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Some Generalizations of Different Type of Integral Inequalities for MT-convex Functions

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Abstract. By using the identity for fractional integrals given in [13], we obtained some new estimates on generalizations of Hermite-Hadamard, Ostrowski and Simpson type inequalities for MT-convex functions via Riemann Liouville fractional integral.

1. Introduction

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

In [5], Dragomir and Agarwal established the following result connected with the right part of (1):

Theorem 1.2. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o, a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

The following result is known in the literature as the Ostrowski inequality (see [21, page 468] or [22]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

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Theorem 1.3. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a differentiable mapping in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $\left|f'(x)\right| \leq M$ for all $x \in [a, b]$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall\, x \in [a, b]. \tag{2}$$

The following inequality is well known in the literature as Simpson's inequality:

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^5, \tag{3}$$

where the mapping $f : [a, b] \to \mathbb{R}$ is supposed to be four time differentiable on the interval (a, b) and having the fourth derivative bounded on (a, b), that is $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. This inequality gives an error bound for the classical Simpson quadrature formula, which, actually, is one of the most used quadrature formulae in practical applications.

In [34] (see also [31, 33]), Tunç and Yidirim defined the following so-called MT-convex function, which may be regarded as a special case of $h-$convex function (see [35]):

Definition 1.4. *A function f: $I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of MT(I), if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{4}$$

Theorem 1.5. *Let $f \in MT(I), a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \tag{5}$$

and

$$\frac{2}{b-a}\int_a^b \tau(x) f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{6}$$

where $\tau(x) = \frac{\sqrt{(b-x)(x-a)}}{b-a}, x \in [a, b]$.

For other recent results concerning Hermite-Hadamard, Ostrowski and Simpson type inequalities through various classes of convex functions, see [2, 4, 6, 10, 11, 14, 15, 18, 24, 26–28, 32] and the references cited therein.

Fractional calculus [8, 20, 25] was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. We recall some definitions and preliminary facts of fractional calculus theory which will be used in this paper.

Definition 1.6. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}(f)$ and $J_{b^-}^{\alpha}(f)$ of older $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, b > x,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\mathrm{d}u$. *Here* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$. *In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.*

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard, Grüss, Simpson, or Ostrowski type inequalities for functions of different classes, see [1, 3, 17, 29, 30, 36] where further references are listed.

The main aim of this paper is to establish some generalizations of Hermite-Hadamard, Ostrowski and Simpson type inequalities for MT-convex functions via Riemann Liouville fractional integral. An interesting feature of our results is that they provide new estimates on these types of integral inequalities for MT-convex functions.

2. Generalized Integral Inequalities Via Fractional Integrals

Let $f \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o, the interior of I, throughout this section we will take

$$\begin{aligned} S_f(x,\lambda,\alpha,a,b) =&(1-\lambda)\left[\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right]f(x)+\lambda\left[\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}\right] \\ &-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)], \end{aligned} \tag{7}$$

where $a,b\in I$ with $a<b$, $x\in[a,b]$, $\lambda\in[0,1]$, $\alpha>0$ and Γ is the Euler Gamma function.

In order to prove the main results in this section, we need the following identity mentioned in [13].

Lemma 2.1. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o such that $f'\in L_1[a,b]$, where $a,b\in I$ with $a<b$. Then for all $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$ we have:*

$$S_f(x,\lambda,\alpha,a,b) =\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1(t^\alpha-\lambda)f'(tx+(1-t)a)\mathrm{d}t+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1(\lambda-t^\alpha)f'(tx+(1-t)b)\mathrm{d}t. \tag{8}$$

With Lemma 2.1, we can obtain the following results.

Theorem 2.2. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable function on I^o such that $f'\in L_1[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|$ is MT-convex function on $[a,b]$ and $|f'(x)|\le M, x\in[a,b]$, then we have the following inequality for fractional integrals with $\alpha>0$ and $\lambda\in[0,1]$:*

$$|S_f(x,\lambda,\alpha,a,b)|\le\frac{MA(\alpha,\lambda)[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}, \tag{9}$$

where

$$A(\alpha,\lambda) =2\lambda\left(\beta\left(\lambda^{\frac{1}{\alpha}};\frac{3}{2},\frac{1}{2}\right)+\beta\left(\lambda^{\frac{1}{\alpha}};\frac{1}{2},\frac{3}{2}\right)\right)+\beta\left(\alpha+\frac{1}{2},\frac{1}{2}\right)-\lambda\pi-2\left(\beta\left(\lambda^{\frac{1}{\alpha}};\alpha+\frac{3}{2},\frac{1}{2}\right)+\beta\left(\lambda^{\frac{1}{\alpha}};\alpha+\frac{1}{2},\frac{3}{2}\right)\right),$$

