



## Statistical $(C, 1)(E, 1)$ Summability and Korovkin's Theorem

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**Abstract.** Korovkin-type approximation theory usually deals with convergence analysis for sequences of positive operators. This approximation theorem was extended to more general space of sequences via different way such as statistical convergence, summation processes. In this work, we introduce a new type of statistical product summability, that is, statistical  $(C, 1)(E, 1)$  summability and further apply our new product summability method to prove Korovkin type theorem. Furthermore, we present a rate of convergence which is uniform in Korovkin type theorem by statistical  $(C, 1)(E, 1)$  summability.

### 1. Introduction and Preliminaries

Let  $C_M[a, b]$  denotes the space of all continuous functions on  $[a, b]$  and bounded by the number  $M$ . Also,  $B[a, b]$  is the space of all bounded functions with the norm  $\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$ . If the sequence of positive linear operators  $A_n : C_M[a, b] \rightarrow B[a, b]$  satisfy the three conditions

$$\begin{aligned}\lim_{n \rightarrow \infty} \|A_n(1, x) - 1\|_\infty &= 0, \\ \lim_{n \rightarrow \infty} \|A_n(t, x) - x\|_\infty &= 0, \\ \lim_{n \rightarrow \infty} \|A_n(t^2, x) - x^2\|_\infty &= 0,\end{aligned}$$

then for any function  $f \in C_M[a, b]$ , we have

$$\lim_{n \rightarrow \infty} \|A_n(f, x) - f(x)\|_\infty = 0.$$

This theorem is known as Korovkin theorem [9] and has an important role in approximation theory.

Korovkin theorem and its generalizations in weighted spaces were extended to more wide space of sequences by using convergence methods and summation process. For more details, we refer to readers [6, 11] and references therein.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [5] and Steinhaus [28] independently in 1951 as:

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Let  $\mathbb{N}$  be the set of positive integers,  $K \subseteq \mathbb{N}$  and  $K_n = \{j : j \leq n \text{ and } j \in K\}$ . Then the natural density of  $K$  is defined by  $\delta(K) := \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$  if the limit exists, where  $|K_n|$  denotes the cardinality of the set  $K_n$ . So, a sequence  $x = (x_j)$  is said to be statistical convergent to the number  $L$  if for every  $\varepsilon > 0$ , the set  $\{j : j \in \mathbb{N} \text{ and } |x_j - L| \geq \varepsilon\}$  has natural density zero, that is, if, for each  $\varepsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ j : j \leq n \text{ and } |x_j - L| \geq \varepsilon \right\} \right| = 0.$$

Since the statistical convergence is more general than the classical convergence, that is a sequence can be convergent in statistical mean even if it isn't convergent in classical mean, it has become an active area of research. Several researchers have studied in this direction, we refer some of the papers in this area as [10], [17], [18], [12], [21], [26] etc. On the other hand, a sequence  $x = (x_k)$  is said to be  $(C, 1)$  summable to  $s$  if  $s_n = \frac{1}{n+1} \sum_{k=0}^n x_k \rightarrow s$  as  $n \rightarrow \infty$ . In the paper [19], Moricz introduced statistical summability  $(C, 1)$  considering two concepts, statistical convergence and  $(C, 1)$  summability. That is, if the sequence  $s_n$  is statistical convergent to the number  $L$ , then  $x = (x_k)$  is said to be statistical  $(C, 1)$  summable to  $L$  and this process is stronger than statistical convergence under some assumptions. Very recently, Mohiuddine et al. [13] and [14] obtained Korovkin type approximation theorem by using the test functions  $1, e^{-x}, e^{-2x}$  and  $1, \sin x, \cos x$ , respectively, through statistical summability  $(C, 1)$ . Among the others we can mention some of the papers on the applications of statistical convergence to approximation theorems as; approximation theorems by generalized statistical convergence in [2], statistical summability of the generalized de la Vallée Poussin mean in [3, 20], generalized statistical convergence in [4], weighted statistical convergence in [22], weighted  $A$ -statistical convergence in [16],  $A$ -statistical approximation in [23], statistical approximation results for Kantorovich-type operators in [24], generalized equi-statistical convergence in [27], statistical approximation for function of two variables in [1, 15], etc.

