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Soft Ditopological Spaces

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Abstract. We introduce the concept of a soft ditopological space as the "soft generalization" of the concept of a ditopological space as it is defined in the papers by L.M. Brown and co-authors, see e.g. L. M. Brown, R. Ertürk, Ş. Dost, Ditopological texture spaces and fuzzy topology, I. Basic Concepts, Fuzzy Sets and Systems **147** (2) (2004), 171–199. Actually a soft ditopological space is a soft set with two independent structures on it - a soft topology and a soft co-topology. The first one is used to describe openness-type properties of a space while the second one deals with its closedness-type properties. We study basic properties of such spaces and accordingly defined continuous mappings between such spaces.

1. Introduction

The concept of a soft set introduced in 1999 by D Molodtsov [19] gave rise to a large amount of publications, exploiting soft sets both from theoretical point of view and in the prospectives of their applications. Actually in modern times it happens very often when a new mathematical concept, especially if it is assumed to have practical applications, arises interest of many researchers. Especially this concerns young people since it allows to enter the real scientific life in a relatively short way. In particular this happened with the soft sets. Among different areas of theoretical mathematics where soft sets are exploited probably the largest amount of papers are related to general topology. Soft topological and fuzzy soft topological spaces and their properties were studied in [1, 3, 10, 14, 18, 20, 23–25, 30]. An alternative approach to the concept of topology in the framework of soft sets was developed in [21, 22]. Since the subject of this work is also related to soft topology, we feel it is important to explain more clearly our position in this field.

First we conclude, that for applications of soft sets in topological setting it is more natural to work in the framework of ditopologies, than in the framework of topologies. The concept of a ditopology was introduced by L.M. Brown and studied in a series of papers by L.M. Brown and co-authors, see e.g. [5–8] Ditopologies are related to the concept of a bitopology introduced by J.L. Kelly [16]. However, as different from bitopologies, in ditopologies two conceptional different structures on a set are exploited: one for description of properties related to openness of sets, while the other describes the properties related to closedness of sets. These structures need not have any interrelations between them, although in the

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trivial case they can collapse into a usual topology. The idea of a ditopology seems especially suitable for the soft variation of topology since it allows to avoid the operation of complementation which is often "inconvenient" in the framework of soft set theory.

The second distinction of our work to compare with most publications on soft topology is the interpretation of the sets E and A in the definition of a soft set, see Definition 2.1. We realize the set E as the set of potential parameters while the set A is interpreted as the set of actual parameters. In the papers written on soft topology which are known to us usually the authors either assume that the image of a parameter not belonging to A is zero (or an empty set), or that sets E and A coincide. On the other hand we assume that in the parameters not belonging to A, the values of the soft set are not defined. It makes an essential difference in interpretation and in the methods of research both in case of soft sets and soft topology, and especially in case of fuzzy soft sets and fuzzy soft topology.

The structure of our paper is as follows. In the second section, Preliminaries, we recall some definitions that are used throughout the paper. In the third section we develop the theory of soft topology based on open soft sets. In this part of our work the concepts and results have much in common with the concepts and results which can be found in papers written by other authors, see e.g. [1, 3, 10, 14, 20, 23, 30] and therefore in most cases the proofs are omitted. However, also here we always follow the idea that we cannot use complementation as a tool to get the property of closedness as well as the assumption that the sets of potential and actual parameters may be different and that the image of a parameter contained in $E \setminus A$ is not defined. In the fourth section we develop soft topology on the basis of closed sets, excluding opportunity to operate with open sets at the same time. The theory which is being developed in this section can be called *soft cotopology*. Finally in Section 5 the synthesis of concepts and results from the previous two sections is done. Here we consider the case when two independent soft structures on a given set are defined – one of them is realizing the property of openness, and the other is interpreting the property of closedness. This leads us to the concept of a soft ditopological space. Some properties of such spaces are described. In the last section we sum up basic results of this work and discuss some prospectives for the future work.

2. Preliminaries

Here we recall the basic concepts and results on soft sets. Most of them can be found in [2, 11–13, 15, 17, 19]. However, as it was emphasized in the Introduction, our definition of a soft set distinguishes from the definition of a soft set in other works mentioned above, in the way how we interpret the set E of potential parameters and its subset A of actual parameters.

Definition 2.1. *Let U be a universe, E be a set of parameters and $A \subseteq E$ A mapping $F_A : A \to 2^U$ is called a soft set. That is $F_A(e) \subseteq U$ if $e \in A$ and the value $F_A(e)$ will not be defined for $e \in E \setminus A$.*

Definition 2.2. *The complement of F_A is a soft set $F^c_A : A \to 2^U$ defined by $F^c_A(e) = U \setminus F_A(e)$ for every $e \in A$.*

Definition 2.3. *The intersection $G_C = \tilde{\bigcap}_{i\in I} F_{i_{A_i}}$ of a family of soft sets $\{F_{i_{A_i}} \mid i \in I\}$ where $A_i \subset E$ and $F_{i_{A_i}} : A_i \to 2^U$ is a soft set $G_C : C \to 2^U$ where $C = \bigcap_{i\in I} A_i$ and $G_C(e) = \bigcap_{i\in I} F_{i_{A_i}}(e)$ for $e \in C$.*

Definition 2.4. *Let $\{F_{i_{A_i}} \mid i \in I\}$ be family of soft sets where $A_i \subset E$ and $F_{i_{A_i}} : A_i \to 2^U$. For every $e \in E$ let $I_e = \{i \in I \mid e \in A_i\}$. Then the union $\tilde{\bigcup}_{i\in I} F_{i_{A_i}}$ of the family of soft sets $\{F_{i_{A_i}} \mid i \in I\}$ is defined as the soft set $G_C : C \to 2^U$ such that $C = \bigcup_{i\in I} A_i$ and $G_C(e) = \bigcup_{i\in I_e} F_{i_{A_i}}(e)$ for $e \in C$.*

Definition 2.5. *A soft set F_A is called a soft subset of G_B denoted by $F_A \tilde{\subseteq} G_B$ if $A \subseteq B$ and $F_A(e) \subseteq G_B(e)$ for every $e \in A$.*

Definition 2.6. *A soft set F_E is called the whole soft set if $F_E(e) = U$ for every $e \in E$; we denote it by $\tilde{U}_E$. A soft set F_A is called the whole soft set relative to A if $F_A(e) = U$ for every $e \in A$; we denote it by $\tilde{U}_A$.*

Definition 2.7. *A soft set F_E is called the null soft set if $F_E(e) = \emptyset$ for every $e \in E$; we denote it by ϕ. A soft set F_A is called the null soft set relative to A if $F_A(e) = \emptyset$ for every $e \in A$; we denote it by ϕ_A.*

The proof of the next five statement is easy and can be done as the proof of the analogous statement in e.g. [30]:

Theorem 2.8. *Given a family of soft sets $F_{i_{A_i}} : A_i \to 2^U$ the following De Morgan-type relations hold:*

1. $(\tilde{\bigcap}_{i\in I} F_{i_{A_i}})^c \tilde{\subseteq} \tilde{\bigcup}_{i\in I}(F^c_{i_{A_i}})$.
2. $(\tilde{\bigcup}_{i\in I} F_{i_{A_i}})^c \tilde{\supseteq} \tilde{\bigcap}_{i\in I}(F^c_{i_{A_i}})$.

Proposition 2.9. *Let $F_A \tilde{\subseteq} \tilde{U}_E$. Then the following hold:*

1. $\phi_E \tilde{\cap} F_A = \phi_A,\ \phi_E \tilde{\cup} F_A = F_A$.
2. $\tilde{U}_E \tilde{\cap} F_A = F_A,\ \tilde{U}_E \tilde{\cup} F_A = \tilde{U}_A$.

Proposition 2.10. *Let $F_A, G_B \tilde{\subseteq} \tilde{U}_E$. Then the following hold:*

1. $F_A \tilde{\subseteq} G_B$ *iff* $F_A \tilde{\cap} G_B = F_A$.
2. $F_A \tilde{\subseteq} G_B$ *iff* $F_A \tilde{\cup} G_B = G_B$.