$\beta(\cdot\,;\cdot\,,\cdot)$ is the incomplete Beta function defined by

$$\beta(a;x,y)=\int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, 0<a\le1, x,y>0.$$

Proof. From Lemma 2.1, property of the modulus and using the MT-convexity of $|f'|$, we have

$$\begin{aligned} |S_f(x,\lambda,\alpha,a,b)|\le&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1|t^\alpha-\lambda||f'(tx+(1-t)a)|\mathrm{d}t+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1|\lambda-t^\alpha||f'(tx+(1-t)b)|\mathrm{d}t \\ \le&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1|t^\alpha-\lambda|\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|\right]\mathrm{d}t \\ &+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1|t^\alpha-\lambda|\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|\right]\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M(x-a)^{\alpha+1}}{2(b-a)} \int_0^1 |t^\alpha - \lambda| \left(t^{1/2}(1-t)^{-1/2} + t^{-1/2}(1-t)^{1/2}\right) \mathrm{d}t \\
&\quad + \frac{M(b-x)^{\alpha+1}}{2(b-a)} \int_0^1 |t^\alpha - \lambda| \left(t^{1/2}(1-t)^{-1/2} + t^{-1/2}(1-t)^{1/2}\right) \mathrm{d}t \\
&= \frac{M[(x-a)^{\alpha+1} + (b-x)^{\alpha+1}]}{2(b-a)} \int_0^1 |t^\alpha - \lambda| \left(t^{1/2}(1-t)^{-1/2} + t^{-1/2}(1-t)^{1/2}\right) \mathrm{d}t.
\end{aligned}
$$

By computation

$$
\begin{aligned}
&\int_0^1 |t^\alpha - \lambda| \left(t^{1/2}(1-t)^{-1/2} + t^{-1/2}(1-t)^{1/2}\right) \mathrm{d}t \\
&= \int_0^{\lambda^{\frac{1}{\alpha}}} (\lambda - t^\alpha) \left(t^{1/2}(1-t)^{-1/2}\right) \mathrm{d}t + \int_{\lambda^{\frac{1}{\alpha}}}^1 (t^\alpha - \lambda) \left(t^{1/2}(1-t)^{-1/2}\right) \mathrm{d}t \\
&\quad + \int_0^{\lambda^{\frac{1}{\alpha}}} (\lambda - t^\alpha) \left(t^{-1/2}(1-t)^{1/2}\right) \mathrm{d}t + \int_{\lambda^{\frac{1}{\alpha}}}^1 (t^\alpha - \lambda) \left(t^{-1/2}(1-t)^{1/2}\right) \mathrm{d}t \\
&= \lambda \int_0^{\lambda^{\frac{1}{\alpha}}} t^{1/2}(1-t)^{-1/2} \mathrm{d}t - \int_0^{\lambda^{\frac{1}{\alpha}}} t^{\alpha+1/2}(1-t)^{-1/2} \mathrm{d}t + \int_{\lambda^{\frac{1}{\alpha}}}^1 t^{\alpha+1/2}(1-t)^{-1/2} \mathrm{d}t - \lambda \int_{\lambda^{\frac{1}{\alpha}}}^1 t^{1/2}(1-t)^{-1/2} \mathrm{d}t \\
&\quad + \lambda \int_0^{\lambda^{\frac{1}{\alpha}}} t^{-1/2}(1-t)^{1/2} \mathrm{d}t - \int_0^{\lambda^{\frac{1}{\alpha}}} t^{\alpha-1/2}(1-t)^{1/2} \mathrm{d}t + \int_{\lambda^{\frac{1}{\alpha}}}^1 t^{\alpha-1/2}(1-t)^{1/2} \mathrm{d}t - \lambda \int_{\lambda^{\frac{1}{\alpha}}}^1 t^{-1/2}(1-t)^{1/2} \mathrm{d}t \\
&= \int_0^1 t^{\alpha+1/2}(1-t)^{-1/2} \mathrm{d}t - \lambda \int_0^1 t^{1/2}(1-t)^{-1/2} \mathrm{d}t + \int_0^1 t^{\alpha-1/2}(1-t)^{1/2} \mathrm{d}t - \lambda \int_0^1 t^{-1/2}(1-t)^{1/2} \mathrm{d}t \\
&\quad + 2\lambda \int_0^{\lambda^{\frac{1}{\alpha}}} t^{1/2}(1-t)^{-1/2} \mathrm{d}t - 2 \int_0^{\lambda^{\frac{1}{\alpha}}} t^{\alpha+1/2}(1-t)^{-1/2} \mathrm{d}t \\
&\quad + 2\lambda \int_0^{\lambda^{\frac{1}{\alpha}}} t^{-1/2}(1-t)^{-1/2} \mathrm{d}t - 2 \int_0^{\lambda^{\frac{1}{\alpha}}} t^{\alpha-1/2}(1-t)^{1/2} \mathrm{d}t \\
&= 2\lambda \left(\beta\left(\lambda^{\frac{1}{\alpha}}; \frac{3}{2}, \frac{1}{2}\right) + \beta\left(\lambda^{\frac{1}{\alpha}}; \frac{1}{2}, \frac{3}{2}\right)\right) + \beta\left(\alpha + \frac{3}{2}, \frac{1}{2}\right) + \beta\left(\alpha + \frac{1}{2}, \frac{3}{2}\right) - \lambda\pi \\
&\quad - 2\left(\beta\left(\lambda^{\frac{1}{\alpha}}; \alpha + \frac{3}{2}, \frac{1}{2}\right) + \beta\left(\lambda^{\frac{1}{\alpha}}; \alpha + \frac{1}{2}, \frac{3}{2}\right)\right),
\end{aligned}
$$

