



New Types of Continuity in 2-Normed Spaces

Hüseyin Çakalli^a, Sibel Ersan^b

^aFaculty of Arts and Sciences, Maltepe University, Marmara Eğitim Köyü, TR 34857, Maltepe, İstanbul-Turkey

^bFaculty of Engineering And Natural Sciences, Maltepe University, Marmara Eğitim Köyü, TR 34857, Maltepe, İstanbul-Turkey

Abstract. A sequence (x_n) of points in a 2-normed space X is statistically quasi-Cauchy if the sequence of difference between successive terms statistically converges to 0. In this paper we mainly study statistical ward continuity, where a function f defined on a subset E of X is statistically ward continuous if it preserves statistically quasi-Cauchy sequences of points in E . Some other types of continuity are also discussed, and interesting results related to these kinds of continuity are obtained in 2-normed space setting.

1. Introduction

We need to refer to the concept of a 2-norm if there is a physical situation or an abstract concept where the norm topology does not work but the 2-norm topology does work, especially in the cases when one needs two-inputs for a particular output, but with one main input and other input is required to complete the process. So that one may expect that the concept of continuity and any concept involving continuity in 2-normed spaces will find very important roles not only in pure mathematics, but also in other branches of sciences involving mathematics, especially in computer science, information theory, economics, and biological science.

The concept of 2-normed spaces was introduced and studied by Siegfried Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of papers in German language published in *Mathematische Nachrichten* (see for example [15, 23–25]). This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. of USA in 1969 entitled 2-Banach spaces ([42]). In the same year Gähler published another paper on this theme in the same journal ([26]). A.H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with S. Gähler and S.C. Gupta ([27]) in 1975 also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by A.H. Siddiqi ([40]). Since then, this concept has been studied by many authors ([16, 27, 33, 35]).

The idea of statistical convergence was first given under the name almost convergence by Zygmund in Warsaw in 1935 [43]. Statistical convergence was formally introduced by Fast [19] and later was reintroduced by Schoenberg [39], and also independently by Buck [4]. This concept has become an active area of active

2010 *Mathematics Subject Classification.* Primary 40A05; Secondary 46A45, 46A50, 46A70

Keywords. Statistical convergence, quasi-Cauchy sequences, continuity

Received: 18 June 2015; Revised: 06 August 2015; Accepted: 07 August 2015

Communicated by Ljubiša D.R. Kočinac

Email addresses: huseyincakalli@maltepe.edu.tr; hcakalli@gmail.com (Hüseyin Çakalli), sibelersan@maltepe.edu.tr; sibelersan@gmail.com (Sibel Ersan)

research. It has been applied in various areas ([13, 20, 22, 32, 34]). Gürdal and Pehlivan studied statistical convergence in a 2-normed space ([28–30]).

Using the idea of sequential continuity and sequential compactness, many kinds of continuity and compactness were introduced and investigated ([1, 5, 6, 9, 12]). Statistical convergence was studied in topological space setting in [10] and [14]. Statistical ward continuity of a real function and statistical ward compactness of a subset E of \mathbb{R} were introduced by Çakalli in [7] in the sense that a real function f is called statistically ward continuous on E if the sequence $(f(x_n))$ is statistically quasi-Cauchy whenever $x = (x_n)$ is a statistically quasi-Cauchy sequence of points in E .

The aim of this paper is to investigate statistical ward continuity in 2-normed spaces, and prove interesting theorems.

2. Preliminaries

First of all, some definitions and notation will be given in the following. Throughout this paper, \mathbb{N} , and \mathbb{R} will denote the set of all positive integers, and the set of all real numbers, respectively. First we recall the definition of a 2-normed space.

Definition 2.1. ([23]) Let X be a real linear space with $\dim X > 1$ and $\|\cdot, \cdot\| : X^2 \rightarrow \mathbb{R}$ a function. Then $(X, \|\cdot, \cdot\|)$ is called a linear 2-normed space if

1. $\|x, y\| = 0 \Leftrightarrow x$ and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for $\alpha \in \mathbb{R}$ and $x, y, z \in X$. The function $\|\cdot, \cdot\|$ is called the 2-norm on X . Throughout this paper by X we will mean a 2-normed space with a 2-norm $\|\cdot, \cdot\|$.

