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Topological Groups of Bounded Homomorphisms on a Topological Group

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Abstract. We consider a few types of bounded homomorphisms on a topological group. These classes of bounded homomorphisms are, in a sense, weaker than the class of continuous homomorphisms. We show that with appropriate topologies each class of these homomorphisms on a complete topological group forms a complete topological group.

1. Introduction

When one deals with the concept of a bounded set in a topological vector space, there are some tools which are absolutely handy: for example, the scalar multiplication and absorbing zero neighborhoods. But when we want to consider the notion of a bounded set in a topological group, the situation is completely different. There are neither scalar field nor absorbing neighborhoods. One may define a bounded subset in a topological group by taking group multiplication instead of scalar multiplication in a topological vector space; but this approach does not match our intuition for a bounded subset, since, for example, the multiplicative group S^1 is not bounded in this definition.

Following [1], a subset B of a topological group G is called *bounded* if for each neighborhood U of the identity element e_G of G there is a positive integer n such that $B \subset U^n$.

We give here a result concerning this kind of boundedness in the products of topological groups.

Theorem 1.1. *Let $\{G_\alpha : \alpha \in \Lambda\}$ be a set of abelian topological groups and $G = \prod_{\alpha\in\Lambda} G_\alpha$ with the product topology. Then $B \subseteq G$ is bounded if and only if there are finitely many sets $B_{\alpha_1}, \ldots, B_{\alpha_k}$ such that for each $i \le k$, B_{α_i} is a bounded set in G_{α_i} and $B \subset \prod_{i\le k} B_{\alpha_i} \times \prod_{\alpha\in\Lambda\setminus\{\alpha_1,\ldots,\alpha_k\}} G_\alpha$.*

Proof. Suppose $B \subseteq G$ is bounded. Put

$$B_{\alpha_i} = \{x \in G_{\alpha_i} : \exists\, \mathbf{y} = (y_\alpha) \in B \text{ and } x \text{ is } \alpha_i\text{-th coordinate of } \mathbf{y}\}.$$

Each B_{α_i}, $i \le k$, is bounded. For, if U_i, $i \le k$, is a neighborhood of the identity in G_{α_i} put

$$U = U_{\alpha_1} \times \ldots \times U_{\alpha_k} \times \prod_{\alpha\ne\alpha_i} G_\alpha.$$

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U is a neighborhood of the identity in G. Therefore, there is a positive integer n with $B \subseteq U^n$, so that $B_{\alpha_i} \subseteq U_i^n$. It is easy to see that $B \subseteq B_{\alpha_1} \times \ldots B_{\alpha_k} \times \prod_{\alpha\in\Lambda\setminus\{\alpha_1,\ldots,\alpha_k\}} G_\alpha$.

Suppose now that for a set $B \subseteq G$, there exist sets $B_{\alpha_1}, \ldots, B_{\alpha_k}$ such that B_{α_i} is a bounded set in G_{α_i}, $i \le k$, and $B \subset \prod_{i\le k} B_{\alpha_i} \times \prod_{\alpha\in\Lambda\setminus\{\alpha_1,\ldots,\alpha_k\}} G_\alpha$. Let U be an arbitrary neighborhood of the identity element of G; without loss of generality, we may assume that U is of the form

$$U = U_1 \times \ldots \times U_k \times \prod_{\alpha\in\Lambda\setminus\{\alpha_1,\ldots,\alpha_k\}} G_\alpha,$$

where U_i is a neighborhood of the identity element in G_{α_i}, $i \le k$. For each $i \le k$ there is a positive integer n_i such that $B_{\alpha_i} \subset U_i^{n_i}$. Put $n = \max\{n_1, \ldots, n_k\}$. Evidently $B \subset U^n$, i.e. B is bounded in G. □

Bounded homomorphisms and their algebraic and topological structures on any topological algebraic structure are of interest for their own rights and also for their applications in other areas of mathematics. For examples, bounded operators on a topological vector space with suitable topologies form topological algebras; also, there is a spectral theory for these classes of bounded operators with some useful applications (see [3, 6, 7] for ample information). Therefore, it will be of interest to consider different types of bounded homomorphisms on a topological group and to investigate which topological and algebraic properties of the underlying topological group can be transferred to the mentioned classes of homomorphisms. In this note, we examine a few notions of bounded homomorphisms on a topological group. These classes of bounded homomorphisms contain continuous homomorphisms. We equip each class of such bounded homomorphisms by an appropriate topology. It turns out that they constitute complete topological groups provided that the underlying topological group is assumed to be complete and its singletons are bounded.