## 2. Korovkin Approximation Theorem Through Statistical Summability $(C, 1)$ $(E, 1)$

Let us recall [8] that a sequence  $x = (x_k)$  is said to be  $(E, 1)$  summable to  $s$  if  $s_k = \frac{1}{2^k} \sum_{r=0}^k \binom{k}{r} x_r \rightarrow s$  as  $k \rightarrow \infty$ . The  $(C, 1)$  transform of the  $(E, 1)$  transform  $E_n^1$  defines the  $(C, 1)$   $(E, 1)$  transform of the sequence  $(x_k)$ , i.e., the product summability  $(C, 1)$   $(E, 1)$  is obtained by superimposing  $(C, 1)$  summability on  $(E, 1)$  summability. Thus, if

$$(C, E)_1^n = \frac{1}{n+1} \sum_{k=0}^n E_k^1 = \frac{1}{n+1} \sum_{k=0}^n \frac{1}{2^k} \sum_{r=0}^k \binom{k}{r} x_r \rightarrow s, \text{ as } n \rightarrow \infty,$$

then the  $(x_k)$  is said to be summable  $(C, 1)$   $(E, 1)$  to the definite number  $s$ .

Firstly, we introduce a new type of statistical product summability with the help of  $(C, 1)$  and  $(E, 1)$  summability as follows:

**Definition 2.1.** For a sequence  $x = (x_k)$ , let us write  $t_n = \frac{1}{n+1} \sum_{k=0}^n \frac{1}{2^k} \sum_{r=0}^k \binom{k}{r} x_r$ . If the sequence  $(t_n)$  is statistically convergence to any finite number  $L$ , i.e.  $st - \lim_{n \rightarrow \infty} t_n = L$ , then  $x = (x_k)$  is statistically summable  $(C, 1)$   $(E, 1)$  and we write that  $L = (CE)_1(st) - \lim x$ .

In the following example, we show that the statistical summability  $(C, 1)$   $(E, 1)$  is more powerful than the individual method statistical summability  $(C, 1)$  and so our new method mean (product mean) gives better approximation than the individual one.

**Example 2.2.** Let a sequence  $x = (x_n)$  be defined by  $x_n = (-3)^n, n \in \mathbb{N}$ . Then, a sequence  $(x_n)$  is not statistical summable  $(C, 1)$  but statistical summable  $(C, 1)$   $(E, 1)$  since

$$\frac{1}{n+1} \sum_{k=0}^n (-3)^k = \frac{\frac{3}{4}(-3)^n + \frac{1}{4}}{n+1}$$

and

$$\frac{1}{n+1} \sum_{k=0}^n \frac{1}{2^k} \sum_{r=0}^k \binom{k}{r} (-3)^k = \frac{1}{n+1} \sum_{k=0}^n (-1)^k = \frac{\frac{1}{2}(-1)^n + \frac{1}{2}}{n+1},$$

respectively. Indeed,  $(x_n)$  is neither convergent nor statistically convergent.

On the other hand, let  $C[a, b]$  be the space of all real-valued continuous functions  $f$  on  $[a, b]$  and  $A$  be a linear operator which maps  $C[a, b]$  into itself. We say that  $A$  is positive if, for every non-negative  $f \in C[a, b]$ , we have  $A(f; x) \geq 0$  ( $x \in [a, b]$ ). We know that  $C[a, b]$  is a Banach space with the norm given by

$$\|f\|_\infty := \sup_{x \in [a, b]} |f(x)|.$$

Now, we are ready to prove Korovkin type approximation theorem with the help of statistical summable  $(C, 1)(E, 1)$ .

