n(1,\lambda),$$

respectively. The above relations give the classical Euler polynomials $E_n(x)$ and the Euler members E_n when $\lambda = 1$.

Let $x \in \mathbb{R}$. For arbitrary real or complex parameters α and λ, the Apostol-Genocchi polynomials $G_n(x,\lambda)$ of order α are defined by [8, 11, 17]

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{(\alpha)} e^{x.t} = \sum_{n=0}^{\infty} G_n^{\alpha}(x,\lambda)\frac{t^n}{n!}, \quad (\left|t + \log\lambda\right| \pi, 1^{\alpha} := 1).$$

The Apostol-Genocchi polynomials and the Apostol-Genocchi numbers are given by

$$G_n(x,\lambda) = G_n^{(1)}(x,\lambda), \qquad G_n(\lambda) = G_n(0,\lambda),$$

respectively. When $\lambda = 1$, the above relation give the classical Genocchi polynomials $G_n(x)$ and the classical Genocchi numbers G_n. We should note that the above polynomials have recently been studied and investigated in the papers [7, 11, 17, 20, 22, 23]. A unified Apostol-Bernoulli, Apostol-Euler, Apostal-Genocchi polynomials are defined by Simsek et al. [2] as:

$$\begin{aligned} f_{a,b}(x;t,a,b) &= \frac{2^{1-t}t^k e^{xt}}{\beta^b e^t - a^b} = \sum_{n=0}^{\infty} Y_{n,\beta}(x;k,a,b)\frac{t^n}{n!}, \quad \left(\left|t + b\log(\frac{\beta}{a})\right| < 2\pi\right) \\ (x &\in \mathbb{R}, k \in \mathbb{N}_0,\ a, b \in \mathbb{R}^+, \beta \in \mathbb{C}), \end{aligned}$$

where the associated numbers are given by

$$Y_{n,\beta}(0;k,a,b) = Y_{n,\beta}(k,a,b).$$

The following unified Apostol-Bernoulli, Euler and Genocchi polynomials are defined by Ozarslan in [13] as

$$\begin{aligned} f_{a,b}^{(\alpha)}(x;t,a,b) &= \left(\frac{2^{1-t}t^k}{\beta^b e^t - a^b}\right)^{(\alpha)} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}, \\ k &\in \mathbb{N}_0, a, b \in \mathbb{R}\setminus\{0\}, \alpha, \beta \in \mathbb{C}. \end{aligned} \tag{1}$$

For the convergence of the series involved in (1) we have

i. If $a^b > 0$ and $k \in \mathbb{N}$, then $\left|t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi$, $1^{\alpha} := 1$, $x \in \mathbb{R}$, $\beta \in \mathbb{C}$;

ii. If $a^b > 0$ and $k = 0$, then $0 < \mathrm{Im}\left(t + b\log\left(\frac{\beta}{a}\right)\right) < 2\pi$, $1^{\alpha} := 1$, $x \in \mathbb{R}$, $\beta \in \mathbb{C}$;

iii. If $a^b < 0$, then $\left|t + b\log\left(\frac{\beta}{a}\right)\right| < \pi$, $1^{\alpha} := 1$, $x \in \mathbb{R}$, $k \in \mathbb{N}_0$, $\beta \in \mathbb{C}$ (for details on this subject see [13]).

Remark 1.1. Setting $k = a = b = 1$ and $\beta = \lambda$ in (1), we get

$$P_{n,\lambda}^{(\alpha)}(x;1,1,\lambda) = B_n^{(\alpha)}(x,\lambda),$$

where $B_n^{(\alpha)}(x,\lambda)$ are the generalized Apostol-Bernoulli polynomials of order α.

Remark 1.2. Choosing $k+1=-a=b=1$ and $\beta=\lambda$ in (1), we get

$$P_{n,\lambda}^{(\alpha)}(x;0,-1,1)=E_n^{(\alpha)}(x,\lambda),$$

where $E_n^{(\alpha)}(x,\lambda)$ are the generalized Apostol-Euler polynomials of order α.

Remark 1.3. Letting $k=-2a=b=1$ and $2\beta=\lambda$ in (1), we get

$$P_{n,\frac{\lambda}{2}}^{(\alpha)}\left(x;1,\frac{-1}{2},1\right)=G_n^{(\alpha)}(x,\lambda),$$

where $G_n^{(\alpha)}(x,\lambda)$ are the generalized Apostol-Genocchi polynomials of order α.

