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Closure Operators in Semiuniform Convergence Spaces

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Abstract. In this paper, the characterization of closed and strongly closed subobjects of an object in category of semiuniform convergence spaces is given and it is shown that they induce a notion of closure which enjoy the basic properties like idempotency,(weak) hereditariness, and productivity in the category of semiuniform convergence spaces. Furthermore, T_1 semiuniform convergence spaces with respect to these two new closure operators are characterized.

1. Introduction

Semiuniform convergence spaces which form a strong topological universe, i.e., a cartesian closed and hereditary topological category such that products of quotients are quotients are introduced in [30], [31], [32], and [14]. It is well known, [30] or [32], that the construct ***Conv*** of Kent convergence spaces can be bicoreflectively embedded in ***SUConv*** of semiuniform convergence spaces, and consequently, each semiuniform convergence spaces has an underlying Kent convergence space, namely its bicoreflective ***Conv***-modification. The strong topological universe ***SUConv*** contains all (symmetric) limit spaces as well as uniform convergence spaces [16] as a generalization of Weil's uniform spaces [36] and thus all (symmetric) topological spaces and all uniform spaces. Since topological and uniform concepts are available in ***SUConv***, it is shown, in [31], that semiuniform convergence spaces are the suitable framework for studying continuity, Cauchy continuity, uniform continuity, completeness, total boundedness, compactness, and connectedness as well as convergence structures in function spaces such as simple convergence, continuous convergence, and uniform convergence. There are other known attempts to embed topological and uniform spaces into a common topological supercategory (e.g. quasiuniform spaces by L. Nach [18], syntopogeneous spaces by A. Császár [17], generalized topological spaces (=super topological spaces) by D.B. Doitchinov [22], merotopopic spaces (=seminearness spaces) by M. Katétov [26], and nearness spaces by H. Herrlich [23]) that have not even led to cartesian closed topological categories.
Closure operators are one of the principal topics both in Categorical Topology and Categorical Algebra. In a category equipped with a notion of subobject and closure one may pursue topological concepts in a context no longer confined to TOP-like categories.

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Closure operators have always been one of the main concepts in topology. For example, Herrlich [24] characterized coreflections in the category TOP of topological spaces by means of Kuratowski closure operators finer than the ordinary closure. Hong [25] and Salbany [34] used closure operators to produce epireflective subcategories of TOP.
A comprehensive study of closure operators in an ($\mathcal{E}$,$\mathcal{M}$)-category and their relations with subcategories along with a variety of examples both in topology and algebra is presented in [19] and [21]. Dikranjan and Giuli [19] introduced a closure operator of a topological category and used this to characterize the epimorphisms of the full subcategories of the given topological category.
Baran, in [2] and [3], introduced the notion of (strong) closedness in set-based topological categories and used these notions in [2], [5], [6], [7], [8], [10] and [11] to generalize each of the notions of compactness, connectedness, Hausdorffness, perfectness, regular, completely regular, and normal objects to arbitrary set-based topological categories. Moreover, it is shown, in [9], [10], and [12] that they form appropriate closure operators in the sense of Dikranjan and Giuli [19] in some well-known topological categories.

The aim of this paper is to give the characterization of both closed and strongly closed subobjects of an object in category of semiuniform convergence spaces and to show that they form appropriate closure operators (in the sense of [19]) which enjoy the basic properties like idempotency, (weak) hereditariness, and productivity in in category of semiuniform convergence spaces.

2. Preliminaries

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e. faithful and amnestic), has small (i.e. sets) fibers, and for which every $\mathcal{U}-$source has an initial lift or, equivalently, for which each $\mathcal{U}-$sink has a final lift [1] or [29]. Note that a topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be normalized if constant objects, i.e. , subterminals, have a unique structure.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding).

A filter α on a set B is said to be proper (improper) iff α does not contain (resp., α contains) the empty set, ϕ. Let $F(B)$ denote the set of filters on B. Let $M \subset B$ and $[M] = \{A \subset B : M \subset A\}$ and $[x] = [\{x\}]$. Note that $\alpha \cup \beta$ is the filter $[\{U \cap V \mid U \in \alpha,\ V \in \beta\}]$, $\alpha \cap \beta = [\{U \cup V \mid U \in \alpha,\ V \in \beta\}]$, and $\alpha \times \beta = [\{U \times V : U \in \alpha, V \in \beta\}]$. If $\alpha, \beta \in F(B \times B)$, then $\alpha^{-1} = \{U^{-1} : U \in \alpha\}$, where $U^{-1} = \{(x, y) : (y, x) \in U\}$. If $U \circ V = \{(x, y)$:there exist $z \in B$ with $(x, z) \in V$ and $(z, y) \in U\} \neq \phi$ for every $U \in \alpha$ and every $V \in \beta$, then $\alpha \circ \beta$ is the filter generated by $\{U \circ V : U \in \alpha, V \in \beta\}$.