Observe that in any 2-normed space $(X, \|\cdot, \cdot\|)$ we have $\|\cdot, \cdot\|$ is nonnegative, $\|x - z, x - y\| = \|x - z, y - z\|$, and $\forall x, y \in X, \alpha \in \mathbb{R} \|x, y + \alpha x\| = \|x, y\|$. The set of semi-norms $\{p_z\}$ forms a locally convex topological vector space, and the topology formed by this family of semi-norms gives the required topology on X . Since $\dim X \geq 2$, for each $x \in X$ there exists a $y \in X$ such that x and y are linearly independent, and hence by (1), $p_y(x) = \|(x, y)\| \neq 0$. Thus the locally convex topological vector space induced by the set of semi-norms $\{p_z\}$ separates points in X , i.e. X is a Hausdorff space.

A classical example is the 2-normed space $X = \mathbb{R}^2$ with the 2-norm $\|\cdot, \cdot\|$ defined by $\|\mathbf{a}, \mathbf{b}\| = |a_1 b_2 - a_2 b_1|$ where $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$. This is the area of the parallelogram determined by the vectors \mathbf{a} and \mathbf{b} . An infinite dimensional example is the 2-normed space $X = \ell_\infty$ with the 2-norm $\|\cdot, \cdot\|$ defined by $\|\mathbf{a}, \mathbf{b}\| = \sup_{i \in \mathbb{N}} \sup_{j \in \mathbb{N}} |a_i b_j - a_j b_i|$, $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in \ell_\infty$ where ℓ_∞ denotes the set of bounded sequences of points in \mathbb{R} ([2, 17, 18]).

A sequence (x_n) of points in X is said to converge to $L \in X$ in the 2-normed space X if $\lim_{n \rightarrow \infty} \|x_n - L, z\| = 0$ for every $z \in X$. This is denoted by $\lim_{n \rightarrow \infty} \|x_n, z\| = \|L, z\|$. A sequence (x_n) of points in X is said to be a Cauchy sequence with respect to the 2-norm X if for each $\varepsilon > 0$, $\lim_{n, m \rightarrow \infty} \|x_n - x_m, z\| = 0$ for each $z \in X$. A sequence (x_k) is statistically converges to L in a 2-normed space X if for each $\varepsilon > 0$, $\lim_{k \rightarrow \infty} \frac{1}{k} |\{k \in \mathbb{N} : \|x_k - L, z\| \geq \varepsilon\}| = 0$ for each nonzero z in X ([28, 29]).

3. Results

In this section we investigate the notion of statistical ward continuity of a function. First we give a definition of a statistically quasi-Cauchy sequence in a 2-normed space.

Definition 3.1. A sequence (x_k) of points in X is statistically quasi-Cauchy if $st - \lim_{k \rightarrow \infty} \Delta x_k = 0$, that is, for each $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \|\Delta x_k, z\| \geq \epsilon\}| = 0, \forall z \in X,$$

where $\Delta x_k = x_{k+1} - x_k$.

We note that any quasi-Cauchy sequence is statistically quasi-Cauchy not only in the real case and in the metric setting, but also in a 2-normed space. A statistically convergent sequence is statistically quasi-Cauchy. However the converse is not always true, i.e. there are statistically quasi-Cauchy sequences which are not statistically convergent. Any Cauchy sequence is statistically quasi-Cauchy, but the converse is not always true.

Now we give the definition of statistical ward compactness of a subset of X .

Definition 3.2. A subset E of X is called statistically ward compact if any sequence of points in E has a statistically quasi-Cauchy subsequence.

First, we note that any finite subset of X is statistically ward compact, the union of two statistically ward compact subsets of X is statistically ward compact, and the intersection of any family of statistically ward compact subsets of X is statistically ward compact. Furthermore, any subset of a statistically ward compact set is statistically ward compact. Any compact subset of X is also statistically ward compact.