Recently, Garg et al. in [5, 19]) introduced the following generalization of the Hurwitz-Lerch zeta function $\Phi(z,s;a)$:

$$\Phi_{\mu,\nu}^{(P,\sigma)}(z,s;a)=\sum_{n=0}^{\infty}\frac{(\mu)_{pn}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s}$$

$\left(m\in\mathbb{C},a,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-,p,\sigma\in\mathbb{R},p<\sigma\text{ when }s,z\in\mathbb{C}\ (|z|=1);\ p=\sigma\text{ and }R(s-m+\nu)>0,\text{ when }|z|=1\right)$.
It is obvious that

$$\Phi_{(\mu,1)}^{(1,1)}(z,s;a)=\Phi_\mu^*(z,s;a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s} \tag{2}$$

(for details on this subject, see [5, 19]).

The multiple power sums are defined by Luo in [10, 11] as follows:

$$S_k^{(l)}(m,\lambda)=\sum_{\substack{0\leq\nu_1\leq\ldots\leq\nu_m=l\\ \nu_1+\nu_2+\ldots+\nu_m=n}}\binom{l}{\nu_1,\nu_2,\ldots,\nu_m}\lambda^{\nu_1+2\nu_2+\ldots+m\nu_m}(\nu_1+2\nu_2+\ldots+m\nu_m)^k. \tag{3}$$

From (3), we have

$$\left(\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}\right)^{(l)}=\lambda^{(-l)}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-l)^{n-k}S_k^{(l)}(m,\lambda)\right\}\frac{t^n}{n!}. \tag{4}$$

From (4), for $l=1$, we have

$$\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}=\frac{1}{\lambda}\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}S_k(m,\lambda)\right\}\frac{t^n}{n!}. \tag{5}$$

The generalized Stirling numbers of second kinds $S(n,v,a,b,\beta)$ of order v are defined in [16] as follows:

$$\sum_{n=0}^{\infty}S(n,v,a,b,\beta)\frac{t^n}{n!}=\frac{(\beta^b e^t-a^b)^v}{v!} \tag{6}$$

By using (1), we easily have the following relations

i. $P_{n,\beta}^{(\alpha_1+\alpha_2)}(x+y,k,a,b)=\sum_{k=0}^{n}\binom{n}{k}P_{k,\beta}^{(\alpha_1)}(x;k,a,b)P_{n-k,\beta}^{(\alpha_2)}(y;k,a,b);$

ii. $P_{n,\beta}^{(\alpha_1+\alpha_2)}(x,k,a,b)=\sum_{l=0}^{n}\binom{n}{l}P_{l,\beta}^{(\alpha_1)}(0;k,a,b)P_{n-l,\beta}^{(\alpha_2)}(x,k,a,b).$

In last ten years many mathematicians studied the Apostol-type Bernoulli polynomials. Srivastava in [17] and Srivastava et al. in [18, 19, 23] investigated and proved some relations and theorems for Bernoulli-type polynomials and Apostol-Bernoulli-type polynomials. Luo in [10, 11] proved the multiplication theorems for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order and multiple alternating sums. Luo et al. in [9] and Liu et al. in [8] gave some symmetry relations between for the Apostol-Bernoulli polynomials and Apostol-Euler polynomials.

Firstly, Karende et al. in [6] introduced the unification of the Bernoulli and Euler polynomials. Ozden et al. in [13–15] introduced and investigated the unified Apostol-Bernoulli, Apostol-Euler, Apostol-Genocchi polynomials.

They applied the Mellin transformation to this unified polynomial $Y_{n,\beta}(x;k,a,b)$ and obtained the unified zeta function $J_\beta(n;k,a,b)$. Ozarslan in [13] defined uniform form of the Apostol-Bernoulli, Euler and Genocchi polynomials $P_{n,\beta}^{(\alpha)}(x,k,a,b)$ of order α. He gave the explicit representation of this unified family in terms of a Gaussian hypergeometric function. Also, he gave the recurrence relations and symmetry properties for the unified Apostol-type polynomials. At most, B.S. H-Desouky et al. in [3, 4] defined and investigated the unified family $M_n^{(r)}(x,k,\bar{\alpha}_r)$ of generalized Apostol-Bernoulli, Euler and Genocchi polynomials. They proved some recurrence relations and the addition formula for this unified family $M_n^{(r)}(x,k,\bar{\alpha}_r)$.