Proposition 2.11. *Let $F_A, G_B, H_C, S_D \tilde{\subseteq} \tilde{U}_E$. Then the following hold:*

1. *If $A \subseteq B$ and $F_A \tilde{\cap} G_B = \phi_{A\cap B}$ then $F_A \tilde{\subseteq} G^c_B$.*
   If $A = B$ and $F_A \tilde{\cap} G_A = \phi_A$ iff $F_A \tilde{\subseteq} G^c_A$.
2. $F_A \tilde{\cup} F^c_A = \tilde{U}_A,\ F_A \tilde{\cap} F^c_A = \phi_A$.
3. $F_A \tilde{\subseteq} G_B$ *iff* $G^c_B \tilde{\subseteq} F^c_A$.
4. *If $F_A \tilde{\subseteq} G_B$ and $G_B \tilde{\subseteq} H_C$ then $F_A \tilde{\subseteq} H_C$.*
5. *If $F_A \tilde{\subseteq} G_B$ and $H_C \tilde{\subseteq} S_D$ then $F_A \tilde{\cap} H_C \tilde{\subseteq} G_B \tilde{\cap} S_D$.*
6. *If $F_A \tilde{\subseteq} G^c_B$ then $F_A \tilde{\cap} G_B = \phi_A$.*

Definition 2.12. *Let U, V be universe sets, E, P be parameter sets and let $S(U,E), S(V,P)$ be families of all soft sets defined on (U,E) and (V,P) respectively. Following e.g. [15] we define a soft function $f = (\varphi, \psi) : S(U,E) \to S(V,P)$ induced by mappings $\varphi : U \to V, \psi : E \to P$ by setting*

$$f(F_A)(p) = \varphi(\cup_{e\in\psi^{-1}(p)} F(e)),\ \forall p \in \psi(A)$$

for each $F_A \in S(U,V)$. The preimage of a soft set $G_B \in S(V,P)$ under a soft function $f : S(U,E) \to S(V,P)$ is defined by

$$f^{-1}(G_B)(e) = \varphi^{-1}(G_B(\psi(e))),\ \forall e \in \psi^{-1}(B).$$

A soft mapping $f = (\varphi, \psi)$ is called injective if both φ and ψ are injective. A soft mapping $f = (\varphi, \psi)$ is called surjective if both φ and ψ are surjective.

The proof of the next three theorems is straightforward and can be found example in [15]

Theorem 2.13. *Let $f = (\varphi, \psi) : S(U,E) \to S(V,P)$ be a soft function, $F_A, G_B \tilde{\subseteq} \tilde{U}_E$ and $F_{i_{A_i}}$ be a family of soft sets on (U,E). Then,*

1. $f(\phi_A) = \phi_{\psi_A}, f(\tilde{U}_E) \tilde{\subseteq} \tilde{V}_P$.
2. $f(\tilde{\bigcup}_{i\in I} F_{i_{A_i}}) = \tilde{\bigcup}_{i\in I} f(F_{i_{A_i}})$.
3. $f(\tilde{\bigcap}_{i\in I} F_{i_{A_i}}) \tilde{\subseteq} \tilde{\bigcap}_{i\in I} f(F_{i_{A_i}})$.
4. *If $F_A \tilde{\subseteq} G_B$ then $f(F_A) \tilde{\subseteq} f(G_B)$.*

Theorem 2.14. *Let $f = (\varphi, \psi) : S(U,E) \to S(V,P)$ be a soft function, $F_A, G_B \tilde{\subseteq} \tilde{V}_P$ and $F_{i_{A_i}}$ be a family of soft sets on (V,P). Then,*

1. $f^{-1}(\phi_P) = \phi_E, f^{-1}(\tilde{V}_P) = \tilde{U}_E$.
2. $f^{-1}(\tilde{\bigcup}_{i\in I} F_{i_{A_i}}) = \tilde{\bigcup}_{i\in I} f^{-1}(F_{i_{A_i}})$.
3. $f^{-1}(\tilde{\bigcap}_{i\in I} F_{i_{A_i}}) = \tilde{\bigcap}_{i\in I}(f^{-1}(F_{i_{A_i}}))$.

Theorem 2.15. *Let $f = (\varphi, \psi) : S(U,E) \to S(V,P)$ be a soft function and $F_A \tilde{\subseteq} \tilde{V}_P$. Then,*

1. $f(f^{-1}(F_A)) \tilde{\subseteq} F_A$.
2. $f^{-1}(F^c_A) = (f^{-1}(F_A))^c$.
3. $F_A \tilde{\subseteq} f^{-1}(f(F_A))$.

3. Soft Topological Spaces Defined by Open Soft Sets

3.1. Soft topology

Here we recall some concepts, results and constructions in soft topology, which can be found in [2, 3, 15, 19, 20, 23]. However, as different from most of these works, we make a clear distinction between the set E of potential parameters and its subset A of actual parameters. Besides, here in our considerations we are allowed to use only the property of openness for soft sets and must avoid handling of closedness property.

Definition 3.1. *Let U be a universe, E be a set of parameters. A family τ of subsets of $\tilde{U}_E$ is called a soft topology if the following holds:*

1. $\phi_A, \tilde{U}_E \in \tau$ $(\forall A \subseteq E)$.
2. *If* $\{F_{i_{A_i}} \tilde{\subseteq} \tilde{U}_E \mid i \in I\} \subseteq \tau$ *then* $\tilde{\bigcup}_{i\in I} F_{i_{A_i}} \in \tau$.
3. *If* $F_A, G_B \in \tau$ *then* $F_A \tilde{\cap} G_B \in \tau$.

Every member of τ is called an open soft set and the pair $(\tilde{U}_E, \tau)$ is called a soft topological space.

Given two soft topologies τ_1 and τ_2 on $\tilde{U}_E$, a soft topology τ_2 is called coarser than the soft topology τ_1 if for any $F_A \in \tau_2$ it holds $F_A \in \tau_1$.

The proof of the next two theorems is straightforward and can be verified, e.g. as the proof of the similar statements in [23]

Theorem 3.2. *If $(\tilde{U}_E, \tau_1)$ and $(\tilde{U}_E, \tau_2)$ are two soft topological spaces, then $(\tilde{U}_E, \tau_1 \cap \tau_2)$ is a soft topological space.*

Theorem 3.3. *If $(\tilde{U}_E, \tau)$ is a soft topological space then for every $e \in E$ $(U(e), \tau(e))$ is a topological space.*

Definition 3.4. *Let $x \in U$ and $A \subseteq E$. A soft set x_A defined by $x_A(e) = x$ for every $e \in A$ is called a soft point in $\tilde{U}_E$. A soft set x_A is said to be in a soft set F_B (denoted by $x_A \tilde{\in} F_B$) if $x \in F_B(e)$ for every $e \in A$.*

Definition 3.5. *Given a soft topological space $(\tilde{U}_E, \tau)$, a soft set $G_B \tilde{\subseteq} \tilde{U}_E$ is called a τ-neighborhood of a soft set $x_A \tilde{\in} \tilde{U}_E$ if there exists an open soft set H_C such that $x_A \tilde{\in} H_C \tilde{\subseteq} G_B$. The family of all τ-neighborhoods of x_A is denoted by $\mathfrak{N}(x_A)$.*

Obviously $\tilde{U}_E$ is a τ-neighborhood for every soft point x_A and if $G_B \in \mathfrak{N}(x_A)$ and $G_B \tilde{\subseteq} H_C$, then $H_C \in \mathfrak{N}(x_A)$.

Definition 3.6. *Given a soft topological space $(\tilde{U}_E, \tau)$, let $F_A, G_B \tilde{\subseteq} \tilde{U}_E$. Then G_B is called a τ-neighborhood of F_A if there exists an open soft set H_C such that $F_A \tilde{\subseteq} H_C \tilde{\subseteq} G_B$. The family of all τ-neighborhoods of F_A is denoted by $\mathfrak{N}(F_A)$.*

Definition 3.7. *Let $(\tilde{U}_E, \tau)$ be a soft topological space and $F_A \tilde{\subseteq} \tilde{U}_E$. The soft interior of F_A is defined by:*

$$\mathrm{int} F_A = \tilde{\bigcup_{i\in I}} \{G_{B_i} \tilde{\subseteq} \tilde{U}_E : G_{B_i} \in \tau \text{ and } G_{B_i} \tilde{\subseteq} F_A\}.$$

The proof of the next two theorems can be done patterned e.g. after the proof of the analogous statements in [10, 23, 30]

Theorem 3.8. *Let $(\tilde{U}_E, \tau)$ be a soft topological space, $F_A \tilde{\subseteq} \tilde{U}_E$. Then,*

1. $\mathrm{int} F_A \tilde{\subseteq} F_A$.
2. $\mathrm{int} F_A$ *is the largest open soft set contained in* F_A.
3. F_A *is an open soft set if and only if* $\mathrm{int} F_A = F_A$.
4. $\mathrm{int}(\mathrm{int} F_A) = \mathrm{int} F_A$.
5. $\mathrm{int}\phi_A = \phi_A$ $(\forall A \subseteq E)$, $\mathrm{int}\tilde{U}_E = \tilde{U}_E$.