For terminology and notations used in this paper, we refer the reader to [4], as well as for an abstract and comprehensive taste of topological groups and related notions. All topological groups in this note are assumed to be *abelian*. It is well known that a topological group has a local base at the identity element consisting of symmetric neighborhoods. Throughout the paper we consider only such bases. The identity element of a group G will be denoted by e_G.

2. Results

Definition 2.1. Let G and H be two topological groups. A homomorphism $T : G \to H$ is said to be

(1) nb-*bounded* if there exists a neighborhood U of e_G such that $T(U)$ is bounded in H;

(2) bb-*bounded* if for every bounded set $B \subset G$, $T(B)$ is bounded in H.

The set of all nb-bounded (bb-bounded) homomorphisms from a topological group G to a topological group H is denoted by $\mathrm{Hom}_{\mathrm{nb}}(G, H)$ ($\mathrm{Hom}_{\mathrm{bb}}(G, H)$). We write $\mathrm{Hom}(G)$ instead of $\mathrm{Hom}(G, G)$.

Remark 2.2. For topological groups G and H the following holds:

$$\mathrm{Hom}_{\mathrm{nb}}(G, H) \subset \mathrm{Hom}_{\mathrm{bb}}(G, H).$$

Let $T : G \to H$ be an nb-bounded homomorphism. Then it is bb-bounded. For, suppose $B \subset G$ is a bounded set. Since T is nb-bounded there is a neighborhood U of e_G such that $T(U)$ is bounded in H. Boundedness of B implies $B \subset U^n$ for some natural number n. We prove that $T(B)$ is bounded in H. Let V be a neighborhood of e_H. Boundedness of $T(U)$ implies that there is $m \in \mathbb{N}$ such that $T(U) \subset V^m$. Then

$$T(B) \subset T(U^n) = (T(U))^n \subset V^{mn},$$

i.e. $T(B)$ is bounded in H.

Note that the converse is not true as the following example shows.

Example 2.3. Let $G = \mathbb{C} - \{0\}$ be the group of non-zero complex numbers, and $H = G^{\mathbb{N}}$, the group of all sequences of elements of G, with the pointwise multiplication and the product topology. Consider the identity homomorphism 1_H on H. It is easy to see that 1_H is bb-bounded, but it is not nb-bounded since H is not locally bounded.

Remark 2.4. One may verify that each continuous homomorphism is bb-bounded. Let G and H be topological groups, $T : G \to H$ be a continuous homomorphism, and B a bounded subset of G. Suppose a neighborhood V of e_H is given. There exists a neighborhood U of e_G such that $T(U) \subset V$. Also, since B is bounded in G, there is an $n \in \mathbb{N}$ with $B \subset U^n$. Thus,

$$T(B) \subset T(U^n) = (T(U))^n \subset V^n.$$

Nevertheless, unlike in the case of bounded operators on topological vector spaces, there is no more relation between continuous homomorphisms on a topological group and bounded ones, which is explained in the example below.

Example 2.5. Let G be S^1, the multiplicative group of all complex numbers of modulus 1, with the trivial (anti-discrete) topology, and H be S^1 with the topology inherited from $\mathbb{C}$. Consider the identity homomorphism i from G into H. Clearly, i is not continuous. But it is nb-bounded, as well as bb-bounded.

Remark 2.6. Note that if G is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, we have two notions for a bounded set; one of them is related to the group structure of G and another considers the scalar multiplication. It is easy to see that these aspects of bounded sets have the same meaning.

Let G be a group. For homomorphisms T and S on G, define TS and T^{-1} by

$$TS(x) := T(x)S(x) \text{ and } T^{-1}(x) := (T(x))^{-1}, \ \ x \in G.$$

It is easy to see that with these operations, the class of all homomorphisms on a group G forms a group.

Now, assume G is a topological group. The class of all nb-bounded homomorphisms on G equipped with the topology of uniform convergence on some neighborhood of e_G is denoted by $\mathsf{Hom}_{\mathsf{nb}}(G)$. Observe that a net (S_α) of nb-bounded homomorphisms converges uniformly on a neighborhood U of e_G to a homomorphism S if for each neighborhood V of e_G there exists an α_0 such that for each $\alpha \geq \alpha_0$, $(S_\alpha S^{-1})(U) \subset V$.