where we have used the incomplete Beta function defined by

$$
\beta(a; x, y) = \int_0^a t^{x-1}(1-t)^{y-1} \mathrm{d}t, 0 < a \leq 1, x, y > 0,
$$

which is mentioned in [12] (for more details see [7] and [9]), the proof is completed. □

Remark 2.3. *In Theorem 2.2, if we choose $x = (a+b)/2$, we get the following generalized Simpson type inequality for fractional integrals*

$$
\begin{aligned}
&\left| S_f\left(\frac{a+b}{2}, \lambda, \alpha, a, b\right) \right| \\
&= \left| \frac{(1-\lambda)(b-a)^{\alpha-1}}{2^{\alpha-1}} f\left(\frac{a+b}{2}\right) + \frac{\lambda(b-a)^{\alpha-1}}{2^{\alpha-1}} \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{b-a} \left[J^\alpha_{(\frac{a+b}{2})^-} f(a) + J^\alpha_{(\frac{a+b}{2})^+} f(b) \right] \right| \\
&\leq \frac{MA(\alpha, \lambda)(b-a)^\alpha}{2^{\alpha+1}}.
\end{aligned}
$$

Remark 2.4. *In Theorem 2.2, if we choose $\lambda = 1$, we get the following generalized trapezoid type inequality for fractional integrals*

$$\begin{aligned}|S_f(x,1,\alpha,a,b)| &= \left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right| \\ &\le \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\left[\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right],\end{aligned}$$

which is the same of the inequality in [19, Theorem 3.1].

Remark 2.5. *In Theorem 2.2, if we choose $\lambda = 1$ and $x = \frac{a+b}{2}$, we get the following trapezoid type inequality for fractional integrals*

$$\begin{aligned}\left|S_f\left(\frac{a+b}{2},1,\alpha,a,b\right)\right| &= \left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]\right| \\ &\le \frac{M(b-a)^\alpha}{2^{\alpha+1}}\left[\pi - \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)}\right].\end{aligned}$$

Remark 2.6. *In Theorem 2.2, if we choose $\lambda = 1$ and $\alpha = 1$, we get the following generalized trapezoid type inequality for MT-convex functions*

$$|S_f(x,1,1,a,b)| = \left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le \frac{M\pi[(x-a)^2+(b-x)^2]}{4(b-a)},$$

which is the same of the inequality in [19, Theorem 2.1].