Theorem 3.9. *Let $(\tilde{U}_E, \tau)$ be a soft topological space, $F_A, G_B \tilde{\subseteq} \tilde{U}_E$. Then*

1. *If* $F_A \tilde{\subseteq} G_B$ *then* $\mathrm{int} F_A \tilde{\subseteq} \mathrm{int} G_B$.
2. $\mathrm{int}(F_A \tilde{\cap} G_B) = \mathrm{int} F_A \tilde{\cap} \mathrm{int} G_B$.
3. $\mathrm{int}(F_A \tilde{\cup} G_B) \tilde{\supseteq} \mathrm{int} F_A \tilde{\cup} \mathrm{int} G_B$.

3.2. τ-continuous soft mappings and open soft mappings

In this section we reconsider basic concepts related to mappings of soft topological spaces in a form appropriate for us. We omit the proofs since they are almost verbatim the ones which can be found in the papers [2, 3, 15, 19, 20, 23].

Definition 3.10. *(cf e.g.[30]) Let* $(\tilde{U_E}, \tau_1), (\tilde{V_P}, \tau_2)$ *be two soft topological spaces and let* $f = (\varphi, \psi) : S(U, E) \to S(V, P)$*, where* $\varphi : U \to V, \psi : E \to P$ *be mappings, be defined as in 2.12. We interpret* f *as the soft function* $f = (\varphi, \psi) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$ *and call it* τ*-continuous at* x_A *if for each* τ*-neighborhood* $G_{\psi(A)}$ *of* $f(x_A)$*, there exists a* τ*-neighborhood* H_A *of* x_A *such that* $f(H_A)\tilde{\subseteq}G_{\psi(A)}$*. Further, we call* f τ*-continuous on* $\tilde{U_E}$ *if it is* τ*-continuous at each soft point of* $\tilde{U_E}$*.*

The proof of the next four statements can be done patterned after the proof of the analogous statements in e.g. [30] and [3]:

Theorem 3.11. *The following conditions are equivalent for a soft function* $f = (\varphi, \psi) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$ *:*

1. f *is* τ*-continuous at* x_A*,*
2. *For every* τ*-soft neighborhood* $G_{\psi(A)}$ *of* $f(x_A)$*, there exists a* τ*-neighborhood* H_A *of* x_A *such that* $H_A\tilde{\subseteq}f^{-1}(G_{\psi(A)})$*.*
3. *For any* τ*-neighborhood* $G_{\psi(A)}$ *of* $f(x_A)$*,* $f^{-1}(G_{\psi(A)})$ *is a* τ*-neighborhood of* x_A*.*

Theorem 3.12. *A function* $f = (\varphi, \psi) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$ *is* τ*-continuous iff the preimage of every open soft set of* τ_2 *is an open soft set of* τ_1*.*

Theorem 3.13. *Let* U, V, W *be universe sets,* E, P, K *be parameter sets and* $f = (\varphi_1, \psi_1) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$*,* $g = (\varphi_2, \psi_2) : (\tilde{V_P}, \tau_2) \to (\tilde{W_K}, \tau_3)$ *be soft functions where* $\varphi_1 : U \to V, \psi_1 : E \to P$ *and* $\varphi_2 : V \to W, \psi_2 : P \to K$ *are mappings. If* f, g *are* τ*-continuous then*

$$g \circ f = (\psi_2 \circ \psi_2, \varphi_2 \circ \varphi_2) : (\tilde{U_E}, \tau_1) \to (\tilde{W_K}, \tau_3)$$

is τ*-continuous.*

Theorem 3.14. *Let* $f = (\varphi, \psi) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$ *be a soft function.* f *is* τ*-continuous if and only if for any* $F_A\tilde{\subseteq}\tilde{V_P}$*,* $f^{-1}(\text{int}F_A)\tilde{\subseteq}\text{int}f^{-1}(F_A)$*.*

Example 3.15. *Let* $\varphi : U \to U, \psi : E \to E$ *be identity mappings. Then the soft mapping* $i = (\varphi, \psi) : (\tilde{U_E}, \tau_1) \to (\tilde{U_E}, \tau_2)$ *is* τ*-continuous if and only if* $\tau_2 \subseteq \tau_1$*.*

Definition 3.16. *(cf e.g. [3]) A soft function* $f = (\varphi, \psi) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$ *is called open if the image of every open soft set from* τ_1 *is open in* τ_2*.*

Theorem 3.17. $f = (\varphi, \psi) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$ *is an open soft function if and only if* $f(\text{int}F_A)\tilde{\subseteq}\text{int}[f(F_A)]$ *for every* $F_A\tilde{\subseteq}\tilde{U_E}$*.*

Theorem 3.18. *Let* U, V, W *be universe sets,* E, P, K *be parameter sets and* $f = (\varphi_1, \psi_1) : (\tilde{U_E}, \tau_1) \to (\tilde{V_P}, \tau_2)$*,* $g = (\varphi_2, \psi_2) : (\tilde{V_P}, \tau_2) \to (\tilde{W_K}, \tau_3)$ *be soft functions where* $\varphi_1 : U \to V, \psi_1 : E \to P$ *and* $\varphi_2 : V \to W, \psi_2 : P \to K$ *are mappings. If* f, g *are open then* $g \circ f$ *is open, too.*

3.3. τ-separation axioms for soft topological spaces

Separation Axioms for soft topological spaces were investigated by M.Shabir and M. Naz in [23] and B. Pazar Varol and H. Aygun in [20]. In these papers separation axioms are defined on the basis of classical points. We revise the definitions and theorems from [23], [20] for soft points defined and come to the following

Definition 3.19. *Let* $(\tilde{U_E}, \tau)$ *be a soft topological space, and let* $x_A, y_A (x \neq y, x, y \in U, A \subseteq E)$ *be two different soft points of* $\tilde{U_E}$*. The space* $(\tilde{U_E}, \tau)$ *is called*

- *τ-soft T_0-space if there exists a τ-neighborhood G_A of x_A such that $y_A \tilde{\notin} G_A$ or there exists a τ-neighborhood G_A of y_A such that $x_A \tilde{\notin} G_A$.*
- *τ-soft T_1-space if there exist τ-neighborhoods G_A, H_A of x_A, y_A respectively such that $y_A \tilde{\notin} G_A$ and $x_A \tilde{\notin} H_A$.*
- *τ-soft T_2-space if there exist τ-neighborhoods G_A, H_A of x_A, y_A respectively such that $G_A \tilde{\cap} H_A = \phi_A$.*

Theorem 3.20. *Let $(\tilde{U}_E, \tau)$ be a soft topological space. If x_A^c is an open soft set for each $x \in U, A \subseteq E$ then $(\tilde{U}_E, \tau)$ is a τ-soft T_1-space.*

Proof. Let x_A^c be an open soft set and $y_A \neq x_A$. Then $y_A \tilde{\in} x_A^c$ and $x_A \tilde{\notin} x_A^c$ and similarly $x_A \tilde{\in} y_A^c$ and $y_A \tilde{\notin} y_A^c$. This shows that $(\tilde{U}_E, \tau)$ is a τ-soft T_1-space. □

Obviously if $(\tilde{U}_E, \tau)$ is a τ-soft T_2-space then it is a τ-soft T_1-space and if $(\tilde{U}_E, \tau)$ is a τ-soft T_1-space then it is τ-soft T_0-space. The converses generally are not true as shown by the next two examples:

Example 3.21. *Let $U = \{x, z\}$ be the universe set, $E = \{e_1, e_2, e_3, e_4\}$ be the parameter set, $A = \{e_1, e_2\}$ and $\tau = \{\phi_A, \phi_E, \tilde{U}_E, F_A, G_A\}$ be the soft topology where, $F_A = \{e_1 = \{x\}, e_2 = \{x, z\}\}$ and $G_A = \{e_1 = \{x\}\}$. One can easily see that $(\tilde{U}_E, \tau)$ is a τ-soft T_0-space. However there does not exit an open soft set containing z_A but not containing x_A, and hence $(\tilde{U}_E, \tau)$ is not a τ-soft T_1-space.*

Example 3.22. *Let $U = \{x, y\}$ be the universe set $E = \{e_1, e_2, e_3\}$ be the parameter set and $\tau = \{\phi_A, \phi_B, \phi_C, \phi_E, \tilde{U}_E, F_A, G_A, D_A, T_A, H_B, K_B, I_C, L_C, \}$ be the soft topology where*

$$
\begin{aligned}
F_A &= \{e_1 = \{x\}, e_2 = \{x, y\}\},\\
G_A &= \{e_1 = \{x, y\}, e_2 = \{y\}\},\\
D_A &= \{e_1 = \{x\}, e_2 = \{y\}\},\\
T_A &= \{e_1 = \{y\}, e_2 = \{x\}\},\\
H_B &= \{e_1 = \{x\}\},\\
K_B &= \{e_1 = \{y\}\},\\
I_C &= \{e_2 = \{y\}\},\\
L_C &= \{e_2 = \{x\}\}.
\end{aligned}
$$

Then $(\tilde{U}_E, \tau)$ is a τ-soft T_1-space. But it is not a τ-soft T_2-space since for $x_A \neq y_A$, there do not exist τ-neighborhoods F_A, G_A of x_A, y_A such that $F_A \tilde{\cap} G_A = \phi_A$.