A function f on a subset E of X is sequentially continuous at x_0 if for any sequence (x_n) of points in E converging to x_0 , we have $(f(x_n))$ converges to $f(x_0)$. f is sequentially continuous on E if it is sequentially continuous at every point of E . This is equivalent to the statement that f preserves convergent sequences of points in E .

Definition 3.3. A function f on a subset E of X is said to be statistically sequentially continuous at x_0 if for any sequence (x_n) of points in E statistically converging to x_0 , we have $(f(x_n))$ statistically converges to $f(x_0)$ (see also [10, 14]).

Theorem 3.4. If $st - \lim_{n \rightarrow \infty} x_n = x_0$ implies that $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ for an $x_0 \in X$, then f is a constant function.

Proof. Let x_0 be a fixed element of X . We can construct a sequence as

$$x_n = \begin{cases} t, & \text{if } n = k^2 \text{ for a positive integer } k \\ x_0, & \text{if otherwise} \end{cases}.$$

It is easy to see that $st - \lim_{n \rightarrow \infty} x_n = x_0$. The sequence $(f(x_n))$ defined by

$$f(x_n) = \begin{cases} f(t), & \text{if } n = k^2 \text{ for a positive integer } k \\ f(x_0), & \text{if otherwise} \end{cases}$$

is convergent. Thus the subsequence

$$(w_n) = (f(x_{n^2})) = (f(x_1), f(x_4), f(x_9), \dots, f(x_{n^2}), \dots)$$

of the sequence $(f(x_n))$ is also convergent to $f(x_0)$. Since for all $n \in \mathbb{N}$, $w_n = f(x_{n^2}) = f(t)$, it follows that $f(t) = f(x_0)$. This completes the proof. \square

Definition 3.5. A function defined on a subset E of X is called statistically ward continuous if it preserves statistically quasi-Cauchy sequences, i.e. $(f(x_n))$ is statistically quasi-Cauchy whenever (x_n) is.

We note that the composition of two statistically ward continuous functions is statistically ward continuous. Now we prove that the sum of two statistically ward continuous functions is statistically ward continuous.

Proposition 3.6. *The sum of two statistically ward continuous functions is statistically ward continuous.*

Proof. Let f and g be statistically ward continuous functions on a subset E of X . To prove that $f + g$ is statistically ward continuous on E , take any statistically quasi-Cauchy sequence (x_k) in E . Then $(f(x_k))$ and $(g(x_k))$ are statistically quasi-Cauchy sequences. Let $\varepsilon > 0$ be given. Since $(f(x_k))$ and $(g(x_k))$ are statistically quasi-Cauchy, we have $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |\Delta(f(x_k))| \geq \frac{\varepsilon}{2}\}| = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |\Delta(g(x_k))| \geq \frac{\varepsilon}{2}\}| = 0$. Hence $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |\Delta(f(x_k) + g(x_k))| \geq \varepsilon\}| = 0$ which follows from the following inclusion $\{k \in I_n : |\Delta((f + g)(x_k))| \geq \varepsilon\} \subset \{k \leq n : |\Delta(f(x_k))| \geq \frac{\varepsilon}{2}\} \cup \{k \leq n : |\Delta(g(x_k))| \geq \frac{\varepsilon}{2}\}$. \square

Concerning statistically quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on X :

1. (x_n) is statistical quasi-Cauchy $\Rightarrow (f(x_n))$ is statistically quasi-Cauchy.
2. (x_n) is statistical quasi-Cauchy $\Rightarrow (f(x_n))$ is convergent.
3. (x_n) is convergent $\Rightarrow (f(x_n))$ is convergent.
4. (x_n) is convergent $\Rightarrow (f(x_n))$ is statistically quasi-Cauchy.
5. (x_n) is statistically sequentially convergent $\Rightarrow (f(x_n))$ is statistically sequentially convergent.
6. (x_n) is statistical quasi-Cauchy $\Rightarrow (f(x_n))$ is quasi-Cauchy.
7. (x_n) is statistical quasi-Cauchy $\Rightarrow (f(x_n))$ is statistically convergent.