Remark 2.7. *In Theorem 2.2, if we choose $\lambda = 0$, we get the following Ostrowski type inequality for fractional integrals*

$$\begin{aligned}|S_f(x,0,\alpha,a,b)| &= \left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x) - \frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right| \\ &\le \frac{M\beta(\alpha+\frac{1}{2},\frac{1}{2})[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}.\end{aligned}$$

Remark 2.8. *In Theorem 2.2, if we choose $\lambda = 0$ and $x = \frac{a+b}{2}$, we get the following midpoint type inequality for fractional integrals*

$$\left|S_f\left(\frac{a+b}{2},0,\alpha,a,b\right)\right| = \left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]\right| \le \frac{M\beta(\alpha+\frac{1}{2},\frac{1}{2})(b-a)^\alpha}{2^{\alpha+1}}.$$

Remark 2.9. *In Theorem 2.2, if we choose $\lambda = 0$ and $\alpha = 1$, we get the following Ostrowski type inequality for MT-convex functions*

$$|S_f(x,0,1,a,b)| = \left|f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le \frac{M\pi[(x-a)^2+(b-a)^2]}{4(b-a)},$$

which is the same of the inequality in [32, Theorem 2].

Theorem 2.10. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is MT-convex function on $[a,b]$, $q > 1$, $p^{-1}+q^{-1} = 1$ and $|f'(x)| \le M, x \in [a,b]$, then we have the following inequality for fractional integrals with $\alpha > 0$ and $\lambda \in [0,1]$:*

$$|S_f(x,\lambda,\alpha,a,b)| \le \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{b-a}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}(B(\alpha,\lambda))^{\frac{1}{p}}, \tag{10}$$

where

$$B(\alpha,\lambda)=\frac{2}{\alpha}\int_0^\lambda(\lambda-s)^p s^{\frac{1}{\alpha}-1}\mathrm{d}s-\frac{1}{\alpha}\int_0^1(\lambda-s)^p s^{\frac{1}{\alpha}-1}\mathrm{d}s.$$

Proof. From Lemma 2.1, property of the modulus and using the Hölder inequality, we have

$$\begin{aligned}|S_f(x,\lambda,\alpha,a,b)|\le&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1|t^\alpha-\lambda||f'(tx+(1-t)a)|\mathrm{d}t+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1|\lambda-t^\alpha||f'(tx+(1-t)b)|\mathrm{d}t\\ \le&\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1|t^\alpha-\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1|t^\alpha-\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)|\le M$, then we have

$$\int_0^1|f'(tx+(1-t)a)|^q\mathrm{d}t\le\int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right]\mathrm{d}t=\frac{\pi}{4}\left[|f'(x)|^q+|f'(a)|^q\right]\le\frac{\pi}{2}M^q,$$

$$\int_0^1|f'(tx+(1-t)b)|^q\mathrm{d}t\le\int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t=\frac{\pi}{4}\left[|f'(x)|^q+|f'(b)|^q\right]\le\frac{\pi}{2}M^q$$

and

$$\begin{aligned}\int_0^1|t^\alpha-\lambda|^p\mathrm{d}t=&\frac{1}{\alpha}\int_0^1|\lambda-s|^p s^{\frac{1}{\alpha}-1}\mathrm{d}s=\frac{1}{\alpha}\int_0^\lambda(\lambda-s)^p s^{\frac{1}{\alpha}-1}\mathrm{d}s+\frac{1}{\alpha}\int_\lambda^1(s-\lambda)^p s^{\frac{1}{\alpha}-1}\mathrm{d}s\\ =&\frac{2}{\alpha}\int_0^\lambda(\lambda-s)^p s^{\frac{1}{\alpha}-1}\mathrm{d}s-\frac{1}{\alpha}\int_0^1(\lambda-s)^p s^{\frac{1}{\alpha}-1}\mathrm{d}s.\end{aligned}$$

Hence we have

$$|S_f(x,\lambda,\alpha,a,b)|\le\frac{(x-a)^{\alpha+1}}{b-a}\left(B(\alpha,\lambda)\right)^{\frac{1}{p}}\left(\frac{\pi M^q}{2}\right)^{\frac{1}{q}}+\frac{(b-x)^{\alpha+1}}{b-a}\left(B(\alpha,\lambda)\right)^{\frac{1}{p}}\left(\frac{\pi M^q}{2}\right)^{\frac{1}{q}},$$

which completes the proof. □

Remark 2.11. *In Theorem 2.10, if we choose $x=(a+b)/2$, we get the following generalized Simpson type inequality for fractional integrals*

$$\begin{aligned}\left|S_f\left(\frac{a+b}{2},\lambda,\alpha,a,b\right)\right|=&\left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}f\left(\frac{a+b}{2}\right)+\frac{\lambda(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]\right|\\ \le&\frac{M(b-a)^\alpha}{2^\alpha}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(B(\alpha,\lambda)\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 2.12. *In Theorem 2.10, if we choose $\lambda=1$, we get the following generalized trapezoid type inequality for fractional integrals*

$$\begin{aligned}|S_f(x,1,\alpha,a,b)|=&\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right|\\ \le&\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{2}-\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right]^{\frac{1}{q}},\end{aligned}$$

which is the same of the inequality in [19, Theorem 3.2].