Theorem 3.23. *Let $(\tilde{U}_E, \tau_1), (\tilde{V}_P, \tau_2)$ be two soft topological spaces and $f = (\varphi, \psi) : (\tilde{U}_E, \tau_1) \to (\tilde{V}_P, \tau_2)$ be an injective τ-continuous function. If $(\tilde{V}_P, \tau_2)$ is a τ-soft T_2-space, then $(\tilde{U}_E, \tau_1)$ is a τ-soft T_2-space.*

Proof. Let $x_A \neq y_A$ be two soft points of $\tilde{U}_E$. Then $f(x_A) \neq f(y_A)$ since f is an injective function. There exist τ-neighborhoods $F_{\psi(A)}, G_{\psi(A)}$ of $f(x_A), f(y_A)$ respectively such that $F_{\psi(A)} \tilde{\cap} G_{\psi(A)} = \phi_{\psi(A)}$. Hence $f^{-1}(F_{\psi(A)}), f^{-1}(G_{\psi(A)})$ are τ-neighborhoods of x_A, y_A respectively such that $f^{-1}(F_{\psi(A)}) \tilde{\cap} f^{-1}(G_{\psi(A)}) = \phi_A$. This shows that $(\tilde{U}_E, \tau_1)$ is a τ-soft T_2-space. □

Theorem 3.24. *Let $(\tilde{U}_E, \tau_1), (\tilde{V}_P, \tau_2)$ be two soft topological spaces and $f = (\varphi, \psi) : (\tilde{U}_E, \tau_1) \to (\tilde{V}_P, \tau_2)$ be a bijective open soft function. If $(\tilde{U}_E, \tau_1)$ is a τ-soft T_2-space then $(\tilde{V}_P, \tau_2)$ is a τ-soft T_2-space.*

Definition 3.25. *A soft topological space $(\tilde{U}_E, \tau)$ is called τ-soft regular if for every $x_A \tilde{\in} \tilde{U}_E$ and for every $F_A \neq \phi_E \tilde{\subseteq} \tilde{U}_E$, such that $x_A \tilde{\in} F_A^c \in \tau$, there exist $G_A \in \mathfrak{N}(x_A), H_A \in \mathfrak{N}(F_A)$ such that $V_A \tilde{\cap} H_A = \phi_A$.*
If a soft topological space $(\tilde{U}_E, \tau)$ is both τ-soft regular and a τ-soft T_1-space then it is called a τ-soft T_3-space.

The next example shows that a τ-soft regular space need not be a τ-soft T_1-space:

Example 3.26. *Let $U = \{x, y, z\}$, be the universe set $E = \{e_1, e_2, e_3\}$ be the parameter set and $\tau = \{\phi_A, \phi_E, \tilde{U_E}, F_A, G_A, H_A\}$ be the soft topology where,*

$$F_A = \{e_1 = \{x\}\},$$
$$G_A = \{e_1 = \{y, z\}\},$$
$$\{e_1 = \{x, y, z\}\}.$$

Then $(\tilde{U_E}, \tau)$ is a τ-soft regular space which is not a τ-soft T_1-space.

Definition 3.27. *A soft topological space $(\tilde{U_E}, \tau)$ is called τ-soft normal space if for any soft sets $F_A, G_A \tilde{\subseteq} \tilde{U_E}$ where $F^c_A, G^c_A \in \tau$ and $F_A \tilde{\cap} G_A = \phi_A$, there exist $V_A \in \mathfrak{N}(F_A)$, $W_A \in \mathfrak{N}(G_A)$ such that $V_A \tilde{\cap} W_A = \phi_A$. In case $(\tilde{U_E}, \tau)$ is both τ-soft normal and a τ-soft T_1-space then it is called a τ-soft T_4-space.*

4. Soft Cotopological Spaces and Closed Soft Sets

Definition 4.1. *Let U be a universe and E be the parameter set. A family κ of subsets of $\tilde{U_E}$ is called a soft cotopology if the following holds:*

1. $\phi_A, \tilde{U_E} \in \kappa$ $(\forall A \subseteq E)$.
2. *If* $\{K_{i_{A_i}} \tilde{\subseteq} U_E \tilde{:} i \in I\} \subseteq \kappa$ *then* $\tilde{\bigcap}_{i \in I} K_{i_{A_i}} \in \kappa$.
3. *If* $K_A, L_B \in \kappa$ *then* $K_A \tilde{\cup} L_B \in \kappa$.

Every member of κ is called a closed soft set and the pair $(\tilde{U_E}, \kappa)$ is called a soft cotopological space.

Definition 4.2. *Let κ_1, κ_2 be two soft cotopologies on $\tilde{U_E}$. Then κ_2 is called coarser than κ_1 (denoted by $\kappa_2 \subseteq \kappa_1$) if $F_A \in \kappa_1$ whenever $F_A \in \kappa_2$.*

Theorem 4.3. *If $(\tilde{U_E}, \kappa)$ is a soft cotopological space then for every $e \in E$ $(U(e), \kappa(e))$ is a cotopological space in the sense of* [7].

Theorem 4.4. *If $(\tilde{U_E}, \kappa_1)$ and $(\tilde{U_E}, \kappa_2)$ are soft cotopological spaces then $(\tilde{U_E}, \kappa_1 \cap \kappa_2)$ is a soft cotopological space.*

To describe the local structure of a soft cotopological space we explore Wang's idea of the so called *remote neighborhood* [29].

Definition 4.5. *Let $(\tilde{U_E}, \kappa)$ be a soft cotopological space, $M_B \tilde{\subseteq} \tilde{U_E}$ and $x_A \tilde{\in} \tilde{U_E}$. A soft set M_B is called a soft remote neighborhood of x_A if there exists a closed soft set K_C such that $x_A \tilde{\notin} K_C \tilde{\supseteq} M_B$. The family of all soft remote neighborhoods of x_A is denoted by $\mathfrak{R}_N(x_A)$.*

Definition 4.6. *Let $(\tilde{U_E}, \kappa)$ be a soft cotopological space, $F_A, S_B \tilde{\subseteq} \tilde{U_E}$. Then S_B is called a soft remote neighborhood of F_A if there exists a closed soft set K_C such that F_A is not a subset of K_C and $K_C \tilde{\supseteq} S_B$.*

Theorem 4.7. *Let $(\tilde{U_E}, \kappa)$ be a soft cotopological space, $F_B, G_C \tilde{\subseteq} \tilde{U_E}$ and let x_A be a soft point of $\tilde{U_E}$. Then the following holds:*

1. ϕ_E *is a soft remote neighborhood of every soft point of* $\tilde{U_E}$.
2. *If* $G_C \in \mathfrak{R}_N(x_A)$, $F_B \tilde{\subseteq} G_C$ *then* $F_B \in \mathfrak{R}_N(x_A)$.

Definition 4.8. *Let $(\tilde{U_E}, \kappa)$ be a soft cotopological space, $F_A \tilde{\subseteq} \tilde{U_E}$. A soft point x_A of $\tilde{U_E}$ is said to be a soft adherence point of F_A if $M_A \tilde{\cup} F^c_A \neq \tilde{U_A}$ for any soft remote neighborhood M_A of x_A. The family of all soft adherence points of F_A is called the closure of F_A and it is denoted by $\mathrm{cl}F_A$.*

Theorem 4.9. *Let $(\tilde{U_E}, \kappa)$ be a soft cotopological space and $F_A \tilde{\subseteq} \tilde{U_E}$. Then*

$$\mathrm{cl}F_A = \tilde{\cap}\{K_A \tilde{\subseteq} \tilde{U_E} : K_A \in \kappa \text{ and } K_A \tilde{\supseteq} F_A\}.$$

Proof. Let $x_A \tilde{\in} \tilde{\cap}\{K_A \tilde{\subseteq} \tilde{U}_E : K_A \in \kappa$ and $K_A \tilde{\supseteq} F_A\}$ and we assume that $x_A \tilde{\notin} \mathrm{cl}F_A$. Then there exists a closed soft set L_A not containing x_A such that $L_A \tilde{\cup} F^c_A = \tilde{U}_A$. Then $L^c_A \tilde{\subseteq} F^c_A$ and so $F_A \tilde{\subseteq} L_A$. Hence there exists a closed soft set L_A such that $x_A \tilde{\notin} L_A \tilde{\supseteq} F_A$. This shows that $x_A \tilde{\notin} \tilde{\cap}\{K_A \tilde{\subseteq} \tilde{U}_E : K_A \in \kappa$ and $K_A \tilde{\supseteq} F_A\}$. This is a contradiction. Conversely let $x_A \tilde{\in} \mathrm{cl}F_A$ and $x_A \tilde{\notin} \tilde{\cap}\{K_A \tilde{\subseteq} \tilde{U}_E : K_A \in \kappa$ and $K_A \tilde{\supseteq} F_A\}$. Then there exists a closed soft set K_A such that $x_A \tilde{\notin} K_A \tilde{\supseteq} F_A$. Hence K_A is a soft remote neighborhood of x_A and $F^c_A \tilde{\cup} K_A = \tilde{U}_A$. It shows that $x_A \tilde{\notin} \mathrm{cl}F_A$. This is a contradiction. □

From here one can esily establish the following useful properties of closure operator.

