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Random Compact Operators

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Abstract. Random compact operators are useful to study random differentiation and random integral equations. In this paper, we define the random norm of R-bounded operators and study random norms of differentiation operators and integral operators. The definition of random norm of R-bounded operators led us to study the random operator theory.

1. Preliminaries

In this section, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [1–3], and then we consider random normed algebras. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. For example, an element for Δ^+ is the distribution function ε_a given by

$$\varepsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

The maximal element for Δ^+ in this order is the distribution function ε_0.

Definition 1.1. ([3]) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Łukasiewicz t-norm).

If T is a t-norm, then, for all $x \in [0, 1]$ and $n \in N \cup \{0\}$, $x_T^{(n)}$ is defined by 1 if $n = 0$ and $T(x_T^{(n-1)}, x)$ if $n \geq 1$.

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We say the t-norm T has Σ property and write $T \in \Sigma$ whenever, for any $\lambda \in (0,1)$, there exists $\gamma \in (0,1)$ (which does not depend on n) such that

$$T^{n-1}(1-\gamma,\cdots,1-\gamma) > 1-\lambda \tag{1}$$

for each $n \geq 1$.

Definition 1.2. ([3]) A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:
(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Example 1.3. Every normed space $(X, \|.\|)$ defines a random normed space (X,μ,T_M), where

$$\mu_x(t) = \frac{t}{t+\|x\|}$$

for all $t > 0$, and $\mu_x(t) = 0$ for $t \leq 0$ in which T_M is the minimum t-norm. This space is called the induced random normed space. Note that, if $T = T_P$, the product t-norm, then the last example is a RN-space.

Definition 1.4. Let (X,μ,T) be an RN-space.
(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon) > 1-\lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_m-x_n}(\epsilon) > 1-\lambda$ whenever $n \geq m \geq N$.
(3) An RN-space (X,μ,T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X. A complete RN-space is called Banach random space.

Definition 1.5. Let (X,μ,T) be an RN-space. We define the open ball $B_x(r,t)$ and the closed ball $B_x[r,t]$ with center $x \in X$ and radius $0 < r < 1$ for any $t > 0$ as follows:

$$B_x(r,t) = \{y \in X \ : \ \mu_{x-y}(t) > 1-r\},$$

$$B_x[r,t] = \{y \in X \ : \ \mu_{x-y}(t) \geq 1-r\}.$$

Theorem 1.6. ([5]) *Let (X,μ,T) be an RN-space. Every open ball $B_x(r,t)$ is an open set.*

Different kinds of topologies can be introduced in a random normed space [3]. The *(r,t)-topology* is introduced by a family of neighborhoods

$$\{B_x(r,t)\}_{x\in X,\ \ t>0,\ \ r\in(0,1)}.$$

In fact, every random norm μ on X generates a topology ((r,t)-topology) on X which has as a base the family of open sets of the form

$$\{B_x(r,t)\}_{x\in X,\ \ t>0,\ \ r\in(0,1)}.$$

Remark 1.7. Since $\left\{B_x\left(\frac{1}{n},\frac{1}{n}\right) : n = 1,2,3,\cdots\right\}$ is a local base at x, the (r,t)-topology is first countable.

Theorem 1.8. ([5]) *Every RN-space (X,μ,T) is a Hausdorff space.*

Definition 1.9. Let (X,μ,T) be an RN-space. A subset A of X is said to be *R-bounded* if there exist $t > 0$ and $r \in (0,1)$ such that $\mu_{x-y}(t) > 1-r$ for all $x, y \in A$.

Ones can find others definitions of boundedness at [1].

Definition 1.10. The RN-space (X, μ, T) is said to be randomly compact (simply R-compact) if every sequence $\{p_m\}_m$ in X has a convergent subsequence $\{p_{m_k}\}$. A subset A of a RN-space (X, μ, T) is said to be R-compact if every sequence $\{p_m\}$ in A has a subsequence $\{p_{m_k}\}$ convergent to a vector $p \in A$.