This paper is organized as follows. In Section 2, we give some explicit relation for the Unified Apostol type polynomials. In section 3, we prove the relation between Hurwitz-Lerch zeta function and the unified Apostol-type polynomials and give some symetry relations for these unified Apostol-type polynomials.

2. Some Explicit Relations for the Unified Family of Generalized Apostol-Type Polynomials

In this section, we aim to obtain the explicit relations of the polynomials $P_{n,\beta}^{(\alpha)}(x,k,a,b)$. By the motivation of the M.El-Mikkay and F.Altan's article [12], we prove some relations for these polynomials and give the relations between the unified family of generalized Apostol-type polynomials and the Stirling numbers of second kind $S(n,\upsilon,a,b,\beta)$ of order υ.

For $\alpha=1$, we write again the equation (1) as

$$F(x;k,a,b,\beta,t)=\sum_{n=0}^{\infty}P_{n,\beta}^{(1)}(x,k,a,b)\frac{t^n}{n!}=\frac{2^{1-k}t^k}{\beta^b e^t-a^b}e^{xt}. \tag{7}$$

We can obtain the following equation easily from (7)

$$F(x+1;k,a,b,\beta,t)=e^tF(x;k,a,b,\beta,t), \tag{2.1.a}$$

$$(\beta^b e^t+a^b)F(x;k,a,2b,\beta,2t)=F(2x;k,a,b,\beta,t), \tag{2.1.b}$$

$$(\beta^b e^t - a^b)F(x;k,a,2b,\beta,2t) = F(2x;k,-a,2b+1,\beta,t), \tag{2.1.c}$$

$$F(x;k,a,b,\beta,t)F(y;k,a,b,\beta,t) = F^{(2)}(0;k,a,b,\beta,t)e^{(x+y)t}, \tag{2.1.d}$$

$$\left(\beta^b e^t + a^b\right)F(x;k,a^2,b,\beta^2,t) = 2^k F(2x;k,a,b,\beta,t) \tag{2.1.e}$$

and

$$F(x;k,a,b,\beta,t)F(y;k,a,b,\beta,t) = F(k;a,b,\beta,t)F(x+y;k,a,b,\beta,t). \tag{2.1.f}$$

Proposition 2.1. *The unified Apostol-type Bernoulli polynomials satisfy the following relation*

$$\beta^b P_{n,\beta}^{(\alpha)}(x+1;k,a,b) - a^b P_{n,\beta}^{(\alpha)}(x;k,a,b) = 2^{1-k}\frac{n!}{(n-k)!}P_{n-k,\beta}^{(\alpha-1)}(x;k,a,b). \tag{8}$$

Proof. From (1), we have

$$\beta^b\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{(\alpha)} e^{(x+1)t} - a^b\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{(\alpha)} e^{xt} = 2^{1-k}t^k\sum_{n=0}^{\infty}P_{n,\beta}^{(\alpha-1)}(x,k,a,b)\frac{t^n}{n!}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left\{\beta^b P_{n,\beta}^{(\alpha)}(x+1,k,a,b) - a^b P_{n,\beta}^{(\alpha)}(x,k,a,b)\right\}\frac{t^n}{n!}\\
= \quad & 2^{1-k}\sum_{n=k}^{\infty}\frac{(n+k)!}{n!}P_{n,\beta}^{(\alpha-1)}(x,k,a,b)\frac{t^{n+k}}{(n+k)!}.
\end{aligned}$$

□

Comparing of the coefficient of $\frac{t^n}{n!}$ of both sides, we have (8).