Remark 2.13. *In Theorem 2.10, if we choose $\lambda = 1$ and $x = \frac{a+b}{2}$, we get the following trapezoid type inequality for fractional integrals*

$$\begin{aligned}\left|S_f\left(\frac{a+b}{2},1,\alpha,a,b\right)\right| &= \left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{(\frac{a+b}{2})^-}f(a) + J^{\alpha}_{(\frac{a+b}{2})^+}f(b)\right]\right| \\ &\le \frac{M(b-a)^{\alpha}}{2^{\alpha}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p)\Gamma(\frac{1}{\alpha})}{\alpha\Gamma(1+p+\frac{1}{\alpha})}\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 2.14. *In Theorem 2.10, if we choose $\lambda = 1$ and $\alpha = 1$, we get the following generalized trapezoid type inequality for MT-convex functions*

$$|S_f(x,1,1,a,b)| = \left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le \frac{M}{(1+p)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)},$$

which is the same of the inequality in [19, Theorem 2.2].

Remark 2.15. *In Theorem 2.10, if we choose $\lambda = 0$, we get the following Ostrowski type inequality for fractional integrals*

$$\begin{aligned}|S_f(x,0,\alpha,a,b)| &= \left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x) - \frac{\Gamma(\alpha+1)}{b-a}[J^{\alpha}_{x^-}f(a) + J^{\alpha}_{x^+}f(b)]\right| \\ &\le \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{b-a}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 2.16. *In Theorem 2.10, if we choose $\lambda = 0$ and $x = \frac{a+b}{2}$, we get the following midpoint type inequality for fractional integrals*

$$\begin{aligned}\left|S_f\left(\frac{a+b}{2},0,\alpha,a,b\right)\right| &= \left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{(\frac{a+b}{2})^-}f(a) + J^{\alpha}_{(\frac{a+b}{2})^+}f(b)\right]\right| \\ &\le \frac{M(b-a)^{\alpha}}{2^{\alpha}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}.\end{aligned}$$

Remark 2.17. *In Theorem 2.10, if we choose $\lambda = 0$ and $\alpha = 1$, we get the following Ostrowski type inequality for MT-convex functions*

$$|S_f(x,0,1,a,b)| = \left|f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le \frac{M}{(1+p)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)},$$

which is the same of the inequality in [32, Theorem 3].

Theorem 2.18. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I^o such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is MT-convex function on $[a,b]$, $q \ge 1$ and $|f'(x)| \le M, x \in [a,b]$, then we have the following inequality for fractional integrals with $\alpha > 0$ and $\lambda \in [0,1]$:*

$$|S_f(x,\lambda,\alpha,a,b)| \le \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{b-a}\left(\frac{A(\alpha,\lambda)}{2}\right)^{\frac{1}{q}}\left(\frac{2\alpha\lambda^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\lambda\right)^{1-\frac{1}{q}}. \tag{11}$$

Proof. From Lemma 2.1, property of the modulus and using the power mean inequality, we have

$$|S_f(x,\lambda,\alpha,a,b)| \le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 |t^{\alpha}-\lambda||f'(tx+(1-t)a)|\mathrm{d}t + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 |\lambda-t^{\alpha}||f'(tx+(1-t)b)|\mathrm{d}t$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 |t^\alpha-\lambda|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 |t^\alpha-\lambda||f'(tx+(1-t)a)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 |t^\alpha-\lambda|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 |t^\alpha-\lambda||f'(tx+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}. \tag{12}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)| \leq M$, then we have

$$\int_0^1 |t^\alpha-\lambda||f'(tx+(1-t)a)|^q\mathrm{d}t \leq \int_0^1 |t^\alpha-\lambda|\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right]\mathrm{d}t$$
$$\leq \frac{M^q}{2}\int_0^1 |t^\alpha-\lambda|\left(t^{1/2}(1-t)^{-1/2}+t^{-1/2}(1-t)^{1/2}\right)\mathrm{d}t = \frac{M^q}{2}A(\alpha,\lambda), \tag{13}$$

$$\int_0^1 |t^\alpha-\lambda||f'(tx+(1-t)b)|^q\mathrm{d}t \leq \int_0^1 |t^\alpha-\lambda|\left[\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right]\mathrm{d}t \leq \frac{M^q}{2}A(\alpha,\lambda) \tag{14}$$

and

$$\int_0^1 |t^\alpha-\lambda|\mathrm{d}t = \int_0^{\lambda^{\frac{1}{\alpha}}} (\lambda-t^\alpha)\mathrm{d}t + \int_{\lambda^{\frac{1}{\alpha}}}^1 (t^\alpha-\lambda)\mathrm{d}t = \frac{2\alpha\lambda^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\lambda. \tag{15}$$