Theorem 1.11. ([5]) *Every R-compact subset A of an RN-space (X, μ, T) is closed and R-bounded.*

Theorem 1.12. ([3]) *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Theorem 1.13. ([5]) *Let (X, μ, T) be an RN-space such that every Cauchy sequence in X has a convergent subsequence. Then (X, μ, T) is complete.*

Lemma 1.14. ([5]) *If (X, μ, T) is an RN-space, then*
(1) *The function $(x, y) \longrightarrow x + y$ is continuous.*
(2) *The function $(\alpha, x) \longrightarrow \alpha x$ is continuous.*

Note that, in [6] the authors proved that every RN-space is topological vector space (see also Theorem 2 of [7], and [8, 11]).

Lemma 1.15. *Let (X, μ, T) be RN-space, in which $T \in \Sigma$. If we define $E_{\lambda,\mu} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,\mu}(x) = \inf\{t > 0 : \mu_x(t) > 1 - \lambda\}$$

for each $\lambda \in]0, 1[$ and $x \in X$, then we have the following:
(1) *For any $\kappa \in]0, 1[$, there exists $\lambda \in]0, 1[$ such that*

$$E_{\kappa,\mu}(x_1 - x_k) \le E_{\lambda,\mu}(x_1 - x_2) + E_{\lambda,\mu}(x_2 - x_3) + \cdots + E_{\lambda,\mu}(x_{k-1} - x_k)$$

for any $x_1, ..., x_k \in X$;

(2) *For any sequence $\{x_n\}$ in X, we have, $\mu_{x_n - x}(t) \longrightarrow 1$ if and only if $E_{\lambda,\mu}(x_n - x) \to 0$. Also the sequence $\{x_n\}$ is Cauchy w.r.t. f if and only if it is Cauchy with $E_{\lambda,\mu}$.*

Proof. The proof is the same as in Lemma 1.6 of [9]. □

Note that, λ in Lemma 1.15 (1) does not depend on k (see [9]).

Definition 1.16. A linear operator $\Lambda : (X, \mu, T) \longrightarrow (Y, \nu, T')$ is said to be *R-bounded* if there exists a constant $h \in \mathbb{R} - \{0\}$ such that

$$\nu_{\Lambda x}(t) \ge \mu_{hx}(t) \tag{2}$$

for all $x \in X$ and $t > 0$.

Theorem 1.17. (Continuity and boundedness) *Let (X, μ, T) and (Y, ν, T') be RN-spaces, in which $T, T' \in \Sigma$. Let $\Lambda : X \longrightarrow Y$ be a linear operator. Then:*
(a) *Λ is continuous if and only if Λ is R-bounded;*
(b) *If Λ is continuous at a single point, it is continuous.*

Proof. See Theorem 3.2 of [10]. □

2. Random Norm of Operators

Let (X, μ, T) and (Y, μ, T) be RN-spaces and $\Lambda : X \longrightarrow Y$ be a R-bounded linear operator. Define

$$\eta(\Lambda) = \inf\{h > 0 : \mu_{\Lambda x}(t) \geq \mu_{hx}(t)\}, \tag{3}$$

for each $x \in X$ and $t > 0$. $\eta(\Lambda)$ is called the operator random norm.

Lemma 2.1. *Let (X, μ, T) and (Y, μ, T) be RN-spaces and $\Lambda : X \longrightarrow Y$ be a R-bounded linear operator. Then*

$$\mu_{\Lambda x}(t) \geq \mu_{\eta(\Lambda)x}(t), \tag{4}$$

for each $x \in X$ and $t > 0$.

Proof. Since $\Lambda : X \longrightarrow Y$ is a R-bounded linear operator, then by (3) there exists a non-increasing sequence $\{h_n\}$ converges to $\eta(\Lambda)$ and satisfies at

$$\mu_{\Lambda x}(t) \geq \mu_{h_n x}(t) \tag{5}$$

for each $x \in X$ and $t > 0$. Take the limit on n from the last inequality, we get (4). □

Example 2.2. Let (X, μ, T) be RN-space. The identity operator $I : X \longrightarrow X$ is R-bounded and

$$\eta(I) = \inf\{h > 0 : \mu_{Ix}(t) = \mu_x(t)\} = 1$$

for each $x \in X$ and $t > 0$.

