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On Generalized Lorentz Sequence Space Defined by Modulus Functions

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Abstract. The object of this paper is to introduce generalized Lorentz sequence spaces $L(f, v, p)$ defined by modulus function f. Also we study some topologic properties of this space and obtain some inclusion relations.

1. Introduction

Throughout this work, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, real numbers and complex numbers, respectively. The concept of modulus function was introduced by Nakano [11] . We recall that a function $f : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfies the following properties

1) $f(x) = 0$ if and only if $x = 0$;
2) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$;
3) f is increasing;
4) f is continuous from right at 0.

It follows that f is continuous on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = x/(x + 1)$, then $f(x)$ is bounded. But, for $0 < p < 1$, $f(x) = x^p$ is not bounded.

By the condition 2), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ and so $f(x) = f\left(nx\frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right)$, and hence

$$\frac{1}{n}f(x) \leq f\left(\frac{x}{n}\right)$$

for all $n \in \mathbb{N}$.

The FK-spaces $L(f)$, introduced by Ruckle in [14], is in the form

$$L(f) = \left\{x \in w : \sum_{k=1}^{\infty} f(|x_k|) < \infty\right\},$$

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where f is a modulus function and w is the space of all complex sequences. This space is closely related to the space ℓ_1 which is an $L(f)-$ space with $f(x) = x$ for all real $x \geq 0$. Later on, this space was investigated by many authors in [1], [4], [8], [9], [15].

The notion of *paranorm* is closely related to linear metric spaces. Let X be a linear space. A function $p : X \to \mathbb{R}$ is called *paranorm*, if

i) $p(0) = 0$,
ii) $p(x) \geq 0$ for all $x \in X$,
iii) $p(-x) = p(x)$ for all $x \in X$,
iv) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
v) (λ_n) be a sequence in $\mathbb{C}$, λ be an element in $\mathbb{C}$, $\{x_n\}$ be a sequence in X and x be an element in X. If $|\lambda_n - \lambda| \to 0$ as $n \to \infty$ and $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$ (continuity of multiplication by scalars).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total* [7].

The Lorentz space was introduced by G. G. Lorentz in [5], [6]. This space play an important role in the theory of Banach space. Many authors studied these spaces and explored their many properties.

Let $(E, \|\cdot\|)$ be a Banach space. The Lorentz sequence space $l(p, q, E)$ $\left(or\ l_{p,q}(E)\right)$ for $1 \leq p, q \leq \infty$ is the collection of all sequences $\{a_i\} \in c_0(E)$ such that

$$\|\{a_i\}\|_{p,q} = \begin{cases} \left(\sum\limits_{i=1}^{\infty} i^{q/p-1} \left\|a_{\phi(i)}\right\|^q\right)^{1/q} & for\ 1 \leq p \leq \infty,\ 1 \leq q < \infty \\ \sup_i i^{1/p} \left\|a_{\phi(i)}\right\| & for\ 1 \leq p < \infty,\ q = \infty \end{cases}$$

is finite, where $\left\{\left\|a_{\phi(i)}\right\|\right\}$ is non-increasing rearrangement of $\{\|a_i\|\}$ (We can interpret that the decreasing rearrangement $\left\{\left\|a_{\phi(i)}\right\|\right\}$ is obtained by rearranging $\{\|a_i\|\}$ in decreasing order). This space was introduced by Miyazaki in [10] and examined comprehensively by Kato in [3].

A weight sequence $v = \{v(i)\}$ is a positive decreasing sequence such that $v(1) = 1$, $\lim_{i\to\infty} v(i) = 0$ and $\lim_{i\to\infty} V(i) = \infty$, where $V(i) = \sum\limits_{n=1}^{i} v(n)$ for every $i \in \mathbb{N}$. Popa [13] defined the generalized Lorentz sequence space $d(v, p)$ for $0 < p < \infty$ as follows

$$d(v,p) = \left\{x = \{x_i\} \in w : \|x\|_{v,p} = \sup_{\pi} \left(\sum_{i=1}^{\infty} \left|x_{\pi(i)}\right|^p v(i)\right)^{1/p} < \infty\right\},$$

where π ranges over all permutations of the positive integers and $v = \{v(i)\}$ is a weight sequence. It is know that $d(v, p) \subset c_0$ and hence for each $x \in d(v, p)$ there exists a non-increasing rearrangement $\{x^*\} = \left\{x_i^*\right\}$ of x and

$$\|x\|_{v,p} = \left(\sum_{n=1}^{\infty} \left|x_i^*\right|^p v(i)\right)^{\frac{1}{p}}$$

(see [12], [13]).