In the previous statements (1) is statistical ward continuity, (3) is the ordinary sequential continuity and (5) is a statistical sequential continuity of the function f . It is obvious that (2) \Rightarrow (1) while (1) does not imply (2), (1) \Rightarrow (4) while (4) does not imply (1), (2) \Rightarrow (3) while (3) does not imply (2) and lastly (3) is equivalent to (4). (1) implies (6), but (6) does not imply (1). Now we prove that (1) implies (5) in the following.

Theorem 3.7. *If $f : X \rightarrow X$ is statistically ward continuous on a subset E of X , then it is statistically sequentially continuous on E .*

Proof. Let (x_k) be any statistically convergent sequence of points in E with a statistical limit x_0 . Hence for all $z \in X$

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \|x_k - x_0, z\| \geq \varepsilon\}| = 0.$$

Then the sequence $\xi = (\xi_n)$ defined by

$$\xi_n = \begin{cases} x_k, & \text{if } n = 2k - 1 \text{ for a positive integer } k \\ x_0, & \text{if } n \text{ is even} \end{cases}$$

is also statistically convergent to x_0 . Therefore it is a statistically quasi-Cauchy sequence. As f is statistically ward continuous on E , the transformed sequence $f(\xi) = (f(\xi_n))$ obtained by

$$f(\xi_n) = \begin{cases} f(x_k), & \text{if } n = 2k - 1 \text{ for a positive integer } k \\ f(x_0), & \text{if } n \text{ is even} \end{cases}$$

is also statistically quasi-Cauchy. Now it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \|f(x_k) - f(x_0), z\| \geq \varepsilon\}| = 0$$

for every $z \in X$. It implies that the sequence $(f(x_n))$ statistically converges to $f(x_0)$. This completes the proof of the theorem. \square

The converse of the theorem is not always true. The following example illustrates this situation:

Example 3.8. Consider 2-normed space \mathbb{R}^2 with the 2-norm $\|(a_1, a_2), (b_1, b_2)\| = |a_1b_2 - a_2b_1|$, the sequence $x = (x_k) = (x_k^1, x_k^2) = (\sqrt{k}, \sqrt{k})$, and the function $f(x) = f(x_1, x_2) = ((x_1)^2, (x_2)^2)$. Thus it is easily verified that the sequence (x_k) is statistically quasi-Cauchy, since $\forall z \in X$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \|\Delta x_k, z\| \geq \epsilon\}| &= \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \|x_{k+1} - x_k, z\| \geq \epsilon\}| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left\| (\sqrt{k+1}, \sqrt{k+1}) - (\sqrt{k}, \sqrt{k}), z \right\| \geq \epsilon \right\} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left\| (\sqrt{k+1} - \sqrt{k}, \sqrt{k+1} - \sqrt{k}), z \right\| \geq \epsilon \right\} \right| = 0 \end{aligned}$$

On the other hand, $(f(x_k)) = (f(\sqrt{k}, \sqrt{k})) = (k, k)$ is not statistically quasi-Cauchy since $\forall z \in X$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|\Delta f(x_k), z\| \geq \frac{1}{2} \right\} \right| &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|f(x_{k+1}) - f(x_k), z\| \geq \frac{1}{2} \right\} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|(k+1, k+1) - (k, k), z\| \geq \frac{1}{2} \right\} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|(1, 1), (z_1, z_2)\| \geq \frac{1}{2} \right\} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : z_2 - z_1 \geq \frac{1}{2} \right\} \right|, \text{ for } z_1 = 1, z_2 = 4 \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : 3 \geq \frac{1}{2} \right\} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} n = 1 \neq 0. \end{aligned}$$

Before proving that any statistically ward continuous function on a subset of X is sequentially continuous in the ordinary sense, we need the following lemma.