Example 2.3. Let (X, μ, T) and (Y, μ, T) be RN-spaces. The zero operator $0 : X \longrightarrow Y$ is R-bounded and

$$\eta(0) = \inf\{h > 0 : \mu_{0(x)}(t) = \mu_0(t) = 1\} = 0$$

for each $x \in X$ and $t > 0$.

Example 2.4. (Differentiation operator) Consider the Example 1.3. Let X be the RN-space of all polynomials on $J = [0, 1]$ with random norm given

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \min_{p \in J} \frac{t}{t+|x(p)|}, & \text{if } t > 0. \end{cases}$$

A differentiation operator D is defined on X by

$$Dx(p) = x'(p),$$

where the prime denotes differentiation with respect to p. This operator is linear but not R-bounded. Indeed, let $x_n(p) = p^n$ where $n \in \mathbb{N}$. Then,

$$\mu_x(t) = \min_{p \in J} \frac{t}{t + |x(p)|} = \frac{t}{t+1},$$

for $t > 0$ and

$$Dx_n(p) = np^{n-1}.$$

Then

$$\mu_{Dx}(t) = \min_{p \in J} \frac{t}{t + np^{n-1}} = \frac{t}{t+n},$$

for $t > 0$ and $n \in \mathbb{N}$. Now

$$\eta(D) = \inf\left\{h > 0 : \frac{t}{t+n} \geq \frac{t}{t+h}\right\} = n.$$

Note that, n depended to choice of $x \in X$.

Example 2.5. (Integral operator) Consider the Example 1.3. Let X be the RN-space of all continuous function on $J = [0,1]$ i.e., $C[0,1]$ with random norm given

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \min_{p\in J} \frac{t}{t+|x(p)|}, & \text{if } t > 0. \end{cases}$$

We can define an integral operator

$$S : C[0,1] \to C[0,1]$$

by $y = Sx$ where

$$y(p) = \int_0^1 \kappa(p,\alpha)x(\alpha)d\alpha.$$

Here κ is a given function, which is called the kernel of S and is assumed to be continuous on the closed square $G = J \times J$ in the $p\alpha$-plane, where $J = [0,1]$. This operator is linear and R-bounded. The continuity of κ on the closed square implies that κ is bounded, say, $\kappa(p,\alpha) \leq k$ for all $(p,\alpha) \in G$, where k is a positive real number. Then,

$$\mu_x(t) = \min_{p\in J} \frac{t}{t+|x(p)|} = \frac{t}{t+1},$$

for $t > 0$ and

$$\begin{aligned} \mu_{Sx}(t) &= \min_{p\in J} \frac{t}{t + |\int_0^1 \kappa(p,\alpha)x(\alpha)d\alpha|} \\ &\geq \min_{p\in J} \frac{t}{t + \int_0^1 |k||x(\alpha)|d\alpha} \\ &\geq \min_{p\in J} \frac{t}{t+|k||x(p)|} \\ &\geq \mu_{kx}(t) \end{aligned}$$

for $t > 0$ i.e., the integral operator S is R-bounded.

Theorem 2.6. *Let (X,μ,T) be an RN-space, in which $T \in \Sigma$ and X is finite dimensional on the field $(\mathbb{F},\mu',T)$, then every linear operator on X is R-bounded.*

Proof. Let $\dim X = n$ and $\{e_1,..,e_n\}$ a basis for X. We take any

$$x = \sum_{j=1}^{n} \alpha_j e_j,$$

and consider any linear operator Λ on X. Since Λ is linear,

$$\mu_{\Lambda x}(t) = \mu_{\sum_{j=1}^{n}\alpha_j\Lambda e_j}(t)$$

for $t > 0$. By Theorem 6.1 of [5] and since $T \in \Sigma$, for every $\lambda \in (0,1)$, there exists $\gamma \in (0,1)$ and $K_0 \in \mathbb{F}$ such that