The class of all bb-bounded homomorphisms on G endowed with the topology of uniform convergence on bounded sets is denoted by $\mathsf{Hom}_{\mathsf{bb}}(G)$. Note that a net (S_α) of bb-bounded homomorphisms uniformly converges to a homomorphism S on a bounded set $B \subset G$ if for each neighborhood V of e_G there is an α_0 with $(S_\alpha S^{-1})(B) \subset V$ for each $\alpha \geq \alpha_0$.

The class of all continuous homomorphisms on G equipped with the topology of c-convergence is denoted by $\mathsf{Hom}_{\mathsf{c}}(G)$. A net (S_α) of continuous homomorphisms c-converges to a homomorphism S if for each neighborhood W of e_G, there is a neighborhood U of e_G such that for every neighborhood V of e_G there exists an α_0 with $(S_\alpha S^{-1})(U) \subset VW$ for each $\alpha \geq \alpha_0$.

Note that $\mathsf{Hom}_{\mathsf{c}}(G)$ and $\mathsf{Hom}_{\mathsf{bb}}(G)$ form subgroups of the group of all homomorphisms on G. On the other hand, $\mathsf{Hom}_{\mathsf{nb}}(G)$ is not a group, in general; the identity homomorphism on G in Example 2.3 does not belong to $\mathsf{Hom}_{\mathsf{nb}}(G)$.

Theorem 2.7. *The operations of multiplication and inversion are continuous in* $\mathsf{Hom}_{\mathsf{nb}}(G)$ *(with respect to the topology of uniform convergence on some neighborhood of* e_G*).*

Proof. Suppose (T_α) and (S_α) are two nets of nb-bounded homomorphisms which are convergent uniformly on some neighborhood U of e_G to the nb-bounded homomorphisms T and S, respectively. Let W be an arbitrary neighborhood of e_G. There is a neighborhood V of e_G with $VV \subset W$. There exist some α_1 and α_2

such that $(T_\alpha T^{-1})(U) \subset V$ for each $\alpha \geq \alpha_1$ and $(S_\alpha S^{-1})(U) \subset V$ for each $\alpha \geq \alpha_2$. Choose α_0 with $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. If $\alpha \geq \alpha_0$, then we have

$$(T_\alpha S_\alpha)(TS)^{-1}(U) \subset (T_\alpha T^{-1})(U)(S_\alpha S^{-1})(U) \subset VV \subset W.$$

Also, because of continuity of the inversion we have

$$(T_\alpha T^{-1})^{-1}(U) = (T_\alpha T^{-1}(U))^{-1} \subset W,$$

for sufficiently large α. This completes the proof. □

Theorem 2.8. *The operations of multiplication and inversion are continuous in* $\mathsf{Hom_{bb}}(G)$ *with respect to the topology of uniform convergence on bounded sets.*

Proof. Suppose (T_α) and (S_α) are two nets of bb-bounded homomorphisms which are convergent uniformly on bounded sets to the bb-bounded homomorphisms T and S, respectively. Fix a bounded set $B \subset G$. Let W be an arbitrary neighborhood of e_G. There exists a neighborhood V of e_G with $VV \subset W$. There are some α_1 and α_2 such that $(T_\alpha T^{-1})(B) \subset V$ for each $\alpha \geq \alpha_1$ and $(S_\alpha S^{-1})(B) \subset V$ for each $\alpha \geq \alpha_2$. Choose α_0 with $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. If $\alpha \geq \alpha_0$, then we have

$$(T_\alpha S_\alpha)(TS)^{-1}(B) \subset (T_\alpha T^{-1})(B)(S_\alpha S^{-1})(B) \subset VV \subset W.$$

Since the inversion in G is continuous, for sufficiently large α we have

$$(T_\alpha T^{-1})^{-1}(B) = (T_\alpha T^{-1}(B))^{-1} \subset W$$

which completes the proof. □

Theorem 2.9. *The operations of multiplication and inversion are continuous in* $\mathsf{Hom_c}(G)$*, i.e.* $\mathsf{Hom_c}(G)$ *is a topological group.*