Let $(X, \|\cdot\|)$ be a Banach space, f be a modulus function and $v = \{v(n)\}$ be a weight sequence. We introduce the generalized Lorentz sequence space $L(f, v, p)$ for $0 < p < \infty$ using a modulus function f. The space $L(f, v, p)$ is the collection of all $X-$valued $0-$sequences $\{x_n\}$ $(\{x_n\} \in c_0\{X\})$ such that

$$g(x) = \left(\sum_{n=1}^{\infty} \left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}}$$

is finite, where $\left\{\left\|x_{\phi(n)}\right\|\right\}$ is non-increasing rearrangement of $\{\|x_n\|\}$. If we take $f(x) = x$, then $L(f, v, p) = d(v, p)$ ([13]).

We shall need the following lemmas.

Lemma 1.1. *(Hardy, Littlewood and Pólya [2]). Let $\left\{c_i^*\right\}$ and $\{{}^*c_i\}$ be the non-increasing and non-decreasing rearrangements of a finite sequence $\{c_i\}_{1\le i\le n}$ of positive numbers, respectively. Then for two sequences $\{a_i\}_{1\le i\le n}$ and $\{b_i\}_{1\le i\le n}$ of positive numbers we have*

$$\sum_i a_i^* \cdot^* b_i \le \sum_i a_i \cdot b_i \le \sum_i a_i^* \cdot b_i^*.$$

Lemma 1.2. *(Kato [3]) Let $\left\{x_i^{(\mu)}\right\}$ be an $X-$valued double sequence such that $\lim_{i\to\infty} x_i^{(\mu)} = 0$ for each $\mu \in \mathbb{N}$ and let $\{x_i\}$ be an $X-$valued sequence such that $\lim_{\mu\to\infty} x_i^{(\mu)} = x_i$ (uniformly in i). Then $\lim_{i\to\infty} x_i = 0$ and for each $i \in \mathbb{N}$*

$$\left\|x_{\phi(i)}\right\| \le \varliminf_{\mu\to\infty} \left\|x^{(\mu)}_{\phi_\mu(i)}\right\|,$$

where $\left\{\left\|x_{\phi(i)}\right\|\right\}$ and $\left\{\left\|x^{(\mu)}_{\phi_\mu(i)}\right\|\right\}_i$ are the non-increasing rearrangements of $\{\|x_i\|\}$ and $\left\{\left\|x_i^{(\mu)}\right\|\right\}_i$, respectively.

Lemma 1.3. *Let f be any modulus function and $0 < \delta < 1$. Then*

$$f(x) \le \frac{2f(1)}{\delta} x$$

for all $x \ge \delta$ [9] .

Lemma 1.4. *For any modulus f there exists $\lim_{t\to\infty} \frac{f(t)}{t}$ [9].*

Lemma 1.5. *Let f be any modulus with $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$. Then there is a constant $\beta > 0$ such that*

$$f(t) \ge \beta t$$

for all $t \ge 0$ [9].