$$E_{\lambda,\mu'}(K_0) \geq 1$$

and

$$\begin{aligned} E_{\lambda,\mu}(\Lambda x) &= E_{\lambda,\mu}(\sum_{j=1}^{n} \alpha_j \Lambda e_j) \\ &\leq \sum_{j=1}^{n} E_{\gamma,\mu}(\alpha_j \Lambda e_j) \\ &\leq \sum_{j=1}^{n} |\alpha_j| \max_{1\leq j\leq n} E_{\gamma,\mu}(\Lambda e_j) \\ &\leq \sum_{j=1}^{n} |\alpha_j| M_0 E_{\lambda,\mu'}(K_0) \\ &\leq E_{\lambda,\mu'}(M_0 K_0 \sum_{j=1}^{n} |\alpha_j|) \\ &\leq E_{\lambda,\mu}(M_0 K_0 c x) \end{aligned}$$

in which $M_0 = \max_{1\leq j\leq n} E_{\gamma,\mu}(\Lambda e_j)$. Put $M_0K_0c = h$, by Theorem 1.17, Λ is R-bounded. □

Corollary 2.7. (Continuity, null space) *Let (X, μ, T) and (Y, μ, T) be RN-spaces. Let $\Lambda : X \longrightarrow Y$ be a R-bounded linear operator. Then:*
(a) $x_n \to x$ implies $\Lambda x_n \to \Lambda x$;
(b) The null space $\mathcal{N}(\Lambda) = \{x \in X : \Lambda x = 0\}$ is closed.

Proof. (a) Since $\Lambda : X \longrightarrow Y$ is a R-bounded linear operator, we have

$$\begin{aligned} \mu_{\Lambda x_n - \Lambda x}(t) &= \mu_{\Lambda(x_n - x)}(t) \\ &\geq \mu_{\eta(\Lambda)(x_n - x)}(t) \\ &\to 1, \end{aligned}$$

for every $t > 0$.

(b) Let $x \in \overline{\mathcal{N}(\Lambda)}$, then there exists a sequence $\{x_n\}$ in $\mathcal{N}(\Lambda)$ such that $x_n \to x$. By part (a) of this corollary, we have $\Lambda x_n \to \Lambda x$. Since $\Lambda x_n = 0$, then $\Lambda x = 0$ which implies that $x \in \mathcal{N}(\Lambda)$. Since $x \in \overline{\mathcal{N}(\Lambda)}$ was arbitrary, $\mathcal{N}(\Lambda)$ is closed. □

3. Random Operator Space

Let (X, μ, T) and (Y, μ, T) be RN-spaces. In this section, first, we consider the set $B(X, Y)$ consisting of all R-bounded linear operators from X into Y. We want to show that $B(X, Y)$ can itself be made into a normed space. The whole matter is quite simple. First of all, $B(X, Y)$ becomes a vector space if we define the sum $\Lambda_1 + \Lambda_2$ of two operators $\Lambda_1, \Lambda_2 \in B(X, Y)$ in a natural way by

$$(\Lambda_1 + \Lambda_2)x = \Lambda_1 x + \Lambda_2 x$$

and the product $\alpha\Lambda$ of $\Lambda \in B(X, Y)$ and a scalar α by

$$(\alpha\Lambda)x = \alpha\Lambda x.$$

Note that, if (3) hold, then for every $\lambda \in (0, 1)$ we have

$$\eta(\Lambda) = \inf\{h > 0 : \ E_{\lambda,\mu}(\Lambda x) \leq E_{\lambda,\mu}(hx)\} \tag{6}$$

and therefore

$$E_{\lambda,\mu}(\Lambda x) \leq \eta(\Lambda) E_{\lambda,\mu}(x) \tag{7}$$

for $x \in X$. Then

$$E_{\mu}(\Lambda x) \leq \eta(\Lambda) E_{\mu}(x) \tag{8}$$

for $x \in X$ in which

$$E_{\mu}(\Lambda x) = \sup_{\lambda \in (0,1)} E_{\lambda,\mu}(\Lambda x) < \infty. \tag{9}$$