Lemma 3.9. *If f is statistically sequentially continuous on a subset E of X , then it is sequentially continuous on E .*

Proof. Suppose that f is not sequentially continuous at a point x of E so that there exists a convergent sequence $\mathbf{x} = (x_n)$ of points in E with limit x such that $(f(x_n))$ is not convergent to $f(x)$. Then there exists a $z \in X$, and a positive real number ϵ_0 such that for each $n \in \mathbb{N}$ there is a $k_n \in \mathbb{N}$ with $\|f(x_{k_n}) - f(x), z\| \geq \epsilon_0$. Consider the subsequence (x_{k_n}) of \mathbf{x} . As statistical sequential method is regular, (x_{k_n}) is statistically convergent to x . Thus

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{n \in N : \|f(x_{k_n}) - f(x), z\| \geq \epsilon\}| \neq 0.$$

This implies that the transformed sequence $(f(x_{k_n}))$ is not statistically convergent to $f(x)$. This contradiction completes the proof. \square

The converse of this lemma is not generally true since an infinite dimensional 2-normed space whose topology does not have a countable local base of the origin allows to construct a counterexample. On the other hand, in finite dimensional 2-normed spaces, the converse is also valid, i.e. statistically sequential continuity coincides with ordinary sequential continuity in finite dimensional 2-normed spaces (see [35, Theorem 3.9]).

Theorem 3.10. *If a function f is statistically ward continuous, then it is sequentially continuous in the ordinary sense.*

Proof. If f is statistically ward continuous on a subset E of X , then it follows from Theorem 3.7 that it is statistically sequentially continuous. Therefore by using Lemma 3.9 it is sequentially continuous. \square

Theorem 3.11. *A statistically ward continuous image of any statistically ward compact subset of X is statistically ward compact.*

Proof. Let E be a statistically ward compact subset of X . Take any sequence $\mathbf{y} = (y_n)$ of points in $f(E)$. Write $y_n = f(x_n)$ for each $n \in \mathbb{N}$. As E is statistically ward compact, there is a subsequence $z = (z_k)$ of $\mathbf{x} = (x_n)$ with $st - \lim_{k \rightarrow \infty} \|\Delta z_k, y\| = 0, \forall y \in X$. Write $(t_k) = (f(z_k))$. (t_k) is a statistically quasi-Cauchy subsequence of the sequence $f(x)$. This completes the proof of the theorem. \square

A sequence of functions (f_n) is said to be uniformly convergent to a function f on a subset E of X if for each $\epsilon > 0$, an integer N can be found such that $\|f_n(x) - f(x), z\| < \epsilon$ for $n \geq N$ and for all $x, z \in X$ ([21]). It is well known that the ordinary uniform limit of continuous functions is continuous. It is also true that the uniform limit of statistically ward continuous functions is statistically ward continuous in a 2-normed space.

Theorem 3.12. *If a sequence (f_n) of statistically ward continuous functions on a subset E of X and (f_n) is uniformly convergent to a function f , then f is statistically ward continuous on E .*

Proof. Let z be any fixed point of X and $\epsilon > 0$. Take any statistically quasi-Cauchy sequence (x_k) of points in E . By uniform convergence of (f_n) , there exists a positive integer N such that $\|f_n(x) - f(x), z\| < \frac{\epsilon}{3}, \forall x \in E$ whenever $n \geq N$. Since the function f_N is statistically ward continuous on E , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|f_N(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3} \right\} \right| = 0.$$

Using (4) of Definition 2.1, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|f(x_{k+1}) - f(x_k), z\| \geq \epsilon \right\} \right| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|f(x_{k+1}) - f_N(x_{k+1}), z\| \geq \frac{\epsilon}{3} \right\} \right| \\ & + \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|f_N(x_{k+1}) - f_N(x_k), z\| \geq \frac{\epsilon}{3} \right\} \right| \\ & + \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \|f_N(x_k) - f(x_k), z\| \geq \frac{\epsilon}{3} \right\} \right| = 0. \end{aligned}$$

Thus f is statistically ward continuous on E . This completes the proof of the theorem. \square

Recalling that the set of statistically ward continuous functions on a subset of \mathbb{R} is a closed subset of the set of all continuous functions, we guess the case for the 2-normed spaces, and we see that the situation might be different in a 2-normed space, i.e. the set of statistically ward continuous functions on a subset E of X is a closed subset of the set of ordinary sequentially continuous functions on E when X has a countable base of origin which can be the case provided that X has a countable base as a vector space.