Corollary 2.2. *The following relation is true*

$$P_{n,\beta}(x+1,k,a,b) = \sum_{l=0}^{n}\binom{n}{l}P_{l,\beta}(x,k,a,b).$$

Proof. This corollary can be proved by using (2.1.a). □

Corollary 2.3. *The following relation holds true:*

$$\beta^b\sum_{p=0}^{n}\binom{n}{p}P_{p,\beta}(x,k,a,2b)2^p + a^b P_{n,\beta}(x,k,a,2b)2^n = 2^k P_{n,\beta}(2x,k,a,b).$$

Proof. By using (2.1.b), we have the result. □

Corollary 2.4. *The following relation holds true*

$$\beta^b \sum_{q=0}^{n} \binom{n}{q} P_{q,\beta}(x,k,a,2b)2^q - a^b P_{n,\beta}(x,k,a,2b)2^n = P_{n,\beta}(2x,k,-a,2b+1).$$

Proof. From (2.1.c), we obtain the corollary. □

Corollary 2.5. *There is the following relation*

$$\begin{aligned}\sum_{q=0}^{n} \binom{n}{q} P_{n-q,\beta}(x,k,a,b)P_{q,\beta}(y,k,a,b) &= \sum_{r=0}^{n} \binom{n}{r} P^{(2)}_{r,\beta}(k,a,b)(x+y)^{n-r}, \\ &= \sum_{q=0}^{n} \binom{n}{q} P^{(2)}_{q,\beta}(x+y,k,a,b)P^{(-1)}_{n-q,\beta}(k,a,b).\end{aligned}$$

Proof. From (2.1.d), we have the result. □

Corollary 2.6. *The following relation holds true*

$$\beta^b \sum_{q=0}^{n} \binom{n}{q} \left\{P_{q,\beta^2}(x;k,a^2,b)2^r + a^b P_{n,\beta^2}(x;k,a^2,b)\right\} 2^n = 2^k P_{n,\beta}(2x,k,a,b).$$

Proof. By using (2.1.e), we have the result. □

Corollary 2.7. *The unified Apostol-type Bernoulli polynomials satisfy the following relation*

$$\sum_{q=0}^{n} \binom{n}{q} P_{q,\beta}(x,k,a,b)P_{n-q,\beta}(y,k,a,b) = \sum_{q=0}^{n} \binom{n}{q} P_{q,\beta}(k,a,b)P_{n-q,\beta}(x+y;k,a,b).$$

Proof. By using (2.1.f), we prove easily the corollary. □

Theorem 2.8. *There is the following relation between the λ-Stirling numbers of second kinds and the unified Apostol-type polynomials $P^{(\alpha)}_{n,\beta}(x;k,a,b)$:*

$$\alpha! a^{b\alpha} \sum_{r=0}^{n} \binom{n}{r} P^{(\alpha)}_{n-r,\beta}(x;k,a,b) S\left(r,\alpha,\left(\frac{\beta}{a}\right)^b\right) = 2^{(1-k)\alpha} \frac{n!}{(n-k\alpha)!} x^{n-k\alpha}. \tag{9}$$

Proof. The λ-Stirling numbers of second kinds is defined by Simsek in [16] as

$$\frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k,\lambda)\frac{t^n}{n!}. \tag{10}$$

By using equation (1) and (10), we write

$$\begin{aligned}\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!} &= \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{(\alpha)} e^{xt}\\ &= \frac{2^{(1-k)\alpha}t^{k\alpha}}{a^{b\alpha}\left(\left(\frac{\beta}{a}\right)^b e^t - 1\right)^{\alpha}} e^{xt} = \frac{2^{(1-k)\alpha}t^{k\alpha}e^{xt}}{a^{b\alpha}\alpha!\sum_{n=0}^{\infty} S\left(n,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^n}{n!}},\end{aligned}$$

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}a^{b\alpha}\alpha!\sum_{n=0}^{\infty} S\left(n,\alpha,\left(\frac{\beta}{a}\right)^b\right)\frac{t^n}{n!} = 2^{(1-k)\alpha}t^{k\alpha}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}.$$

By using Cauchy product, comparing the coefficient of $\frac{t^n}{n!}$, we have (9). □

Theorem 2.9. *There is the following relation between the generalized Stirling numbers of second kind $S(n,\upsilon,a,b,\beta)$ of order υ and the unified Apostol-type Bernoulli polynomials $P_{n,\beta}^{(\alpha)}(x;k,a,b)$:*

$$P_{n-k\gamma,\beta}^{(\alpha-\gamma)}(x;k,a,b) = \frac{2^{(k-1)\gamma}\gamma!(n-k\gamma)!}{n!}\sum_{r=0}^{n}\binom{n}{r}P_{n-r,\beta}^{(\alpha)}(x;k,a,b)S(r,\gamma,a,b,\beta). \tag{11}$$