Theorem 4.10. *Let $(\tilde{U}_E, \kappa)$ be a soft cotopological space and $F_A \tilde{\subseteq} \tilde{U}_E$. Then the followings hold:*

1. $F_A \tilde{\subseteq} \mathrm{cl}F_A$.
2. $\mathrm{cl}F_A$ *is the smallest closed soft set containing* F_A.
3. *a soft set F_A is closed if and only if* $F_A = \mathrm{cl}F_A$.
4. $\mathrm{clcl}F_A = \mathrm{cl}F_A$.

From Theorem 4.10 easily follows:

Theorem 4.11. *Given a soft cotopological space $(\tilde{U}_E, \kappa)$, let $F_A, G_B, H_C \tilde{\subseteq} \tilde{U}_E$. Then the following holds:*

1. *If* $F_A \tilde{\subseteq} G_B$ *then* $\mathrm{cl}F_A \tilde{\subseteq} \mathrm{cl}G_B$.
2. $\mathrm{cl}(G_B \tilde{\cup} H_C) = \mathrm{cl}G_B \tilde{\cup} \mathrm{cl}H_C$.
3. $\mathrm{cl}(F_A \tilde{\cap} G_B) \tilde{\subseteq} \mathrm{cl}F_A \tilde{\cap} \mathrm{cl}G_B$.
4. $\mathrm{cl}\tilde{U}_E = \tilde{U}_E, \mathrm{cl}\phi_A = \phi_A (A \tilde{\subseteq} E)$.

Definition 4.12. *Let $(\tilde{U}_E, \kappa)$ be a soft cotopological space, $F_A \tilde{\subseteq} \tilde{U}_E$. A soft point x_A is called a soft accumulation point of F_A if $(M_A \tilde{\cup} x_A) \tilde{\cup} F^c_A \neq \tilde{U}_A$ for every soft remote neighborhood M_A of x_A. The family F'_A of all soft accumulation points of F_A is called the accumulation of F_A.*

One can easily see that every soft accumulation point of a soft set F_A is its soft adherence point.

Theorem 4.13. *Let $(\tilde{U}_E, \kappa)$ be a soft cotopological space and $F_A \tilde{\subseteq} \tilde{U}_E$. Then $F_A \tilde{\cup} F'_A$ is a closed soft set.*

Proof. Assume that $F_A \tilde{\cup} F'_A$ is not a closed soft set. Then there exists a soft point x_A such that $x_A \tilde{\in} \mathrm{cl}(F_A \tilde{\cup} F'_A)$ but $x_A \tilde{\notin} F_A \tilde{\cup} F'_A$. Then $x_A \tilde{\notin} F_A, x_A \tilde{\notin} F'_A$. Hence there exists a soft remote neighborhood M_A of a soft point x_A such that $(M_A \tilde{\cup} x_A) \tilde{\cup} F^c_A = \tilde{U}_A$. Since $x_A \tilde{\notin} F_A$ then $M_A \tilde{\cup} F^c_A = \tilde{U}_A$ and hence $x_A \tilde{\notin} \mathrm{cl}F_A$. Therefore $x_A \tilde{\notin} \mathrm{cl}(F_A \tilde{\cup} F'_A)$. This is a contradiction. □

Theorem 4.14. *Let $(\tilde{U}_E, \kappa)$ be a soft cotopological space and $F_A \tilde{\subseteq} \tilde{U}_E$. Then $\mathrm{cl}F_A = F_A \tilde{\cup} F'_A$.*

Proof. It is known that $F_A \tilde{\subseteq} \mathrm{cl}F_A$ and $F'_A \tilde{\subseteq} \mathrm{cl}F_A$ so $F_A \tilde{\cup} F'_A \tilde{\subseteq} \mathrm{cl}F_A$. On the other hand, noticing that $F_A \tilde{\subseteq} F_A \tilde{\cup} F'_A$ and recalling that $F_A \tilde{\cup} F'_A$ is a closed soft set we conclude that $\mathrm{cl}F_A \tilde{\subseteq} F_A \tilde{\cup} F'_A$. Hence the proof is completed. □

From Theorem 4.14 easily follows the next:

Theorem 4.15. *Let $(\tilde{U}_E, \kappa)$ be a soft cotopological space. A soft set $F_A \tilde{\subseteq} \tilde{U}_E$ is closed if and only if $F'_A \tilde{\subseteq} F_A$.*

4.1. *κ-Continuous and Closed Mappings*

Definition 4.16. *Let $(\tilde{U}_E, \kappa_1), (\tilde{V}_P, \kappa_2)$ be soft cotopological spaces, $x_A \tilde{\in} \tilde{U}_E$. A function $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_1) \to (\tilde{V}_P, \kappa_2)$ where $\varphi : U \to V, \psi : E \to P$ are mappings is called κ-continuous at x_A if for every soft remote neighborhood $M_{\psi(A)}$ of $f(x_A)$ there exists a soft remote neighborhood N_A of x_A such that $f(N_A) \tilde{\supseteq} M_{\psi(A)} \tilde{\cap} f(\tilde{U}_E)$. A function f is said to be κ-continuous on $\tilde{U}_E$ if f is κ-continuous at every soft points of $\tilde{U}_E$.*

Let $(\tilde{U}_E, \kappa_U), (\tilde{V}_P, \kappa_V)$ be soft cotopological spaces, $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_U) \to (\tilde{V}_P, \kappa_V)$ be a soft mapping and let $\tilde{V}_P{}' = f(\tilde{U}_E)$. Further, let κ'_V be the soft cotopology on $\tilde{V}_P{}'$ induced by the cotopology κ_V, that is $K'_A \in \kappa'_V$ iff $K'_A = K_A \tilde{\cap} V_P{}'$ for some $K_A \in \kappa_V$.

Proposition 4.17. *A mapping $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_U) \to (\tilde{V}_P, \kappa_V)$ is κ-continuous if and only if $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_U) \to (\tilde{V}_P{}', \kappa_{V'})$ is κ-continuous.*

Proof. $\Rightarrow$ Let $K'_{\psi(A)}$ be a closed soft set such that $f(x_A)\tilde{\notin}K'_{\psi(A)}$.Let $K'_{\psi(A)} = K_{\psi(A)}\tilde{\cap}V_P{}'$. Then $f(x_A) \notin K_{\psi(A)}$. Since f is κ-continuous there exists a closed soft set M_A not containing x_A such that

$$f(M_A)\tilde{\supseteq}K_{\psi(A)}\tilde{\cap}f(\tilde{U}_E) = K'_{\psi(A)}.$$

Hence $f : (\tilde{U}_E, \kappa_U) \to (\tilde{V}_P{}', \kappa_{V'})$ is κ-continuous.
$\Leftarrow$: Let $K_{\psi(A)}$ be a closed soft set such that $f(x_A)\tilde{\notin}K_{\psi(A)}$. Since $K'_{\psi(A)} = K_{\psi(A)}\tilde{\cap}V_P{}'$, $f(x_A) \notin K'_{\psi(A)} \in \kappa_{V'}$. Then there exists a closed soft set M_A not containing x_A such that $f(M_A)\tilde{\supseteq}K'_{\psi(A)} = K_{\psi(A)}\tilde{\cap}f(\tilde{U}_E)$. □

The previous proposition can be reformulated in the following way:

Proposition 4.18. *A mapping $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_U) \to (\tilde{V}_P, \kappa_V)$ is κ-continuous if and only if for any soft remote neighborhood $K'_{\psi(A)}\tilde{\subseteq}f(\tilde{U}_E)$ of $f(x_A)$ there exists a soft remote neigborhood M_A of x_A such that $f(M_A)\tilde{\supseteq}K'_{\psi(A)}$. Moreover in this case one can assume that $f(M_A)) = K'_{\psi(A)}$.*

Theorem 4.19. *f is κ-continuous at x_A iff for any soft remote neighborhood $K'_{\psi(A)}$ of $f(x_A)$, $f^{-1}(K'_{\psi(A)})$ is a soft remote neighborhood of x_A.*