If we use (13) , (14) and (15) in (12) we obtain the desired result. □

Remark 2.19. *In Theorem 2.18, if we choose $x = (a+b)/2$, we get the following generalized Simpson type inequality for fractional integrals*

$$\left|S_f\left(\frac{a+b}{2},\lambda,\alpha,a,b\right)\right| = \left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}f\left(\frac{a+b}{2}\right)+\frac{\lambda(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]\right|$$
$$\leq \frac{M(b-a)^\alpha}{2^\alpha}\left(\frac{A(\alpha,\lambda)}{2}\right)^{\frac{1}{q}}\left(\frac{2\alpha\lambda^{1+\frac{1}{\alpha}}+1}{\alpha+1}-\lambda\right)^{1-\frac{1}{q}}.$$

Remark 2.20. *In Theorem 2.18, if we choose $\lambda = 1$, we get the following generalized trapezoid type inequality for fractional integrals*

$$|S_f(x,1,\alpha,a,b)| = \left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}[J^\alpha_{x^-}f(a)+J^\alpha_{x^+}f(b)]\right|$$
$$\leq \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{b-a}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{2}-\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right]^{\frac{1}{q}},$$

which is the same of the inequality in [19, Theorem 3.3].

Remark 2.21. *In Theorem 2.18, if we choose $\lambda = 1$ and $x = \frac{a+b}{2}$, we get the following trapezoid type inequality for fractional integrals*

$$\left|S_f\left(\frac{a+b}{2},1,\alpha,a,b\right)\right| = \left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]\right|$$
$$\leq \frac{M(b-a)^\alpha}{2^\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{\pi}{2}-\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right]^{\frac{1}{q}}.$$

Remark 2.22. *In Theorem 2.18, if we choose $\lambda = 1$ and $\alpha = 1$, we get the following generalized trapezoid type inequality for MT-convex functions*

$$|S_f(x,1,1,a,b)| = \left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le M\left(\frac{1}{2}\right)^{1+\frac{1}{q}} \pi^{\frac{1}{q}} \frac{(x-a)^2+(b-x)^2}{(b-a)},$$

which is the same of the inequality in [19, Theorem 2.3].

Remark 2.23. *In Theorem 2.18, if we choose $\lambda = 0$, we get the following Ostrowski type inequality for fractional integrals*

$$\begin{aligned}|S_f(x,0,\alpha,a,b)| &= \left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a}[J_{x^-}^\alpha f(a) + J_{x^+}^\alpha f(b)]\right| \\ &\le \frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{b-a}\left(\frac{\beta(\alpha+\frac{1}{2},\frac{1}{2})}{2}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}.\end{aligned}$$

Remark 2.24. *In Theorem 2.18, if we choose $\lambda = 0$ and $x = \frac{a+b}{2}$, we get the following midpoint type inequality for fractional integrals*

$$\begin{aligned}\left|S_f\left(\frac{a+b}{2},0,\alpha,a,b\right)\right| &= \left|\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{(\frac{a+b}{2})^-}^\alpha f(a) + J_{(\frac{a+b}{2})^+}^\alpha f(b)\right]\right| \\ &\le \frac{M(b-a)^\alpha}{2^\alpha}\left(\frac{\beta(\alpha+\frac{1}{2},\frac{1}{2})}{2}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}.\end{aligned}$$

Remark 2.25. *In Theorem 2.18, if we choose $\lambda = 0$ and $\alpha = 1$, we get the following Ostrowski type inequality for MT-convex functions*

$$|S_f(x,0,1,a,b)| = \left|f(x) - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \le M\left(\frac{1}{2}\right)^{1+\frac{1}{q}} \pi^{\frac{1}{q}} \frac{(x-a)^2+(b-x)^2}{(b-a)},$$

which is the same of inequality in [32, Theorem 4].

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