4. Conclusion

The concept of a 2-normed space was extensively studied by Gähler who asked what the real motivation for studying 2-norm structures is, and if there is a physical situation or an abstract concept where the norm topology does not work but the 2-norm topology does work. If a term in the definition of a 2-norm represents the change of a shape, and the 2-norm stands for the associated area, we can think of some plausible application of the notion of a 2-norm, and then the generalized convergence make sense. This can also be viewed as: suppose for a particular output we need two-inputs but with one main input and other input is required to complete the process. So that one may expect to be a more useful tool in the field of a 2-normed space in modeling various problems, occurring in many areas of science. computer science

and information theory. Such possible applications attract researchers to be involved in investigation on 2-normed spaces. We note that the present work contains not only an investigation of quasi-Cauchy sequences as it has been presented in a very different setting, i.e. in a 2-normed space which is quite different from the real case, and metric case, but also an investigation of some other kinds of sequential continuities. In this paper, we investigate not only statistical ward continuity, but also some other kinds of continuities in a 2-normed space. One may expect these concepts to be useful tools in the field of 2-normed space theory in modelling various problems occurring in many areas of science, computer science, information theory and biological science. For a further study, we suggest to investigate quasi-Cauchy sequences of fuzzy points, and statistical ward continuity for the fuzzy functions in a 2-normed fuzzy space. However, due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see [3, 8, 31] for the definitions and related concepts in fuzzy setting). We note that the study in this paper can be generalized to n -normed spaces without much effort (see for example [36–38] for the definition of an n -normed space), and can be extended to 2-cone normed spaces (see for example [11, 41] for the definitions and related concepts in cone normed spaces).