**Theorem 2.3.** *Let  $\{A_r\}$  be a sequence of positive linear operators from  $C[a, b]$  into  $C[a, b]$ . Then for all  $f \in C[a, b]$  bounded on the whole real line*

$$(CE)_1(st) - \lim_{r \rightarrow \infty} \|A_r(f; x) - f(x)\|_\infty = 0, \tag{1}$$

if and only if

$$(CE)_1(st) - \lim_{r \rightarrow \infty} \|A_r(1; x) - 1\|_\infty = 0, \tag{2}$$

$$(CE)_1(st) - \lim_{r \rightarrow \infty} \|A_r(t; x) - x\|_\infty = 0, \tag{3}$$

$$(CE)_1(st) - \lim_{r \rightarrow \infty} \|A_r(t^2; x) - x^2\|_\infty = 0. \tag{4}$$

Proof. Since each of  $1, t, t^2$  belongs to  $C[a, b]$ , conditions (2)-(4) follow immediately from (1). In order to complete the proof of assertion (1) of Theorem 2.3, we assume that the conditions (2)-(4) hold true. Let  $f \in C[a, b]$  and  $x \in [a, b]$  be fixed. Since  $f \in C[a, b]$  and  $f$  is bounded on the whole real line, we have

$$|f(x)| \leq M, \quad -\infty < x < \infty,$$

and therefore

$$|f(t) - f(x)| \leq 2M, \quad -\infty < x, t < \infty. \tag{5}$$

Also, by the continuity of  $f$ , for every  $\varepsilon > 0$ , there exists a number  $\delta > 0$  such that  $|t - x| < \delta$ , we get

$$|f(t) - f(x)| < \varepsilon. \tag{6}$$

Combining the inequalities (5) and (6) and putting  $\psi(t) = (t - x)^2$ , we get

$$|f(t) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2} \psi, \quad \forall |t - x| < \delta$$

which means

$$-\varepsilon - \frac{2M}{\delta^2} \psi \leq f(t) - f(x) \leq \varepsilon + \frac{2M}{\delta^2} \psi.$$

Since the operators are positive, we can write

$$A_r(1, x) \left( -\varepsilon - \frac{2M}{\delta^2} \psi \right) \leq A_r(1, x) (f(t) - f(x)) \leq A_r(1, x) \left( \varepsilon + \frac{2M}{\delta^2} \psi \right)$$

and by the linearity of  $A_r$ , we have

$$-\varepsilon A_r(1, x) - \frac{2M}{\delta^2} A_r(\psi, x) \leq A_r(f, x) - f(x) A_r(1, x) \leq \varepsilon A_r(1, x) + \frac{2M}{\delta^2} A_r(\psi, x). \tag{7}$$

On the other hand,

$$A_r(f, x) - f(x) = A_r(f, x) - f(x) A_r(1, x) + f(x) [A_r(1, x) - 1]. \tag{8}$$

If we consider inequality (7) and equality (8), we have

$$A_r(f, x) - f(x) < \varepsilon A_r(1, x) + \frac{2M}{\delta^2} A_r(\psi, x) + f(x) [A_r(1, x) - 1]. \tag{9}$$

Let us estimate  $A_r(\psi, x)$ ,

$$\begin{aligned} A_r(\psi, x) &= A_r((t-x)^2, x) \\ &= [A_r(t^2, x) - x^2] - 2x[A_r(t, x) - x] + x^2[A_r(1, x) - 1]. \end{aligned}$$

Using this equality in (9), we obtain

$$\begin{aligned} A_r(f, x) - f(x) &< \varepsilon [A_r(1, x) - 1] + \varepsilon + f(x) [A_r(1, x) - 1] \\ &\quad + \frac{2M}{\delta^2} \{ [A_r(t^2, x) - x^2] - 2x[A_r(t, x) - x] + x^2[A_r(1, x) - 1] \}. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we have

$$\begin{aligned} \|A_r(f; x) - f(x)\|_\infty &\leq \left( \varepsilon + \frac{2Mb^2}{\delta^2} + M \right) \|A_r(1, x) - 1\|_\infty \\ &\quad + \frac{4Mb}{\delta^2} \|A_r(t, x) - x\|_\infty + \frac{2M}{\delta^2} \|A_r(t^2, x) - x^2\|_\infty \\ &\leq K \left\{ \|A_r(1, x) - 1\|_\infty + \|A_r(t, x) - x\|_\infty + \|A_r(t^2, x) - x^2\|_\infty \right\}, \end{aligned} \tag{10}$$

where  $K = \max \left\{ \varepsilon + \frac{2Mb^2}{\delta^2} + M, \frac{4Mb}{\delta^2}, \frac{2M}{\delta^2} \right\}$ .