2. Main Results

Theorem 2.1. *The space $L(f, v, p)$ for $0 < p < \infty$ is a linear space over the field $K = \mathbb{R}$ or $\mathbb{C}$.*

Proof. Let $x, y \in L(f, v, p)$ and let $\left\{\left\|x_{\phi(n)}\right\|\right\}$, $\left\{\left\|y_{\eta(n)}\right\|\right\}$ and $\left\{\left\|x_{\psi(n)} + y_{\psi(n)}\right\|\right\}$ be the non-increasing rearrangements of the sequences $\{\|x_n\|\}$, $\left\{\left\|y_n\right\|\right\}$ and $\left\{\left\|x_n + y_n\right\|\right\}$, respectively. Since v is non-increasing and f is increasing, by the Lemma 1 we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left[f\left(\left\|x_{\psi(n)} + y_{\psi(n)}\right\|\right)\right]^p v(n) &\le \sum_{n=1}^{\infty} \left[f\left(\left\|x_{\psi(n)}\right\| + \left\|y_{\psi(n)}\right\|\right)\right]^p v(n) \\
&\le \sum_{n=1}^{\infty} \left[f\left(\left\|x_{\psi(n)}\right\|\right) + f\left(\left\|y_{\psi(n)}\right\|\right)\right]^p v(n) \\
&\le D\sum_{n=1}^{\infty} \left(\left[f\left(\left\|x_{\psi(n)}\right\|\right)\right]^p v(n) + \left[f\left(\left\|y_{\psi(n)}\right\|\right)\right]^p v(n)\right) \\
&\le D\left\{\sum_{n=1}^{\infty} \left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n) + \sum_{n=1}^{\infty} \left[f\left(\left\|y_{\eta(n)}\right\|\right)\right]^p v(n)\right\} \\
&< \infty,
\end{aligned}$$

where $D = \max\left\{1, 2^{p-1}\right\}$. Let $\alpha \in K$, then there exists $M_\alpha \in \mathbb{N}$ such that $|\alpha| \le M_\alpha$. Hence we get

$$\begin{aligned}\sum_{n=1}^{\infty}\left[f\left(\left\|\alpha x_{\phi(n)}\right\|\right)\right]^p v(n) &\le \sum_{n=1}^{\infty}\left[f\left(M_\alpha \left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\\ &\le M_\alpha^p \sum_{n=1}^{\infty}\left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\\ &< \infty.\end{aligned}$$

This shows that $x + y \in L(f, v, p)$, $\alpha x \in L(f, v, p)$ and so $L(f, v, p)$ is a linear space. □

Theorem 2.2. *The space $L(f, v, p)$ for $1 \le p < \infty$ is paranormed space with the paranorm*

$$g(x) = \left(\sum_{n=1}^{\infty}\left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}},$$

where $\left\{\left\|x_{\phi(n)}\right\|\right\}$ denotes the non-increasing rearrangements of $\{\|x_n\|\}$.

Proof. It is clear that $g(x) = g(-x)$ and $g(0) = 0$. Let $x, y \in L(f, v, p)$. Since f is increasing and weight sequence v is decreasing, by Lemma 1 we have

$$\begin{aligned}g(x+y) &= \left(\sum_{n=1}^{\infty}\left[f\left(\left\|x_{\psi(n)} + y_{\psi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}}\\ &\le \left(\sum_{n=1}^{\infty}\left[f\left(\left\|x_{\psi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left[f\left(\left\|y_{\psi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}}\\ &\le \left(\sum_{n=1}^{\infty}\left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left[f\left(\left\|y_{\eta(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}}\\ &= g(x) + g(y)\end{aligned}$$

where $\left\{\left\|x_{\phi(n)}\right\|\right\}$, $\left\{\left\|y_{\eta(n)}\right\|\right\}$ and $\left\{\left\|x_{\psi(n)} + y_{\psi(n)}\right\|\right\}$ denote the non-increasing rearrangements of $\{\|x_n\|\}$, $\left\{\left\|y_n\right\|\right\}$ and $\left\{\left\|x_n + y_n\right\|\right\}$, respectively.