Theorem 3.1. *Let (X, μ, T) and (Y, μ, T) be RN-spaces, in which $T \in \Sigma$ and X. The vector space $B(X, Y)$ of all R-bounded linear operators from X into Y is itself a normed space with norm defined by* (3) *whenever $E_{\mu}(x) < \infty$.*

Proof. In Example 2.3 we showed that $\eta(0) = 0$. Now, if $\eta(\Lambda) = 0$ we have $\mu_{\Lambda x}(t) = 1$ for each $x \in X$ and $t > 0$, which implies that $\Lambda x = 0$ and $\Lambda = 0$. On the other hand,

$$\begin{aligned} \eta(\alpha\Lambda) &= \inf\{h > 0 : \ \mu_{\alpha\Lambda x}(t) \geq \mu_{hx}(t)\} \\ &= \inf\{h > 0 : \ \mu_{\Lambda x}(t) \geq \mu_{\frac{h}{\alpha} x}(t)\} \\ &= |\alpha| \inf\{h > 0 : \ \mu_{\Lambda x}(t) \geq \mu_{hx}(t)\} \\ &= |\alpha| \eta(\Lambda). \end{aligned}$$

Now, we prove triangle inequality for η. Let $\Lambda, \Gamma \in B(X, Y)$. Then

$$\mu_{(\Lambda+\Gamma)x}(t) \geq \mu_{\eta(\Lambda+\Gamma)x}(t),$$

for each $x \in X$ and $t > 0$. For every $\lambda \in (0, 1)$ there exists $\gamma \in (0, 1)$ such that both

$$E_{\lambda,\mu}((\Lambda + \Gamma)x) \leq \eta(\Lambda + \Gamma) E_{\lambda,\mu}(x)$$

which implies that,

$$E_{\mu}((\Lambda + \Gamma)x) \leq \eta(\Lambda + \Gamma) E_{\mu}(x) \tag{10}$$

and

$$\begin{aligned} E_{\lambda,\mu}((\Lambda + \Gamma)x) &\leq E_{\gamma,\mu}(\Lambda x) + E_{\gamma,\mu}(\Gamma x) \\ &\leq [\eta(\Lambda) + \eta(\Gamma)] E_{\gamma,\mu}(x) \end{aligned}$$

which implies that,

$$E_{\mu}((\Lambda + \Gamma)x) \leq [\eta(\Lambda) + \eta(\Gamma)] E_{\mu}(x) \tag{11}$$

for each $x \in X$. From (10) and (11) we have

$$\eta(\Lambda + \Gamma) \leq \eta(\Lambda) + \eta(\Gamma).$$

□

Theorem 3.2. *Let (X, μ, T) and (Y, μ, T) be RN-spaces, in which $T \in \Sigma$ and X. If Y is complete RN-space then $(B(X, Y), \eta)$ is complete whenever $E_{\mu}(x) < \infty$.*

Proof. We consider an arbitrary Cauchy sequence $\{\Lambda_n\}$ in $(B(X,Y),\eta)$ and show that $\{\Lambda_n\}$ converges to an operator $\Lambda \in B(X,Y)$. Since $\{\Lambda_n\}$ is Cauchy, for every $h>0$, there exists $N\in\mathbb{N}$ such that if $m,n\geq N$ then

$$\eta(\Lambda_n-\Lambda_m)<h,$$

or

$$\eta(\Lambda_n-\Lambda_m)\to 0$$

whenever m,n tend to ∞. For all $x\in X$ and $t>0$ we have

$$\begin{aligned}\mu_{\Lambda_n x-\Lambda_m x}(t) &= \mu_{(\Lambda_n-\Lambda_m)x}(t)\\ &\geq \mu_x\left(\frac{t}{\eta(\Lambda_n-\Lambda_m)}\right)\\ &\to 1\end{aligned} \tag{12}$$

whenever m,n tend to ∞. Then the sequence $\{\Lambda_n x\}$ is Cauchy in complete RN-space (Y,μ,T) and so converges to $y\in Y$ depends on the choice of $x\in X$. This defines an operator $\Lambda: X\to Y$, where $y=\Lambda x$. The operator Λ is linear since

$$\lim\Lambda_n(\alpha x+\beta z)=\lim\alpha\Lambda_n x+\lim\beta\Lambda_n z=\alpha\lim\Lambda_n x+\beta\lim\Lambda_n z,$$

for $x,z\in X$ and scalers α,β.