Proof. $\Rightarrow$: Let $K'_{\psi(A)}$ be a soft remote neighborhood of a soft point $f(x_A)$. Without loss of generality we may assume that $K'_{\psi(A)}$ is an arbitrary closed soft set in $\tilde{V}_P{}'$ not containing $f(x_A)$. Then there exists a closed soft set M_A not containing x_A such that $f(M_A)\tilde{\supseteq}K'_{\psi(A)}$. Now we shall show that $M_A\tilde{\supseteq}f^{-1}(K'_{\psi(A)})$. Suppose that $M_A\tilde{\not\supseteq}f^{-1}(K'_{\psi(A)})$. Then $M_A\tilde{\cup}(f^{-1}(K'_{\psi(A)}))^c \neq \tilde{U}_E$ and $M_A\tilde{\cup}f^{-1}((K'_{\psi(A)})^c) \neq \tilde{U}_E$. Hence $f(M_A)\tilde{\cup}(K'_{\psi(A)})^c \neq V'_P$. This shows that $f(M_A)\tilde{\not\supseteq}K'_{\psi(A)}$. The obtained contradiction means that $M_A\tilde{\supseteq}f^{-1}(K'_{\psi(A)})$. Therefore

$$f^{-1}(K'_{\psi(A)}) = \tilde{\bigcap}\{M_A : x_A \notin f^{-1}(K'_{\psi(A)})\}$$

and hence $f^{-1}(K'_{\psi(A)})$ is a closed soft set not containing x_A.
$\Leftarrow$:Let $K'_{\psi(A)}$ be a soft remote neighborhood of $f(x_A)$. Then by our assumption $f^{-1}(K'_{\psi(A)})$ is a soft remote neighborhood of x_A and $f(f^{-1}(K'_{\psi(A)})) = K'_{\psi(A)}$. Hence f is κ-continuous at x_A. □

Corollary 4.20. *If f is κ-continuous at x_A then for any soft remote neighborhood $K_{\psi(A)}$ of $f(x_A)$, the preimage $f^{-1}(K_{\psi(A)})$ is a soft remote neighborhood of x_A.*

Example 4.21. *Let $U = \{a, c\}, V = \{1, 2\}$ be the universe sets $E = \{e_1, e_2\}, P = \{p_1, p_2\}$ be the parameter sets and $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_1) \to (\tilde{V}_P, \kappa_2)$ be the soft function where $\varphi(a) = 1, \varphi(c) = 2, \psi(e_1) = \psi(e_2) = p_2$. $\kappa_1 = \{\phi_A, \tilde{U}_E, \{e_1 = \{c\}, e_2 = \{c\}\}\}$, $\kappa_2 = \{\phi_B, \tilde{V}_P, \{p_1 = \{1, 2\}, p_2 = \{2\}\}\}$. Then the soft function f is κ-continuous on $\tilde{U}_E$.*

Lemma 4.22. *A soft set $F_A\tilde{\subseteq}\tilde{U}_E$ in a soft cotopological space $(\tilde{U}_E, \kappa)$ is closed if and only if F_A is a soft remote neighborhood of every soft point not belonging to F_A.*

Theorem 4.23. *A soft function $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_U) \to (\tilde{V}_P, \kappa_V)$ is κ-continuous if and only if the preimage of every closed soft set of κ'_V is a closed soft set of κ_U.*

Proof. Notice first that conditions:

- the preimage of every closed soft set of κ'_V is a closed soft set of κ_U;
- the preimage of every closed soft set of κ_V is a closed soft set of κ_U

are equivalent.
⇒: The proof follows from the proof of the first part of Theorem 4.19 taking into account that obviously a soft set F_A is closed if and only if it is a remote neighborhood for each soft point x_A not belonging to it.
⇐: Let $K'_{\psi(A)}$ be a soft closed set of κ'_V not containing $f(x_A)$. Then $f^{-1}(K'_{\psi(A)})$ is a soft remote neighborhood of x_A. Hence $f(f^{-1}(K'_{\psi(A)}) = K'_{\psi(A)}$. This shows that f is κ-continuous. □

Theorem 4.24. *Let $(\tilde{U}_E, \kappa_1), (\tilde{V}_P, \kappa_2), (\tilde{W}_R, \kappa_3)$ be soft cotopological spaces and $f = (\varphi_1, \psi_1) : (\tilde{U}_E, \kappa_1) \to (\tilde{V}_P, \kappa_2)$, $g = (\varphi_2, \psi_2) : (\tilde{V}_P, \kappa_2) \to (\tilde{W}_R, \kappa_3)$ be soft functions where $\varphi_1 : U \to V, \psi_1 : E \to P$ and $\varphi_2 : V \to W, \psi_2 : P \to R$. If f, g are κ-continuous then $g \circ f = (\varphi_1 \circ \varphi_2, \psi_1 \circ \psi_2) : (\tilde{U}_E, \kappa_1) \to (\tilde{W}_R, \kappa_3)$ is κ-continuous, too.*

Proof. Let $K_A \in \kappa'_3$. Since g is $\kappa-$ continuous $g^{-1}(K'_A) \in \kappa_2$ and similarly since f is $\kappa-$ continuous $(g \circ f)^{-1}(K'_A) = f^{-1}(g^{-1}(K'_A)) \in \kappa_1$. Hence $g \circ f$ is $\kappa-$ continuous. □

Definition 4.25. *Let $(\tilde{U}_E, \kappa_1), (\tilde{V}_P, \kappa_2)$ be soft cotopological spaces. A soft function $f : (\tilde{U}_E, \kappa_1) \to (\tilde{V}_P, \kappa_2)$ is called closed if the image of each closed soft set is soft closed.*

Theorem 4.26. *A soft function $f = (\varphi, \psi) : (\tilde{U}_E, \kappa_1) \to (\tilde{V}_P, \kappa_2)$ is closed soft if and only if $\mathrm{cl}(f(F_A)) \tilde{\subseteq} f(\mathrm{cl}F_A)$ for every soft subset F_A of $\tilde{U}_E$,*

Proof. ⇒: Let F_A be a soft subset of $\tilde{U}_E$. It is known that $\mathrm{cl}F_A$ is a closed soft set and hence $\mathrm{cl}(f(F_A)) \tilde{\subseteq} f(\mathrm{cl}F_A)$.
⇐: Let K_A be a closed soft set. By the hypothesis $\mathrm{cl}(f(K_A)) \tilde{\subseteq} f(\mathrm{cl}K_A) = f(K_A)$. This shows that f is a closed soft function. □

4.2. κ-Soft Separation Axioms

Definition 4.27. *A soft cotopological space $(\tilde{U}_E, \kappa)$ is called a κ-soft T_0-space if for any different soft points x_A, y_A there exists $M_A \in \mathfrak{R}_N(x_A)$ such that $y_A \tilde{\in} M_A$ or there exists $M_A \in \mathfrak{R}_N(y_A)$ such that $x_A \tilde{\in} M_A$.*

Theorem 4.28. *A soft cotopological space $(\tilde{U}_E, \kappa)$ is a κ-soft T_0-space if and only if $\mathrm{cl}x_A \neq \mathrm{cl}y_A$ whenever $x \neq y$.*

Proof. ⇒: Let $(\tilde{U}_E, \kappa)$ be a κ-soft T_0-space. Suppose that $\mathrm{cl}x_A = \mathrm{cl}y_A$ for some $x \neq y$ Then $x_A \tilde{\in} \mathrm{cl}x_A = \mathrm{cl}y_A$, $y_A \tilde{\in} \mathrm{cl}y_A = \mathrm{cl}x_A$. Since $x_A \tilde{\in} \mathrm{cl}y_A$, we have $M_A \tilde{\cup} y_A^c \neq \tilde{U}_A$ for any soft remote neighborhood M_A of x_A. Then there exists $e \tilde{\in} A$ such that $M_A(e) \cup (U \setminus y_A(e)) \neq U$. Hence $y \notin M_A(e)$. This shows that $y_A \tilde{\notin} M_A$. We can show that $x_A \tilde{\notin} M_A$ by a similar way. Hence $(\tilde{U}_E, \kappa)$ is not a κ-soft T_0- space. This is a contradiction.