Now we show the continuity of scalar multiplication. Let λ be an element in K, $\left\{\lambda^{(m)}\right\}$ be a sequence in K such that $\left|\lambda^{(m)} - \lambda\right| \to 0$ as $m \to \infty$, x be an element in $L(f, v, p)$ and $\left\{x^{(m)}\right\}$ be a sequence in $L(f, v, p)$ such that $g(x^{(m)} - x) \to 0$ as $m \to \infty$. Using triangle inequality we have

$$g(\lambda^{(m)}x^{(m)} - \lambda x) \le g(\lambda^{(m)}x^{(m)} - \lambda^{(m)}x) + g(\lambda^{(m)}x - \lambda x). \tag{1}$$

By monotonity of modulus function

$$\begin{aligned}g(\lambda^{(m)}x^{(m)} - \lambda^{(m)}x) &= \left(\sum_{n=1}^{\infty}\left[f\left(\left\|\lambda^{(m)}x^{(m)}_{\psi_m(n)} - \lambda^{(m)}x_{\psi_m(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}}\\ &= \left(\sum_{n=1}^{\infty}\left[f\left(\left|\lambda^{(m)}\right|\left\|x^{(m)}_{\psi_m(n)} - x_{\psi_m(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}}\\ &\le A \cdot \left(\sum_{n=1}^{\infty}\left[f\left(\left\|x^{(m)}_{\psi_m(n)} - x_{\psi_m(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}}\\ &= A \cdot g(x^{(m)} - x)\end{aligned}$$

where $A = \left(\left[\left|\sup_m \left|\lambda^{(m)}\right|\right|\right] + 1\right)$ and $\left\{\left\|\lambda^{(m)}x^{(m)}_{\psi_m(n)} - \lambda^{(m)}x_{\psi_m(n)}\right\|\right\}_n$ denotes the non-increasing rearrangement of $\left\{\left\|\lambda^{(m)}x^{(m)}_n - \lambda^{(m)}x_n\right\|\right\}_n$. Thus we get

$$g(\lambda^{(m)}x^{(m)} - \lambda^{(m)}x) \to 0 \tag{2}$$

as $m \to \infty$.

Since $\left|\lambda^{(m)} - \lambda\right| \to 0$ as $m \to \infty$, there exists $T \in \mathbb{N}$ such that $\left|\lambda^{(m)} - \lambda\right| \leq T$ for each $m \in \mathbb{N}$. Let us take any $\varepsilon > 0$. Since $x \in L(f, v, p)$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\sum_{n=n_0}^{\infty}\left[f\left(\left|\lambda^{(m)} - \lambda\right|\left\|x_{\phi(n)}\right\|\right)\right]^p v(n) &\leq \sum_{n=n_0}^{\infty}\left[f\left(T.\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\\ &\leq T^p\sum_{n=n_0}^{\infty}\left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\\ &< \frac{\varepsilon}{2}\end{aligned}$$

and hence we get

$$\sum_{n=n_0}^{\infty}\left[f\left(\left\|\lambda^{(m)}x_{\phi(n)} - \lambda x_{\phi(n)}\right\|\right)\right]^p v(n) < \frac{\varepsilon}{2} \tag{3}$$

for all $m \in \mathbb{N}$. Also by the continuity of f, we have

$$\sum_{n=1}^{n_0-1}\left[f\left(\left\|\lambda^{(m)}x_{\phi(n)} - \lambda x_{\phi(n)}\right\|\right)\right]^p v(n) < \frac{\varepsilon}{2} \tag{4}$$

as $m \to \infty$, where $\left\{\left\|\lambda^{(m)}x_{\phi(n)} - \lambda x_{\phi(n)}\right\|\right\}_n$ is non-increasing rearrangement of $\left\{\left\|\lambda^{(m)}x_n - \lambda x_n\right\|\right\}_n$. Consequently, by (3) and (4) we have

$$\sum_{n=1}^{\infty}\left[f\left(\left\|\lambda^{(m)}x_{\phi(n)} - \lambda x_{\phi(n)}\right\|\right)\right]^p v(n) \to 0 \tag{5}$$

as $m \to \infty$. By (1), (2) and (5), we get $g(\lambda^{(m)}x^{(m)} - \lambda x) \to 0$ as $m \to \infty$.This completes the proof. □

Theorem 2.3. *The space $L(f, v, p)$ for $1 \leq p < \infty$ is complete with respect to its paranorm.*