Lemma 2.1. *Let σ and δ be proper filters on $B \times B$ and $f : B \to C$ be a function. Then*

(*i*) $(f \times f)(\sigma \cap \delta) = (f \times f)(\sigma) \cap (f \times f)(\delta)$ *and* $(f \times f)(\alpha) \cup (f \times f)(\beta) \subset (f \times f)(\alpha \cup \beta)$.

(*ii*) *If $\sigma \subset \delta$, then $(f \times f)(\sigma) \subset (f \times f)(\delta)$, and if δ is proper filter on $C \times C$, then $\delta \subset (ff^{-1} \times ff^{-1})(\delta)$, where $(ff^{-1} \times ff^{-1})(\delta)$ is the proper filter generated by $\{(ff^{-1} \times ff^{-1})(D) : D \in \delta\}$.*

Definition 2.2. *(cf. [30] or [32])*

1. *A semiuniform convergence space is a pair $(B, \mathfrak{I})$, where B is a set $\mathfrak{I}$ is a set of filters on $B \times B$ such that the following conditions are satisfied:*
   (UC_1) $[x] \times [x]$ belongs to $\mathfrak{I}$ for each $x \in B$.
   (UC_2) $\beta \in \mathfrak{I}$ whenever $\alpha \in \mathfrak{I}$ and $\alpha \subset \beta$.
   (UC_3) $\alpha \in \mathfrak{I}$ implies $\alpha^{-1} \in \mathfrak{I}$.
   If $(B, \mathfrak{I})$ is a semiuniform convergence space, then the elements of $\mathfrak{I}$ are called uniform filters.
2. *A map $f : (B, \mathfrak{I}) \to (B', \mathfrak{I}')$ between semiuniform convergence spaces is called uniformly continuous provided that $(f \times f)(\alpha) \in \mathfrak{I}'$ for each $\alpha \in \mathfrak{I}$, where $(f \times f)(\alpha)$ is the proper filter generated by $\{(f \times f)(D) : D \in \alpha\}$.*
3. *The consctruct of semiuniform convergence spaces (and uniformly continuous maps) is denoted by* ***SUConv***.

2.2 A source $\{f_i : (B, \mathfrak{I}) \to (B_i, \mathfrak{I}_i), i \in I\}$ in ***SUConv*** is an initial lift iff $\alpha \in \mathfrak{I}$ precisely when $(f_i \times f_i)(\alpha) \in \mathfrak{I}_i$ for all $i \in I$ (cf. [30], [32] p.33 or [15] p.67).

2.3 An epi sink $\{f_i : (B_i, \mathfrak{I}_i) \to (B, \mathfrak{I})$ in ***SUConv*** is a final lift iff $\alpha \in \mathfrak{I}$ implies that there exist $i \in I$ and $\beta_i \in \mathfrak{I}_i$ such that $(f_i \times f_i)(\beta_i) \subset \alpha$ (cf. [30] , [15] p.67 or [32] p.263).

2.4 The discrete semiuniform convergent structure $\mathfrak{I}_d$ on B is given by $\mathfrak{I}_d = \{[\phi], [x] \times [x] : x \in B\}$.

2.5 The indiscrete semiuniform convergent structure on B is given by $\mathfrak{I} = F(B \times B)$.

Note that ***SUConv*** is a strong topological universe [30].

Recall, in [32] p.31-32, that a generalized convergence space (in [28] and [35] , it is called a filter convergence space and a convergence space, respectively) is pair (B, q), where B is a set and $q \subset F(B) \times B$ such that the following are satisfied:

C_1) $([x], x) \in q$ for each $x \in B$,

C_2) $(\beta, x) \in q$ whenever $(\alpha, x) \in q$ and $\beta \supset \alpha$.

A generalized convergence space (B, q) is called a Kent convergence space [27] (in [28] p.1374, it is called a local filter convergence space) provided that the following is satisfied:

C_3) $(\alpha \cap [x], x) \in q$ whenever $(\alpha, x) \in q$.

C_4) A map $f : (B, q) \to (B', q')$ between generalized convergence spaces is called continuous provided that $((f(\alpha), f(x)) \in q'$ for each $(\alpha, x) \in q$.

C_5) The category of generalized (filter) convergence spaces and continuous maps is denoted by ***GConv*** in [32] (resp., ***FCO*** in [35]) , whereas its full subcategory of Kent (local filter) convergence spaces is denoted by ***Conv*** in [27] (resp., ***LFCO*** in [28] .

Note that every semiuniform convergence spaces $(B, \mathfrak{I})$ has an underlying Kent convergence spaces $(B, q_{\gamma_\mathfrak{I}})$ defined as follows: $q_{\gamma_\mathfrak{I}} = \{(\alpha, x) : \alpha \cap [x] \in \gamma_\mathfrak{I}\}$, where $\gamma_\mathfrak{I} = \{\beta \in F(B) : \beta \times \beta \in \mathfrak{I}\}$ [32] or [30].