Now, we show that Λ is R-bounded and $\Lambda_n\to\Lambda$. For every $m,n\geq N$ we have

$$\begin{aligned}\mu_{\Lambda_n x-\Lambda_m x}(t) &= \mu_{(\Lambda_n-\Lambda_m)x}(t)\\ &\geq \mu_x\left(\frac{t}{\eta(\Lambda_n-\Lambda_m)}\right)\\ &\geq \mu_x\left(\frac{t}{h}\right).\end{aligned} \tag{13}$$

On the other hand, $\Lambda_m x\to\Lambda x$ when m tend to ∞. Using the continuity of the random norm, we obtain from (12), for every $n>N$, $x\in X$ and $t>0$

$$\begin{aligned}\mu_{\Lambda_n x-\Lambda x}(t) &= \lim_{m\to\infty}\mu_{(\Lambda_n-\Lambda_m)x}(t)\\ &\geq \lim_{m\to\infty}\mu_x\left(\frac{t}{\eta(\Lambda_n-\Lambda_m)}\right)\\ &\geq \mu_x\left(\frac{t}{h}\right).\end{aligned} \tag{14}$$

This shows that $(\Lambda_n-\Lambda)$ with $n>N$ is a R-bounded linear operator. Since Λ_n is R-bounded, $\Lambda=\Lambda_n-(\Lambda_n-\Lambda)$ is R-bounded, that is, $\Lambda\in B(X,Y)$. From (14) we have

$$\mu_x\left(\frac{t}{\eta(\Lambda_n-\Lambda)}\right)\geq\mu_x\left(\frac{t}{h}\right).$$

Then

$$\eta(\Lambda_n-\Lambda)\leq h$$

for every $n>N$. Hence

$$\Lambda_n\xrightarrow{\eta}\Lambda.$$

□

A functional is an operator whose range lies on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$. A R-bounded linear functional is a R-bounded linear operator with range in the scalar field of the RN-space (X,μ,T). It is of basic importance that the set of all linear functionals defined on a vector space X can itself be made into a vector space. Let $(\mathbb{F},\mu',T)$ be RN-space ($\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$). The set $X'=B(X,\mathbb{F})$ is said to be random dual space. The random dual space X' is Banach space with the norm η.

4. Compact Operators

Definition 4.1. ***(R-Compact linear operator).*** Let (X, μ, T) and (Y, μ, T') be RN-spaces. An operator $\Lambda : X \longrightarrow Y$ is called a R-compact linear operator if Λ is linear and if for every R-bounded subset M of X, the closure $\overline{\Lambda(M)}$ is R-compact.

Lemma 4.2. *Let (X, μ, T) and (Y, μ, T) be RN-spaces. Then, every R-compact linear operator $\Lambda : X \longrightarrow Y$ is R-bounded, hence continuous.*

Proof. Let U be a R-bounded set, then there exists $r_0 \in (0, 1)$ and $t_0 > 0$ such that

$$\mu_x(t_0) \geq 1 - r_0,$$

for every $x \in U$. On the other hand, $\overline{\Lambda(U)}$ is R-compact and by Theorem 1.11 is R-bounded, then there exists $r_1 \in (0, 1)$ and $t_1 > 0$ such that

$$\mu_{\Lambda x}(t_1) \geq 1 - r_1,$$

for every $x \in U$. By the intermediate value theorem there exists a positive real number h_0 such that

$$\mu_{\Lambda x}(h_0 t_0) \geq \mu_x(t_0),$$

for every $x \in U$ (note that by the last inequality h_0 can not tend to zero), and so $\eta(\Lambda) < \infty$. Hence Λ is R-bounded and by Theorem 1.17 is continuous. □

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