Proof. Let $\left\{x^{(s)}\right\}$ be an arbitrary Cauchy sequence in $L(f, v, p)$ with $x^{(s)} = \left\{x^{(s)}_n\right\}_{n=1}^{\infty}$ for all $s \in \mathbb{N}$. For any $\varepsilon > 0$ and a fixed $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$g(x^{(s)} - x^{(t)}) = \left(\sum_{m=1}^{\infty}\left[f\left(\left\|x^{(s)}_{\pi_{s,t}(m)} - x^{(t)}_{\pi_{s,t}(m)}\right\|\right)\right]^p v(m)\right)^{\frac{1}{p}} < f(\varepsilon)\,(v(n))^{\frac{1}{p}} \tag{6}$$

whenever $s, t \geq n_0$. Here, $\left\{\left\|x^{(s)}_{\pi_{s,t}(m)} - x^{(t)}_{\pi_{s,t}(m)}\right\|\right\}_m$ denotes non-increasing rearrangement of $\left\{\left\|x^{(s)}_m - x^{(t)}_m\right\|\right\}_m$ and we indicate that $\pi_{s,t}(m)$ is a permutation for $\mathbb{N}$. Thus we have

$$\left[f\left(\left\|x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)}\right\|\right)(v(n))^{\frac{1}{p}}\right]^p < \left(f(\varepsilon)\,(v(n))^{\frac{1}{p}}\right)^p$$

whenever $s, t \geq n_0$. Therefore we get

$$\left\|x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)}\right\| < \varepsilon$$

whenever $s, t \geq n_0$. Then $\left\{x_n^{(s)}\right\}$, for a fixed $n \in \mathbb{N}$, is a Cauchy sequence in X .

Then, there exists $x_n \in X$ such that $x_n^{(s)} \to x_n$ as $s \to \infty$. Let $x = \{x_n\}$. Since $\lim_{n\to\infty} x_n^{(s)} = 0$ for each $s \in \mathbb{N}$, by Lemma 2 we have $\lim_{n\to\infty} x_n = 0$. Therefore we can choose the non-increasing rearrangement $\left\{\left\|x_{\pi_t(n)} - x_{\pi_t(n)}^{(t)}\right\|\right\}_n$ of $\left\{\left\|x_n - x_n^{(t)}\right\|\right\}_n$. Also, for an arbitrary $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \left[f\left(\left\|x_{\pi_{s,t}(n)}^{(s)} - x_{\pi_{s,t}(n)}^{(t)}\right\|\right)\right]^p v(n) < \varepsilon^p \tag{7}$$

for $s, t > N$. Let t be an arbitrary positive integer with $t > N$ and fixed. If we put

$$y_n^{(s)} = x_n^{(s)} - x_n^{(t)} \quad \text{and} \quad y_n = x_n - x_n^{(t)} ,$$

then we have

$$\lim_{n\to\infty} y_n^{(s)} = 0 \text{ for each } s \in \mathbb{N} \text{ and } \lim_{s\to\infty} y_n^{(s)} = y_n \text{ (uniformly in } n).$$

Thus by Lemma 2 we get

$$\left\|y_{\phi(n)}\right\| \leq \varliminf_{s\to\infty} \left\|y_{\phi_s(n)}^{(s)}\right\|$$

for each $n \in \mathbb{N}$, that is,

$$\left\|x_{\pi_t(n)} - x_{\pi_t(n)}^{(t)}\right\| \leq \varliminf_{s\to\infty} \left\|x_{\pi_{s,t}(n)}^{(s)} - x_{\pi_{s,t}(n)}^{(t)}\right\| \tag{8}$$

for each $n \in \mathbb{N}$. Hence, by (7), (8) and continuity of f we get

$$\begin{aligned}
g(x - x^{(t)}) &= \left(\sum_{n=1}^{\infty} \left[f\left(\left\|x_{\pi_t(n)} - x_{\pi_t(n)}^{(t)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{n=1}^{\infty} \left[f\left(\varliminf_{s\to\infty} \left\|x_{\pi_{s,t}(n)}^{(s)} - x_{\pi_{s,t}(n)}^{(t)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} \\
&= \varliminf_{s\to\infty} \left(\sum_{n=1}^{\infty} \left[f\left(\left\|x_{\pi_{s,t}(n)}^{(s)} - x_{\pi_{s,t}(n)}^{(t)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} \\
&< \varepsilon.
\end{aligned}$$