Proof. By using (1) and (6), we have

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha-\gamma)}(x;k,a,b)\frac{t^n}{n!} = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{(\alpha)} e^{xt}\left(\frac{\beta^b e^t - a^b}{2^{1-k}t^k}\right)^{\gamma}$$

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha-\gamma)}(x;k,a,b)\frac{t^{n+k\gamma}}{n!} = 2^{(k-1)\gamma}\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}\sum_{n=0}^{\infty} S(r,\gamma,a,b,\beta)\frac{t^n}{n!}.$$

Using the Cauchy product, comparing the coefficient of $\frac{t^n}{n!}$, we have (11). □

3. Some Symmetry Identities for the Unified Generalized Apostol-Type Polynomials

Kurt in [7] proved some symmetry identities for the Apostol-Bernoulli and Apostol-Euler polynomials. Ozarslan in [13] proved some symmetry identities for the unified Apostol-type polynomials. In this section, we give new one symmetry identities for the unified Apostol-type polynomials. Also we prove the relation between for the unified Apostol-type polynomials and Hurwitz-Lerch Zeta Function.

Theorem 3.1. *The following symmetry relations for the unified Apostol-type polynomials hold true:*

$$\begin{aligned}&\sum_{m=0}^{c-1}\left(\frac{\beta}{a}\right)^{bm}\sum_{l=0}^{n}\binom{n}{l}P_{n-l,\beta}(dx;k,a,b).c^{n-k-l}(dm)^l\\ =\;&\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}\sum_{l=0}^{n}\binom{n}{l}P_{n-l,\beta}(cx;k,a,b).d^{n-k-l}(cm)^l.\end{aligned} \tag{12}$$

Proof. We have

$$\begin{aligned}
f(t) &= \frac{2^{1-k}t^k}{\beta^b e^{dt}-a^b} e^{cdxt} \frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{ct}-a^b} \\
&= \frac{1}{d^k}\left(\frac{2^{1-k}(dt)^k}{\beta^b e^{dt}-a^b}\right) e^{cdxt} a^{b(d-1)} \left(\frac{1-\left(\frac{\beta}{a}\right)^{bd} e^{dct}}{1-\left(\frac{\beta}{a}\right)^{b} e^{ct}}\right) \\
&= a^{b(d-1)} d^{(-k)} \sum_{n=0}^{\infty} P_{n,\beta}(cx;k,a,b) \frac{d^n t^n}{n!} \sum_{m=0}^{d-1} \left(\frac{\beta}{a}\right)^{bm} e^{ctm} \\
&= a^{b(d-1)} d^{(-k)} \sum_{r=0}^{\infty} P_{r,\beta}(cx;k,a,b) \frac{d^r t^r}{r!} \sum_{m=0}^{d-1} \left(\frac{\beta}{a}\right)^{bm} \sum_{l=0}^{\infty} (cm)^l \frac{t^l}{l!} \\
&= a^{b(d-1)} d^{(-k)} \sum_{n=0}^{\infty} \sum_{m=0}^{d-1} \left(\frac{\beta}{a}\right)^{bm} \left\{ \sum_{l=0}^{n} \binom{n}{l} P_{n-l,\beta}(cx;k,a,b) d^{n-l} (cm)^l \right\} \frac{t^n}{n!}.
\end{aligned}$$