Finally, replacing  $A_r(\cdot; x)$  by  $L_m(\cdot, x) = \frac{1}{m+1} \sum_{k=0}^m \frac{1}{2^k} \sum_{r=0}^k \binom{k}{r} A_r(\cdot; x)$  in both sides of (10), and for  $\varepsilon' > 0$ , write

$$\begin{aligned} D &= \left\{ m \leq \mathbb{N} : \|L_m(1; x) - 1\|_\infty + \|L_m(t; x) - x\|_\infty + \|L_m(t^2; x) - x^2\|_\infty \geq \frac{\varepsilon'}{K} \right\}, \\ D_1 &= \left\{ m \leq \mathbb{N} : \|L_m(1; x) - 1\|_\infty \geq \frac{\varepsilon'}{3K} \right\}, \\ D_2 &= \left\{ m \leq \mathbb{N} : \|L_m(t; x) - x\|_\infty \geq \frac{\varepsilon'}{3K} \right\}, \\ D_3 &= \left\{ m \leq \mathbb{N} : \|L_m(t^2; x) - x^2\|_\infty \geq \frac{\varepsilon'}{3K} \right\}. \end{aligned}$$

Then,  $D \subset D_1 \cup D_2 \cup D_3$  and so  $\delta(D) \leq \delta(D_1) + \delta(D_2) + \delta(D_3)$ . Therefore, using conditions (2)-(4), we get

$$(CE)_1(st) - \lim_{n \rightarrow \infty} \|A_n(f; x) - f(x)\|_\infty = 0,$$

which completes the proof.

We remark that our Theorem 1 is stronger than that of classical Korovkin approximation theorem as well as Theorem 1 of Gadjiev and Orhan [7]. For this claim, we consider the following example:

**Example 2.4.** Considering the sequence of Bernstein operators

$$B_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1],$$

and using the sequence  $(x_n)$  as mentioned in the beginning of Section 2, we define the sequence of linear operators as  $K_n : C[0, 1] \rightarrow C[0, 1]$  with

$$K_n(f, x) = (1 + |x_n|) B_n(f, x).$$

Since  $B_n(1, x) = 1$ ,  $B_n(t, x) = x$  and  $B_n(t^2, x) = x^2 + \frac{x(1-x)}{n}$ , the sequence  $K_n$  satisfies the conditions (2)-(4). Hence we have

$$(CE)_1(st) - \lim_{n \rightarrow \infty} \|K_n(f; x) - f(x)\|_\infty = 0.$$

Since  $B_n(f, 0) = f(0)$ ,  $K_n(f, 0) = (1 + |x_n|) f(0)$ . Hence

$$\|K_n(f; x) - f(x)\|_\infty \geq |K_n(f, 0) - f(0)| = |x_n| |f(0)|,$$

which means that  $(K_n)$  does not satisfy the classical Korovkin theorem as well as Theorem 1 of Gadjiev and Orhan [7], since  $\limsup_{n \rightarrow \infty} |x_n|$  does not exist.

### 3. Order of Statistical Summability $(C, 1)$ $(E, 1)$ of $(A_n)$

In this section, rate of statistical summability  $(C, 1)$   $(E, 1)$  of a sequence of positive linear operators defined on  $C[a, b]$  is presented. For this purpose, we give following definition.

**Definition 3.1.** Let  $(\alpha_n)$  be a positive non-increasing sequence. Then, the sequence  $x = (x_k)$  is said to be statistical summable  $(C, 1)$   $(E, 1)$  to the number  $L$  with the rate  $o(\alpha_n)$  if for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{\alpha_n} |\{m \leq n : |t_m - L| \geq \varepsilon\}| = 0$$

and we write  $x_k - L = (C, E)_1(st) - o(\alpha_n)$ .