⇐: Assume that $(\tilde{U}_E, \kappa)$ is not a κ-soft T_0-space Then there exist points $x, y \in U$ such that $y_A \notin M_A$ for each $M_A \in \mathfrak{R}_N(x_A)$ and $x_A \notin M_A$ for each $M_A \in \mathfrak{R}_N(y_A)$. We show that $\mathrm{cl}x_A \neq \mathrm{cl}y_A$ in this case. Indeed, let $z_A \in \mathrm{cl}x_A$. Then $M_A \tilde{\cup} x_A^c \neq \tilde{U}_A$ for every $M_A \in \mathfrak{R}_N(z_A)$ Hence $x_A \tilde{\notin} M_A$ and by our assumption $y_A \tilde{\notin} M_A$. Therefore $M_A \tilde{\cup} y_A^c \neq \tilde{U}_A$. This means that $z_A \tilde{\in} \mathrm{cl}y_A$. Hence $\mathrm{cl}x_A \tilde{\subseteq} \mathrm{cl}y_A$. It can be proved that $\mathrm{cl}y_A \tilde{\subseteq} \mathrm{cl}x_A$. This is a contradiction. □

Definition 4.29. *A soft cotopological space $(\tilde{U}_E, \kappa)$ is called a κ-soft T_1-space if for any different soft points $x_A, y_A (\forall x, y \in U)$ there exist $M_A \in \mathfrak{R}_N(x_A)$ such that $y_A \tilde{\in} M_A$ and $N_A \in \mathfrak{R}_N(y_A)$ such that $x_A \tilde{\in} N_A$.*

Theorem 4.30. *A soft cotopological space. $(\tilde{U}_E, \kappa)$ is a κ-soft T_1-space if and only if every soft point $x_A (x \in U, A \subseteq E)$ is a closed soft set.*

Proof. ⇒: Suppose that $x_A \neq \mathrm{cl}x_A$. Then there exists a soft point $z_A \tilde{\in} \mathrm{cl}x_A$ and $z_A \tilde{\notin} x_A$. Hence $z_A \neq x_A$. Since $z_A \tilde{\in} \mathrm{cl}x_A$, for any soft remote neighborhood M_A of z_A, $M_A \tilde{\cup} x_A^c \neq \tilde{U}_A$. Hence $x_A \tilde{\notin} M_A$. This is a contradiction since $(\tilde{U}_E, \kappa)$ is a κ-soft T_1-space.
⇐: Let x_A be a closed soft set. Then $x_A = \mathrm{cl}x_A$. For a different soft point y_A, $y_A \tilde{\notin} \mathrm{cl}x_A = x_A$. Hence x_A is a soft remote neighborhood of y_A containing x_A. We can prove it for y_A similarly. Hence $(\tilde{U}_E, \kappa)$ is a κ-soft T_1-space. □

Every κ-soft T_1-space is a κ-soft T_0-space. But the converse is not true generally as shown the following example.

Example 4.31. *Let U be the set of all real numbers, E be the set of natural numbers and $K_{E_\lambda} = \{(e, [e+\lambda, \infty[) : e \in E_\lambda, \lambda \in \mathbb{N}\}$ and $\kappa = \{(K_E)_\lambda \tilde{\subseteq} \tilde{U_E}\} \cup \{\phi_A, \tilde{U_E}\}$. Then $(\tilde{U_E}, \kappa)$ is a κ-soft T_0-space but it is not a κ-soft T_1-space.*

To define separation properties of T_2, regularity and normality type in an appropriate way we have to apply a stronger version of a soft remote neighborhood introduced in the next definition:

Definition 4.32. *Let $(\tilde{U_E}, \kappa)$ be a soft cotopological space. $S_B \tilde{\subseteq} \tilde{U_E}$ is called a soft strong remote neighborhood of $x_A \tilde{\in} \tilde{U_E}$ if there exists a closed soft set K_C such that $x \notin K_C(e) \supseteq S_B(e)$ for every $e \in C$.*
$S_B \tilde{\subseteq} \tilde{U_E}$ is called a soft strong remote neighborhood of a soft set F_A if there exists a closed soft set $K_C(e) \supseteq S_B(e)$ such that for every $e \in A$, a set $F_A(e)$ is not a subset of $K_C(e)$.

Definition 4.33. *A soft cotopological space $(\tilde{U_E}, \kappa)$ is called a κ-soft T_2-space if for any different soft points x_A, y_A there exist soft strong remote neighborhoods S_A, T_A of x_A, y_A respectively such that $S_A \tilde{\cup} T_A = \tilde{U_A}$.*

Theorem 4.34. *If $(\tilde{U_E}, \kappa)$ is a κ-soft T_2-space then $x_A = \tilde{\bigcap}\{K_A \tilde{\subseteq} \tilde{U_E} : x_A \tilde{\in} K_A \in \kappa\}$.*

Proof. Let x_A be a soft point of $\tilde{U_E}$. For any $y_A \neq x_A$ there exist closed soft sets K_A, L_A such that for every $e \in A, x \notin K_A(e), y \notin L_A(e)$ and $K_A \tilde{\cup} L_A = \tilde{U_A}$. Hence $x_A \tilde{\in} L_A$ and $y_A \tilde{\in} K_A$. This shows that any closed soft set containing x_A does not contain y_A. Therefore $x_A = \tilde{\bigcap}\{K_A \tilde{\subseteq} \tilde{U_E} : x_A \tilde{\in} K_A \in \kappa\}$. □

Theorem 4.35. *If $x_A = \tilde{\bigcap}\{K_A \tilde{\subseteq} \tilde{U_E} : x_A \tilde{\in} K_A \in \kappa\}$ then $(\tilde{U_E}, \kappa)$ is a κ-soft T_0-space.*

Every κ-soft T_2-space is a κ-soft T_1-space. But as shown by the next example the converse is generally not true .

Example 4.36. *Let U be the set of real numbers, $E = \{e_1, e_2, e_3\}$ be the set of parameters, $A \tilde{\subseteq} E$ and $\kappa = \{(K_A)_\lambda \tilde{\subseteq} \tilde{U_E}\} \tilde{\cup} \{\tilde{U_E}\}$ where $\lambda \in \mathbb{N}$ and*
$(K_A)_\lambda = \{(e_i, V) : i \in \{1,2,3\}, e_i \in E$ and $V \subseteq \mathbb{R}$ is a finite set $\}$. Then $(\tilde{U_E}, \kappa)$ is a κ-soft T_1-space since for different soft points x_A, y_A, $K_A = \{(e_i, \{y\}) : e_i \in A\}$ and $L_A = \{(e_i, \{x\}) : e_i \in A\}$ are soft remote neighborhoods of x_A, y_A respectively such that $x_A \in L_A$ and $y_A \in K_A$. However $K_A \tilde{\cup} L_A \neq \tilde{U_A}$ and hence $(\tilde{U_E}, \kappa)$ is not a κ-soft T_2-space

Theorem 4.37. *Let $(\tilde{U_E}, \kappa_1), (\tilde{V_P}, \kappa_2)$ be two cotopological spaces and $f = (\varphi, \psi) : (\tilde{U_E}, \kappa_1) \to (\tilde{V_P}, \kappa_2)$ be an injective κ-continuous soft function. If $(\tilde{V_P}, \kappa_2)$ is a κ-soft T_2-space then $(\tilde{U_E}, \kappa_1)$ is a κ-soft T_2-space.*

Proof. Let x_A, y_A be two different soft points of $\tilde{U_E}$. Then $f(x_A) \neq f(y_A)$. Since $(\tilde{V_P}, \kappa_2)$ is a κ-soft T_2-space, there exist soft strong remote neighborhoods $S_{\psi(A)}, T_{\psi(A)}$ of $f(x_A), f(y_A)$ respectively such that $S_{\psi(A)} \tilde{\cup} T_{\psi(A)} = \tilde{V_{\psi(A)}}$. Since f is κ-continuous $f^{-1}(S_{\psi(A)}), f^{-1}(T_{\psi(A)})$ are soft strong remote neighborhoods of x_A, y_A respectively such that $f^{-1}(S_{\psi(A)}) \tilde{\cup} f^{-1}(T_{\psi(A)}) = \tilde{U_A}$. This shows that $(\tilde{U_E}, \kappa_1)$ is a κ-soft T_2-space. □

Definition 4.38. *A soft cotopological space $(\tilde{U_E}, \kappa)$ is called $\kappa-$ soft regular if for every its soft point x_A and every non-empty closed soft set K_A not containing x_A there exist soft strong remote neighbourhoods S_A, T_A of x_A and K_A respectively, such that $S_A \tilde{\cup} T_A = \tilde{U_A}$. If $(\tilde{U_E}, \kappa)$ is both κ-soft regular and κ-soft T_1-space then it is called κ- soft T_3-space.*

Theorem 4.39. *If $(\tilde{U_E}, \kappa)$ is a κ-soft regular space then for any soft point x_A of $\tilde{U_E}$ and for any soft remote neighborhood M_A of x_A there exists $L_A \in \mathfrak{R}_N(x_A)$ such that $M_A \tilde{\subseteq} L_A$.*

Proof. Let M_A be a soft remote neighborhood of x_A.Then there exists $K_A \in \kappa$ such that $x_A \tilde{\notin} K_A \tilde{\supseteq} M_A$. Since $(\tilde{U_E}, \kappa)$ is a $\kappa-$ soft regular space, there exist soft strong remote neighborhoods S_{1_A}, S_{2_A} of x_A and K_A respectively such that $S_{1_A} \tilde{\cup} S_{2_A} = \tilde{U_A}$. Hence $M_A \tilde{\subseteq} K_A \tilde{\subseteq} S_{1_A} \in \mathfrak{R}_N(x_A)$. □