In a similar manner

$$\begin{aligned}
f(t) &= \frac{2^{(1-k)}t^k}{\beta^b e^{ct}-a^b} e^{cdxt} \left(\frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{dt}-a^b}\right) \\
&= \frac{1}{c^k}\left(\frac{2^{1-k}(ct)^k}{\beta^b e^{ct}-a^b}\right) e^{cdxt} a^{b(d-1)} \left(\frac{1-(\frac{\beta}{a})^{bd} e^{dct}}{1-(\frac{\beta}{a})^{b} e^{dt}}\right) \\
&= \frac{a^{b(d-1)}}{c^k} \sum_{r=0}^{\infty} P_{r,\beta}(dx;k,a,b) c^r \frac{t^n}{n!} \sum_{m=0}^{c-1} \left(\frac{\beta}{a}\right)^{bm} e^{dmt} \\
&= a^{b(d-1)} c^{(-k)} \sum_{m=0}^{c-1} \left(\frac{\beta}{a}\right)^{bm} \sum_{r=0}^{\infty} P_{r,\beta}(dx;k,a,b) c^r \frac{t^r}{r!} \sum_{l=0}^{\infty} (dm)^l \frac{t^l}{l!} \\
&= a^{b(d-1)} c^{(-k)} \sum_{n=0}^{\infty} \sum_{m=0}^{c-1} \left(\frac{\beta}{a}\right)^{bm} \left\{ \sum_{l=0}^{n} \binom{n}{l} P_{n-l,\beta}(dx;k,a,b) c^{n-l} (dm)^l \right\} \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we have result. □

Theorem 3.2. *The unified Apostol-type numbers satisfy the following relation:*

$$\begin{aligned}
& c^k \sum_{m=0}^{d-1} (\frac{\beta}{a})^{mb} \sum_{l=0}^{n} \binom{n}{l} \left\{ P_{n-l,\beta}(k,a,b) d^{n-l} c^l (dx+m)^l \right\} \\
=\; & d^k \sum_{m=0}^{c-1} (\frac{\beta}{a})^{bm} \sum_{l=0}^{n} \binom{n}{l} \left\{ P_{n-l,\beta}(k,a,b) c^{n-l} d^l (xc+m)^l \right\}. \qquad (13)
\end{aligned}$$

Proof. Let

$$\begin{aligned}
f(t) &= \frac{2^{1-k}t^k}{\beta^b e^{dt}-a^b} e^{cdxt} \frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{ct}-a^b} \\
&= \frac{a^{b(d-1)}}{d^k} \left(\frac{2^{1-k}(dt)^k}{\beta^b e^{dt}-a^b}\right) \left(\frac{1-\left(\frac{\beta}{a}\right)^{bd} e^{dct}}{1-\left(\frac{\beta}{a}\right)^{b} e^{ct}}\right) e^{cdxt}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{b(d-1)}}{d^k}\sum_{n=0}^{\infty}P_{n,\beta}(k,a,b)\frac{d^n t^n}{n!}\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}e^{(m+dx)ct}\\
&= \frac{a^{b(d-1)}}{d^k}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}\left(P_{n-l,\beta}(k,a,b)d^{n-l}c^l\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}(m+dx)^l\right)\frac{t^n}{n!}.
\end{aligned}
$$

In a similar manner

$$
f(t)=\frac{2^{(1-k)}t^k}{\beta^b e^{ct}-a^b}e^{cdxt}\left(\frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{dt}-a^b}\right)
$$

$$
=\frac{a^{b(d-1)}}{d^k}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\left\{P_{n-l,\beta}(k,a,b)c^{n-l}d^l\sum_{m=0}^{c-1}\left(\frac{\beta}{a}\right)^{bm}(m+cx)^l\right\}\frac{t^n}{n!}.
$$

From here, we obtain (13). □

Theorem 3.3. *For all $c, d, r \in \mathbb{N}$, $s, p \in \mathbb{N}_0$, we have the following symmmetry relation between Hurwitz-Lerch zeta function and unified Apostol-type polynomials:*

$$
\begin{aligned}
& d^k\sum_{p=0}^{n-k\alpha}\binom{n-k\alpha}{p}d^{n-k\alpha-p}\sum_{s=0}^{p}\binom{p}{s}\left\{\sum_{s=0}^{r}\binom{r}{s}(-\alpha)^{r-s}S_s^{(\alpha)}\left(d,\left(\frac{\beta}{a}\right)^b\right)P_{n-r,\beta}(dy;k,a,b)c^p\right.\\
&\left.\Phi_\alpha^*\left(\left(\frac{\beta}{a}\right)^\alpha,p+k\alpha-n,cx\right)\right\}\\
=\ & c^k\sum_{p=0}^{n-k\alpha}\binom{n-k\alpha}{p}c^{n-k\alpha-p}\sum_{s=0}^{p}\binom{p}{s}\left\{\sum_{s=0}^{r}\binom{r}{s}(-\alpha)^{r-s}S_s^{(\alpha)}\left(c,\left(\frac{\beta}{a}\right)^b\right)d^pP_{n-r,\beta}(cx;k,a,b)\right.\\
&\left.\Phi_\alpha^*\left(\left(\frac{\beta}{a}\right)^\alpha,p+k\alpha-n,dy\right)\right\}. \qquad (14)
\end{aligned}
$$