Proof. Suppose (T_α) and (S_α) are two nets of continuous homomorphisms c-converging to the homomorphisms T and S, respectively. Let W and V be arbitrary neighborhoods of e_G. There exist neighborhoods W_1 and V_1 of e_G such that $W_1W_1 \subset W$ and $V_1V_1 \subset V$. There are a neighborhood U of e_G and some indices α_1 and α_2 with $(T_\alpha T^{-1})(U) \subset V_1W_1$ for each $\alpha \geq \alpha_1$ and $(S_\alpha S^{-1})(U) \subset V_2W_2$ for each $\alpha \geq \alpha_2$. Choose α_0 such that $\alpha_0 \geq \alpha_1$ and $\alpha_0 \geq \alpha_2$. If $\alpha \geq \alpha_0$, then we have

$$(T_\alpha S_\alpha)(TS)^{-1}(U) \subset (T_\alpha T^{-1})(U)(S_\alpha S^{-1})(U) \subset V_1W_1V_2W_2 \subset VW.$$

Now, continuity of the inversion implies

$$(T_\alpha T^{-1})^{-1}(U) = (T_\alpha T^{-1}(U))^{-1} \subset VW,$$

for sufficiently large α. This completes the proof. □

In this part, we investigate whether or not each class of bounded homomorphisms in the assumed topology is uniformly complete.

Lemma 2.10. *Suppose* (S_α) *is a net of continuous homomorphisms which converges to the homomorphism* S *in the* c*-convergence topology. Then,* S *is also continuous.*

Proof. Let an arbitrary neighborhood W of e_G be given. Choose a neighborhood V of e_G such that $V^3 \subset W$. There are a neighborhood U of e_G and an α_0 with $(S_\alpha S^{-1})(U) \subset V^2$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. There exists a neighborhood $U_1 \subset U$ such that $S_\alpha(U_1) \subset V$. From here, together with $(S_\alpha S^{-1})(U_1) \subset V^2$, it follows

$$S(U_1) \subset S_\alpha(U_1)V^2 \subset V^3 \subset W,$$

as desired. □

Lemma 2.11. *If (S_α) is a net of* bb*-bounded homomorphisms which converges to the homomorphism S in the topology of uniform convergence on bounded sets. Then S is* bb*-bounded.*

Proof. Let W be an arbitrary neighborhood of e_G. Fix a bounded set $B \subset G$. There is an α_0 such that $(S_\alpha S^{-1})(B) \subset W$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. There exists a natural n with $S_\alpha(B) \subset W^n$, so that

$$S(B) \subset (S_\alpha(B))W \subset W^n W = W^{n+1},$$

as we wanted. □

Remark 2.12. The class $\mathsf{Hom}_{\mathsf{nb}}(G)$ can contain a Cauchy sequence whose limit is not an *nb*-bounded homomorphism; in other words, $\mathsf{Hom}_{\mathsf{nb}}(G)$ is not uniformly complete in the assumed topology. Let G be as in Example 2.3 and (S_n) be a sequence of homomorphisms on G which are defined as follows:

$$S_n((x_n)) = (x_1, \ldots, x_n, 1, \ldots).$$

Each S_n is nb-bounded. For, if U_n is the neighborhood of e_G defined by

$$U_n = \{(x_n), |x_i - 1| \leq \frac{1}{2}, i = 1, 2, \ldots, n\},$$

then, as it is easy to see, $S_n(U_n)$ is bounded in G. On the other hand, it is not difficult to show that (S_n) is uniformly convergent to the identity homomorphism 1_G on G. But we have seen in Example 2.3 that 1_G is not nb-bounded.

Now, we are going to find conditions under which each class of considered bounded homomorphisms is topologically complete. In the case of bounded operators on topological vector spaces, absorbing neighborhoods and local convexity are two fruitful tools for discovering conditions (see [3]). In the topological group version, it turns out that boundedness of every singleton is a handy tool. By [4, Theorem 7.4], when G is a connected topological group, then it is absorbed by positive powers of any neighborhood of e_G, so that every singleton is bounded. Note that connectedness is not a necessary condition; for example, the additive group $\mathbb{Q}$ of rational numbers is totally disconnected, but every its singleton is bounded. There are examples of abelian topological groups whose singletons are not bounded. For example, let G be an abelian topological group and consider the topological group $H = G \times \mathbb{Z}_2$. Then $G \times \{0\}$ is a zero neighborhood which is not absorbing, so that singletons are not bounded.