Also, since $L(f, v, p)$ is a linear space we have $\{x_n\} = \left\{x_n - x_n^{(N)}\right\} + \left\{x_n^{(N)}\right\} \in L(f, v, p)$. Hence the space $L(f, v, p)$ is complete with respect to its paranorm. □

Theorem 2.4. *Let f and h be two modulus functions. Then*

(*i*) $\limsup \frac{f(t)}{h(t)} < \infty$ implies $L(h, v, p) \subset L(f, v, p)$,
(*ii*) $L(f, v, p) \cap L(h, v, p) \subseteq L(f + h, v, p)$ for $1 \leq p < \infty$.

Proof. (*i*) By the hypothesis there exists $K > 0$ such that $f(t) \leq K.h(t)$ for all $t \geq 0$. Let $x \in L(h, v, p)$. Then we have

$$\left(\sum_{n=1}^{\infty} \left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} \left[K \cdot h\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} < \infty.$$

Hence we get $x \in L(f, v, p)$.

(ii) Let $x \in L(h, v, p) \cap L(f, v, p)$. Hence we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\left[(f+h)\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} &= \left(\sum_{n=1}^{\infty}\left[f\left(\left\|x_{\phi(n)}\right\|\right)+h\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=1}^{\infty}\left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} \\ &\quad + \left(\sum_{n=1}^{\infty}\left[h\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n)\right)^{\frac{1}{p}} < \infty.\end{aligned}$$

Therefore we get $x \in L(f+h, v, p)$ and this completes the proof. □

Theorem 2.5. *Let f be modulus function. Then*

(a) If $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ then $L(f, v, p) \subset d(v, p)$,
(b) $d(v, 1) \subset L(f, v, 1)$.

Proof. (a) Let $x \in L(f, v, p)$. By Lemma 5, there is $\beta > 0$ such that $f(t) \geq \beta t$ for all $t \geq 0$. Hence we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left[\left\|x_{\phi(n)}\right\|\right]^p v(n) &\leq \max\left\{1, \frac{1}{\beta^p}\right\}\sum_{n=1}^{\infty}\left[f\left(\left\|x_{\phi(n)}\right\|\right)\right]^p v(n) \\ &< \infty.\end{aligned}$$

This completes the proof.

(b) Let $x \in d(v, 1)$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty}\left\|x_{\phi(n)}\right\| v(n) < \varepsilon$$

for all $n \geq n_0$. Since f is continuous on $[0, \infty)$, we have for all $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $f(t) < \varepsilon$ for all $t \in [0, \delta]$. Also, by Lemma 3 we have

$$f\left(\left\|x_{\phi(n)}\right\|\right) < \frac{2f(1)}{\delta}\left\|x_{\phi(n)}\right\|$$

for $\left\|x_{\phi(n)}\right\| > \delta$, where $\left\{\left\|x_{\phi(n)}\right\|\right\}$ is the non-increasing rearrangement of $\{\|x_n\|\}$. Hence we get

$$\begin{aligned}\sum_{n=1}^{\infty} f\left(\left\|x_{\phi(n)}\right\|\right) v(n) &= \sum_{\left\|x_{\phi(n)}\right\|\leq\delta} f\left(\left\|x_{\phi(n)}\right\|\right) v(n) \\ &\quad + \sum_{\left\|x_{\phi(n)}\right\|>\delta} f\left(\left\|x_{\phi(n)}\right\|\right) v(n) \\ &< \varepsilon + \frac{2f(1)}{\delta}\sum_{\left\|x_{\phi(n)}\right\|>\delta}\left\|x_{\phi(n)}\right\| v(n) \\ &< \infty\end{aligned}$$

and so we get $x \in L(f, v, 1)$. □

Corollary 2.6. *If* $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ *then* $L(f, v, 1) \subset d(v, 1)$.

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