Proof. Let

$$
\begin{aligned}
f(t) &= \frac{t^{k(\alpha+1)}2^{(1-k)(\alpha+1)}e^{cdxt}(\beta^{bd}e^{cdt}-a^{bd})^\alpha e^{cdyt}}{(\beta^b e^{dt}-a^b)^{\alpha+1}(\beta^b e^{ct}-a^b)^{\alpha+1}}\\
&= \frac{t^{k\alpha}2^{(1-k)\alpha}e^{cdxt}}{c^k(\beta^b e^{dt}-a^b)^{\alpha+1}}\left(\frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{ct}-a^b}\right)^\alpha\frac{(ct)^k 2^{1-k}}{\beta^b e^{ct}-a^b}dyct.
\end{aligned}
$$

By using (1), (2) and (4)

$$
\begin{aligned}
&= \frac{2^{(1-k)\alpha}a^{bd\alpha-b\alpha-b}\beta^{-\alpha b}t^{k\alpha}}{c^k(-1)^{\alpha+1}}\sum_{m=0}^{\infty}\binom{m+\alpha}{m}\left(\frac{\beta^b}{a^b}\right)^m e^{dt(m+cx)}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}(-\alpha)^{n-s}\\
&\times S_s^{(\alpha)}\left(d,\left(\frac{\beta}{a}\right)^b\right)c^r\frac{t^r}{r!}\sum_{n=0}^{\infty}P_{p-r,\beta}(dy,k,a,b)c^n\frac{t^n}{n!}\\
&= \sum_{n=k\alpha}^{\infty}\frac{n!}{(n-k\alpha)!}\frac{2^{(1-k)\alpha}a^{b(d\alpha-\alpha-1)}\beta^{-\alpha b}}{c^k(-1)^{\alpha+1}}\left\{\sum_{p=0}^{n-k\alpha}\binom{n-k\alpha}{p}d^{n-k\alpha-p}\sum_{s=0}^{p}\binom{p}{s}\sum_{s=0}^{r}\binom{r}{s}(-\alpha)^{r-s}\right.\\
&\left.S_s^{(\alpha)}\left(d,\left(\frac{\beta}{a}\right)^\alpha\right)c^pP_{p-r,\beta}(dy,k,a,b)c^p\Phi_\alpha^*\left(\left(\frac{\beta}{a}\right)^b,p+k\alpha-n,cx\right)\right\}\frac{t^n}{n!}.
\end{aligned}
$$

In a similar manner

$$\begin{aligned} f(t) &= \frac{t^{k(\alpha+1)}2^{(1-k)(\alpha+1)}e^{cdyt}(\beta^{bd}e^{cdt}-a^{bd})^{\alpha}e^{cxdt}}{(\beta^{b}e^{ct}-a^{b})^{\alpha+1}(\beta^{b}e^{dt}-a^{b})^{\alpha+1}} \\ &= \sum_{n=k\alpha}^{\infty}\frac{n!}{(n-k\alpha)!}2^{(1-k)\alpha}a^{b(d\alpha-\alpha-1)}\beta^{-\alpha b}\left\{\sum_{p=0}^{n-k\alpha}\binom{n-k\alpha}{p}d^{n-k\alpha-p}\sum_{s=0}^{p}\binom{p}{s}\right. \\ &\quad \left.\sum_{s=0}^{r}\binom{r}{s}(-\alpha)^{r-s}S_{s}^{(\alpha)}\left(c,\left(\frac{\beta}{a}\right)^{b}\right)d^{p}P_{p-r,\beta}(cx,k,a,b)\Phi_{\alpha}^{*}\left(\left(\frac{\beta}{a}\right)^{b},p-k\alpha-n,dy\right)\right\}\frac{t^{n}}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we have (14). $\square$

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