In what follows we assume that every singleton in the underlying topological group G is bounded.

Theorem 2.13. *If G is a complete group, then $\mathsf{Hom}_{\mathsf{c}}(G)$ is complete.*

Proof. Suppose (T_α) is a Cauchy net in $\mathsf{Hom}_{\mathsf{c}}(G)$ and W is an arbitrary neighborhood of e_G. Choose a neighborhood V of e_G with $VV \subset W$. Find a neighborhood U of e_G and an index α_0 such that $(T_\alpha {T_\beta}^{-1}(U)) \subset VV$ for each $\alpha \geq \alpha_0$ and for each $\beta \geq \alpha_0$. Suppose $x \in G$.

First, assume that $x \in U$. Then $T_\alpha {T_\beta}^{-1}(x) \in W$. Thus, $((T_\alpha(x))$ is a Cauchy net in G.

For an arbitrary $x \in G$, there is a positive integer n such that $x \in U^n$. Since the product of Cauchy nets in a topological group is again Cauchy, it follows that $(T_\alpha(x))$ is a Cauchy net in G, so that it converges. Put $T(x) := \lim T_\alpha(x)$ for each $x \in G$. Since this convergence holds in $\mathsf{Hom}_{\mathsf{c}}(G)$, by Lemma 2.10, T is also continuous, and this completes the proof. □

Theorem 2.14. *If G is a complete group, then $\mathsf{Hom}_{\mathsf{bb}}(G)$ is also complete.*

Proof. Suppose (T_α) is a Cauchy net in $\mathsf{Hom}_{\mathsf{bb}}(G)$ and W is an arbitrary neighborhood of e_G. One can find a neighborhood V of e_G such that $VV \subset W$. Since every singleton $x \in G$ is bounded, there is an α_0 such that $T_\alpha {T_\beta}^{-1}(x) \in V$ for each $\alpha \geq \alpha_0$ and each $\beta \geq \alpha_0$. One concludes that $((T_\alpha(x))$ is a Cauchy net in G. Therefore, it converges. Put $T(x) := \lim T_\alpha(x)$, for each $x \in G$. Now fix a bounded set $B \subset G$. For sufficiently large α and β, we have $T_\alpha {T_\beta}^{-1}(B) \subset V$, so that for each $x \in B$,

$$T_\alpha {T_\beta}^{-1}(x) \in V.$$

For sufficiently large β we have $T_\beta(x)T^{-1}(x) \in V$, and therefore

$$T_\alpha T^{-1}(x) = T_\alpha {T_\beta}^{-1}(x)T_\beta(x)T^{-1}(x) \in VV \subset W.$$

Since this convergence holds in $\mathsf{Hom}_{\mathsf{bb}}(G)$, by Lemma 2.11, T is also bb-bounded, as required. □

Remark 2.15. Note that when G is a complete group, then $\mathsf{Hom}_{\mathsf{nb}}(G)$ might fail to be a complete topological group. Consider Example 2.3 and Remark 2.12.

3. Conclusion

We considered here two kinds of bounded homomorphisms in topological groups. It would be interesting to consider other sorts of bounded homomorphisms as well. Recall that a subset B of a topological groups G is $\aleph_0$-*bounded* if for each neighborhood U of e_G there is a countable set C such that $B \subset CU$; $B \subset G$ is *Menger-bounded* (or shortly M-*bounded*) if for each sequence $(U_n : n \in \mathbb{N})$ of neighborhoods of e_G there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of G such that $B \subset \bigcup_{n\in\mathbb{N}} F_nU_n$. If in the last definition F_ns are singletons, then B is said to be *Rothberger-bounded* (shortly R-*bounded* (see [2, 5] for more details about these classes of sets). A homomorphism T between topological groups G and H is said to be:

(*i*) $\omega\mathsf{b}$-*bounded* if for each $\aleph_0$-bounded subset B of G, $T(B)$ is bounded in H;

(*ii*) mb-*bounded* if for each M-bounded subset B of G, $T(B)$ is bounded in H;

(*iii*) rb-*bounded* if for each R-bounded subset B of G, $T(B)$ is bounded in H.

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