By this definition, we immediately have:

**Lemma 3.2.** Let  $(\alpha_n)$  and  $(\beta_n)$  be two positive non-increasing sequences. Let  $x = (x_k)$  and  $y = (y_k)$  be two sequences such that  $x_k - L_1 = (C, E)_1(st) - o(\alpha_n)$  and  $y_k - L_2 = (C, E)_1(st) - o(\beta_n)$ . Then,

- (i)  $s(x_k - L_1) = (C, E)_1(st) - o(\alpha_n)$ , for any scalar  $s$ ,
- (ii)  $(x_k - L_1) \pm (y_k - L_2) = (C, E)_1(st) - o(\gamma_n)$
- (iii)  $(x_k - L_1) \cdot (y_k - L_2) = (C, E)_1(st) - o(\alpha_n \beta_n)$ ,

where  $\gamma_n = \max\{\alpha_n, \beta_n\}$ .

Also, we recall the notion of modulus of continuity. The usual modulus of continuity of  $f \in C[a, b]$  is given as

$$\omega(f, \delta) = \sup_{|x-y| < \delta} |f(x) - f(y)|$$

and has the property

$$|f(x) - f(y)| \leq \omega(f, \delta) \left( \frac{|x-y|}{\delta} + 1 \right). \tag{11}$$

Now, we present following result.

**Theorem 3.3.** Let  $\{A_r\}$  be a sequence of positive linear operators from  $C[a, b]$  into  $C[a, b]$ . Suppose that

$$(a) \|A_r(1, x) - 1\|_\infty = (C, E)_1(st) - o(\alpha_n)$$

$$(b) \omega(f, \delta_r) = (C, E)_1(st) - o(\beta_n), \text{ where } \delta_r(x) = \sqrt{A_r(\phi^2, x)}, \phi(x) = (y - x).$$

Then, for all  $f \in C[a, b]$ , we have

$$\|A_r(f, x) - f\|_\infty = (C, E)_1(st) - o(\gamma_n),$$

where  $\gamma_n = \max\{\alpha_n, \beta_n\}$ .

Proof. Let  $f \in C[a, b]$  and  $x \in [a, b]$ . If we consider the equality (8) and inequality (11), we can write

$$\begin{aligned} |A_r(f, x) - f(x)| &\leq A_r(|f(t) - f(x)|, x) + |f(x)| |A_r(1, x) - 1| \\ &\leq A_r\left(\frac{|x-y|}{\delta} + 1, x\right) \omega(f, \delta) + |f(x)| |A_r(1, x) - 1| \\ &\leq A_r\left(1 + \frac{(x-y)^2}{\delta^2}, x\right) \omega(f, \delta) + |f(x)| |A_r(1, x) - 1| \\ &\leq \left(A_r(1, x) + \frac{1}{\delta^2} A_r(\phi^2, x)\right) \omega(f, \delta) + |f(x)| |A_r(1, x) - 1| \\ &\leq \omega(f, \delta) |A_r(1, x) - 1| + |f(x)| |A_r(1, x) - 1| + \omega(f, \delta) \\ &\quad + \omega(f, \delta) \frac{1}{\delta^2} A_r(\phi^2, x), \end{aligned}$$

and if we choose  $\delta = \delta_r = \sqrt{A_r(\phi^2, x)}$ , we get

$$\begin{aligned} \|A_r(f, x) - f(x)\|_\infty &\leq \|f\|_\infty \|A_r(1, x) - 1\|_\infty + 2\omega(f, \delta_r) + \omega(f, \delta_r) \|A_r(1, x) - 1\|_\infty \\ &\leq K \{ \|A_r(1, x) - 1\|_\infty + \omega(f, \delta_r) + \omega(f, \delta_r) \|A_r(1, x) - 1\|_\infty \}, \end{aligned}$$

where  $K = \max\{\|f\|_\infty, 2\}$ . Now, replacing  $A_r(\cdot, x)$  by  $L_n(\cdot, x) = \frac{1}{n+1} \sum_{k=0}^n \frac{1}{2^k} \sum_{r=0}^k \binom{k}{r} A_r(\cdot, x)$ , we get

$$\|L_n(f, x) - f(x)\|_\infty \leq K \{ \|L_n(1, x) - 1\|_\infty + \omega(f, \delta_r) + \omega(f, \delta_r) \|A_r(1, x) - 1\|_\infty \}.$$

Using the Definition 3.1 and the conditions (a) and (b), we get the desired result.

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