References

- [1] D. Burton, J. Coleman, Quasi-Cauchy sequences, *Amer. Math. Monthly* 117 (2010) 328–333.
- [2] M. Başarir, Ş. Konca, E.E. Kara, Some generalized difference statistically convergent sequence spaces in 2-normed space, *J. Inequal. Appl.* 2013 (2013), Article ID 177, 10 pages.
- [3] T. Beaula, D. Lilly Esthar Rani, On fuzzy ward continuity in an intuitionistic 2-fuzzy 2-normed linear space, *Intern. J. Fuzzy Math. Archive* 6 (2015) 207–213.
- [4] R.C. Buck, Generalized asymptotic density, *Amer. J. Math* 75 (1953) 335–346.
- [5] H. Çakalli, Forward continuity, *J. Comput. Anal. Appl.* 13 (2011) 225–230.
- [6] H. Çakalli, Statistical quasi-Cauchy sequences, *Math. Comput. Modelling* 54 (2011) 1620–1624.
- [7] H. Çakalli, Statistical ward continuity, *Appl. Math. Lett.* 24 (2011) 1724–1728.
- [8] H. Çakalli, P. Das, Fuzzy compactness via summability, *Appl. Math. Lett.* 22 (2009) 1665–1669.
- [9] H. Çakalli, S. Ersan, Strongly lacunary ward continuity in 2-normed spaces, *The Scientific World Journal* 2014 (2014), Article ID 479679, 5 pages.
- [10] H. Çakalli, M.K. Khan, Summability in topological spaces, *Appl. Math. Lett.* 24 (2011) 348–352.
- [11] H. Çakalli, A. Sönmez, Slowly oscillating continuity in abstract metric spaces, *Filomat* 27 (2013) 925–930.
- [12] I. Çanak, M. Dik, New types of continuities, *Abstr. Appl. Anal.* 2010 (2010) 1085–3375.
- [13] A. Caserta, G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in function spaces, *Abstr. Appl. Anal.* 2011 (2011), 11 pages.
- [14] G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in topology, *Topology Appl.* 156 (2008) 28–45.
- [15] C. Diminnie, S. Gähler, A. White, 2-inner product spaces, *Demonstratio Math.* 6 (1973) 525–536.
- [16] H. Dutta, On some 2-Banach spaces, *General Math.* 18:4 (2010) 71–84.
- [17] H. Dutta, Some statistically convergent difference sequence spaces defined over real 2-normed linear spaces, *Applied Sci.* 12 (2010) 37–47.
- [18] H. Dutta, Lj.D.R. Kočinac, On difference sequence spaces defined by Orlicz functions without convexity, *Bull. Iranian Math. Soc.* 41 (2015) 477–489.
- [19] H. Fast, Sur la convergence statistique, *Colloq. Math.* 2 (1951) 241–244.
- [20] A.R. Freedman, J.J. Sember, Densities and summability, *Pacific J. Math.* 95 (1981) 293–305.
- [21] R. Freese, Y.J. Cho, *Geometry of Linear 2-normed spaces*, Nova Science Publishers, 2001.
- [22] J.A. Fridy, On statistical convergence, *Analysis* 5 (1985) 301–313.
- [23] S. Gähler, 2-metrische Räume und ihre topologische Struktur, *Math. Nachr.* 26 (1963) 115–148.
- [24] S. Gähler, Lineare 2-normierte Räume, *Math. Nachr.* 28 (1965) 1–43.
- [25] S. Gähler, Über der Uniformisierbarkeit 2-metrische Räume, *Math. Nachr.* 28 (1965) 235–244.
- [26] S. Gähler, Über 2-Banach-Räume, *Math. Nachr.* 42 (1969) 335–347.
- [27] S. Gähler, A.H. Siddiqi, S.C. Gupta, Contributions to non-archimedean functional analysis, *Math. Nachr.* 69 (1975) 162–171.
- [28] M. Gürdal, S. Pehlivan, The statistical convergence in 2-Banach spaces, *Thai J. Math.* 2 (2004) 107–113.
- [29] M. Gürdal, S. Pehlivan, Statistical convergence in 2-normed spaces, *South. Asian Bull. Math.* 33 (2009) 257–264.
- [30] M. Gürdal, A. Sahiner, Statistical approximation with a sequence of 2-Banach spaces, *Math. Comp. Modelling* 55 (2012) 471–479.
- [31] Lj.D.R. Kočinac, Selection properties in fuzzy metric spaces, *Filomat* 26 (2012) 305–312.
- [32] V.L. Makarov, M.J. Levin, A.M. Rubinov, *Mathematical Economic Theory: Pure and Mixed Types of Economic Mechanisms*, Advanced Textbooks in Economics 33, 1995.
- [33] H.G. Mashadi, On finite dimensional 2-normed spaces, *Soochow J. Math.* 3:27 (2001) 321–329.
- [34] H.I. Miller, A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.* 347 (1995) 1811–1819.
- [35] S. Reddy, Statistical convergence in n -normed spaces, *Int. Math. Forum* 5 (2010) 1185–1193.
- [36] B.S. Reddy, H. Dutta, On equivalence of n -Norms in n -normed spaces, *Pacific J. Sci. Techn.* 11 (2010) 233–238.

- [37] A. Sahiner, M. Gürdal, T. Yigit, Ideal convergence characterization of the completion of linear n -normed spaces, *Comput. Math. Appl.* 61 (2011) 683–689.
- [38] S. Sarabadan, S. Talebi, Statistical convergence and ideal convergence of sequences of functions in 2-normed spaces, *Int. J. Math. Math. Sci.* 2011 (2011), 10 pages.
- [39] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* 66 (1959) 361–375.
- [40] A.H. Siddiqi, 2-normed spaces. *Aligarh Bull. Math.* 7 (1977) 53-70.
- [41] A. Sönmez, H. Çakalli, Cone normed spaces and weighted means, *Math. Comput. Modelling* 52 (2010) 1660–1666.
- [42] A.G.Jr. White, 2-Banach spaces, *Math. Nachr.* 42 (1969) 43–60.
- [43] A. Zygmund, *Trigonometric Series* (with a foreword by Robert A. Fefferman), Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.