Theorem 4.40. *If $(\tilde{U_E}, \kappa)$ is a κ-soft T_3-space then it is a κ-soft T_2-space.*

Proof. Let $x_A \neq y_A$ for some $x, y \in U, A \subseteq E$. Since $(\tilde{U}_E, \kappa)$ is a κ-soft T_1-space y_A is a closed soft set such that $x_A \tilde{\notin} y_A$. On the other hand since $(\tilde{U}_E, \kappa)$ is a κ-soft regular space there exist soft strong remote neighborhoods S_{1_A}, S_{2_A} of x_A, y_A respectively such that $S_{1_A} \tilde{\cup} S_{2_A} = \tilde{U}_A$. Hence the proof is completed. □

Definition 4.41. *A soft cotopological space $(\tilde{U}_E, \kappa)$ is called κ-soft normal, if for any two closed soft sets $K_A, L_A \tilde{\subseteq} U_E$ such that $K_A \tilde{\cap} L_A = \phi_A$ there exist soft strong remote neighborhoods S_{1_A}, S_{2_A} of K_A, L_A respectively such that $S_{1_A} \tilde{\cup} S_{2_A} = \tilde{U}_A$.*
If $(\tilde{U}_E, \kappa)$ is both κ-soft normal and κ-soft T_1-space then it is called κ-soft T_4-space.

5. Soft Ditopological Spaces

Now we are ready to introduce the principal concept of this work - a soft ditopological space, which is actually a synthesis of the two structures studied in the previous sections - a soft topology, related to the property of openness in the space and a soft cotopology, relaying on the property of closedness in the space:

Definition 5.1. *The triple $(\tilde{U}_E, \tau, \kappa)$ is said to be a soft ditopological space if $\tilde{U}_E$ is a soft set, τ is a topology on $\tilde{U}_E$ and κ is a cotopology on $\tilde{U}_E$. A pair $\delta = (\tau, \kappa)$ is called a ditopology on $\tilde{U}_E$ in this case.*

Definition 5.2. *Given two ditopologies $\delta_1 = (\tau_1, \kappa_1)$ and $\delta_2 = (\tau_2, \kappa_2)$ on the same soft set $\tilde{U}_E$, δ_1 is called coarser than δ_2 denoted by $\delta_1 \subseteq \delta_2$ if $\tau_2 \subseteq \tau_1$ and $\kappa_2 \subseteq \kappa_1$.*

Definition 5.3. *Given a soft ditopological space $(\tilde{U}_E, \delta)$, let $x_A \tilde{\in} \tilde{U}_E$. A pair (F_B, M_C), where $F_B, M_C \tilde{\subseteq} \tilde{U}_E$, is called a soft neighborhood of x_A if F_B is a soft τ-neighborhood and M_C is a soft remote neighborhood of x_A. Soft interior and soft closure of a soft set F_A in a soft ditopological space $(\tilde{U}_E, \delta)$ are defined respectively by:*
$\mathrm{int}F_A = \tilde{\bigcup}_{i \in I}\{G_{B_i} \tilde{\subseteq} \tilde{U}_E : G_{B_i} \in \tau \text{ and } G_{B_i} \tilde{\subseteq} F_A\}$,
$\mathrm{cl}F_A = \tilde{\cap}\{K_A \tilde{\subseteq} \tilde{U}_E : K_A \in \kappa \text{ and } K_A \tilde{\supseteq} F_A\}$.

5.1. Soft continuous functions

Definition 5.4. *Let $(\tilde{U}_E, \delta_1), (\tilde{V}_P, \delta_2)$ be two soft ditopological spaces. A function $f = (\varphi, \psi) : (\tilde{U}_E, \delta_1) \to (\tilde{V}_P, \delta_2)$ where $\varphi : U \to V$, $\psi : E \to P$ are mappings is called continuous at a soft point $x_A \tilde{\in} \tilde{U}_E$ if $f : (\tilde{U}_E, \tau_1) \to (\tilde{V}_P, \tau_2)$ is τ-continuous and $f : (\tilde{U}_E, \kappa_1) \to (\tilde{V}_P, \kappa_2)$ is κ-continuous.*

Theorem 5.5. *The followings are equivalent for a function $f : (\tilde{U}_E, \delta_1) \to (\tilde{V}_P, \delta_2)$:*

1. *f is continuous at x_A,*
2. *For any soft neighborhood $(F_{\psi(A)}, M'_{\psi(A)})$ of $f(x_A)$, the pair $(f^{-1}(F_{\psi(A)}), f^{-1}(M'_{\psi(A)}))$ is a soft neighborhood of x_A.*

Proof. The proof follows from Theorem 3.11 and Theorem 4.19. □

Theorem 5.6. *A soft function $f = ((\varphi_1, \psi_1)) : (\tilde{U}_E, \delta_1) \to (\tilde{V}_P, \delta_2)$ is soft continuous if and only if the preimage of any soft set from τ_2 is in τ_1 and the preimage of any soft set from κ'_2 is in κ_1.*

Proof. The proof follows from Theorem 3.12. and Theorem 4.23. □

5.2. Soft separation axioms

We introduce separation axioms for a soft ditopological space $(\tilde{U}_E, \tau, \kappa)$ by requesting corresponding separation properties for its topology τ and cotopology κ:

Definition 5.7. *A soft ditopological space $(\tilde{U}_E, \delta)$ is called a soft T_0-space (soft T_1-space, soft T_2-space, soft regular space, soft T_3-space, soft normal space, soft T_4-space) if $(\tilde{U}_E, \tau)$ is a τ-soft T_0-space (respectively a τ-soft T_1-space, τ-soft T_2-space, τ-soft regular space, τ-soft T_3-space, τ-soft normal space, τ-soft T_4-space) and $(\tilde{U}_E, \kappa)$ is a κ-soft T_0-space (respectively a κ-soft T_1-space, κ-soft T_2-space, κ-soft regular space, κ-soft T_3-space, κ-soft normal space, κ-soft T_4-space)*

From theorems 3.20 and 4.30 it follows

Theorem 5.8. *If for any soft point $x_A (x \in U, A \subseteq E)$ of a soft ditopological space $(\tilde{U_E}, \delta)$, x_A^c is an open and x_A is a closed soft set then $(\tilde{U_E}, \delta)$ is a τ-soft T_1-space.*

From the definitions it easily follows that every soft T_1-ditopological space is a soft T_0-ditopological space. However as shown by the next example the converse generally is not true:

Example 5.9. *Let U be the real numbers, E be the set of natural numbers, $A \subseteq E$ and $F_{E_\lambda} = \{(e,]-\infty, e+\lambda[) : e \in E_\lambda\}$ and $\tau = \{(F_E)_\lambda \tilde{\subseteq} \tilde{U_E}\} \cup \{\phi_A, \tilde{U_E}\}$, $K_{E_\lambda} = \{(e, [e+\lambda, \infty[) : e \in E_\lambda, \lambda \in \mathbb{N}\}$ and $\kappa = \{(K_E)_\lambda \tilde{\subseteq} \tilde{U_E}\} \cup \{\phi_A, \tilde{U_E}\}$. Then $(\tilde{U_E}, \tau, \kappa)$ is a soft T_0-ditopological space but it is not soft T_1-ditopological space.*

Remark 5.10. *Every soft T_2-ditopological space is a soft T_1-ditopological space.*

Theorem 5.11. *Let $(\tilde{U_E}, \tau_1, \kappa_1), (\tilde{V_P}, \tau_2, \kappa_2)$ be ditopological spaces and f be an injection soft function. If f is soft continuous and $(\tilde{V_P}, \tau_2, \kappa_2)$ is a soft T_2-space then $(\tilde{U_E}, \tau_1, \kappa_1)$ is a soft T_2-space.*

Proof. The proof is obvious by Theorem 3.23. and Theorem 4.37. □

6. Conclusion

In this paper we have introduced the concept of a soft ditopological space as a 'soft version" of the concept of a ditopological space in the sense of L.M. Brown [7] on one hand and as a synthesis of the concepts of a soft topology and a soft cotopology, the last one also introduced here. As the main prospectives for the future work in this field we consider the following:

1. To develop categorical foundations for soft ditopological spaces. In particular to describe products, coproducts, quotient spaces, etc. To describe properties of the category of soft topological spaces as a subcategory of the category of soft ditopological spaces.
2. To introduce the concept of an L-fuzzy soft ditopological space where L is a fixed cl-monoid [4] and to develop the corresponding theory.
3. To define the graded versions of the concepts of a soft ditopological space and an L-fuzzy soft ditopological space (on the lines of the papers [9, 26–28]) and to develop the corresponding theory.
4. To study possible applications of soft ditopological spaces in real-world problems.

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