3. Closed Subsets of Semiuniform Spaces

In this section, we characterize the (strongly) closed subsets of a semiuniform convergence space.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p [2], i.e., two disjoint copies of B identified at p, or in other words, the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**, the category of sets). More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram. A point x in $B \vee_p B$ will be denoted by $x_1 (x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal $p-$axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed $p-$axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\nabla_p : B \vee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$ [2], [3].

Note that the maps S_p and ∇_p are the unique maps arising from the above pushout diagram for which $S_p i_1 = (id, id) : B \to B^2$, $S_p i_2 = (p, id) : B \to B^2$, and $\nabla_p i_j = id$, $j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $p : B \to B$ is the constant map at p [10].

The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of B. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, ..., p, x, p, ...)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, ..., p, x, p, ...)$, and $\nabla_p^\infty : \vee_p^\infty B \longrightarrow B$ by $\nabla_p^\infty(x_i) = x$ for all $i \in I$, [2], [3].

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, id, is in the j-th place [10].

Let $\mathcal{U} : \mathcal{E} \to$ ***Set*** be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let M be a nonempty subset of B. We denote by X/M the final lift of the epi $\mathcal{U}$-sink $q : \mathcal{U}(X) = B \to B/M = (B \backslash M) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash M$ and identifying M with a point $*$.

Let p be a point in B.

Definition 3.1. *(cf. [2] or [3])*

1. *X is T_1 at p iff the initial lift of the $U-$source $\{S_p : B \vee_p B \longrightarrow U(X^2) = B^2$ and $\nabla_p : B \vee_p B \longrightarrow UD(B) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.*

2. *$\{p\}$ is closed iff the initial lift of the $U-$source $\{A_p^\infty : \vee_p^\infty B \longrightarrow U(X^\infty) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \longrightarrow UD(B) = B\}$ is discrete.*
3. *$M \subset X$ is closed iff $\{*\}$, the image of M, is closed in X/M or $M = \emptyset$.*
4. *$M \subset X$ is strongly closed iff X/M is T_1 at $\{*\}$ or $M = \emptyset$.*
5. *If $B = M = \emptyset$, then we define M to be both closed and strongly closed.*

Remark 3.2. *1. In **Top**, the category of topological spaces, the notion of closedness coincides with the usual ones [2], and M is strongly closed iff M is closed and for each $x \notin M$ there exist a neighborhood of M missing x. If a topological space is T_1, then the notions of closedness and strong closedness coincide [9].*

2. In general, for an arbitrary topological category, the notions of closedness and strong closedness are independent of each other [3].

Theorem 3.3. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space and let $p \in B$. $(B, \mathfrak{I})$ is T_1 at p iff for each $x \neq p$, $[x] \times [p] \notin \mathfrak{I}$ and $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$.*

Proof. Suppose that $(B, \mathfrak{I})$ is T_1 at p. If $[x]\times[p] \in \mathfrak{I}$ for some $x \neq p$, then let $\alpha = [x_1]\times[x_2]$. Clearly, $(\pi_1 S_p \times \pi_1 S_p)(\alpha) = [x] \times [p] \in \mathfrak{I}$, $(\pi_2 S_p \times \pi_2 S_p)(\alpha) = [x] \times [x] \in \mathfrak{I}$, where $\pi_i : B^2 \to B$, $i = 1, 2$, are the projection maps, and $(\nabla_p \times \nabla_p)([x_1] \times [x_2]) = [x] \times [x] \in \mathfrak{I}_d$, the discrete semiuniform convergence structure on B. Since $(B, \mathfrak{I})$ is T_1 at p, we get a contradiction. Hence, $[x] \times [p] \notin \mathfrak{I}$ for all $x \neq p$.

If $([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$ for some $x \neq p$, then let $\alpha = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$. By Lemma 2.1 (i), $(\pi_1 S_p \times \pi_1 S_p)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$, $(\pi_2 S_p \times \pi_2 S_p)(\alpha) = [x] \times [x] \in \mathfrak{I}$, $(\nabla_p \times \nabla_p)(\alpha) = [x] \times [x] \in \mathfrak{I}_d$, a contradiction since $(B, \mathfrak{I})$ is T_1 at p. Hence, we must have $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$ for all $x \neq p$.

Conversely, suppose that for each $x \neq p$, $[x] \times [p] \notin \mathfrak{I}$ and $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$. We need to show that $(B, \mathfrak{I})$ is T_1 at p ,i.e., by 2.2, 2.4, and Definition 3.1, we must show that the semiuniform convergence structure $\mathfrak{I}_W$ on $B \vee_p B$ induced by $S_p : B \vee_p B \to U((B^2, \mathfrak{I}^2)) = B^2$ and $\nabla_p : B \vee_p B \longrightarrow U((B, \mathfrak{I}_d)) = B$ is discrete, where $\mathfrak{I}^2$ and $\mathfrak{I}_d$ are the product semiuniform convergence structure on B^2 and the discrete semiuniform convergence structure on B, respectively. Let α be any filter in $\mathfrak{I}_W$, i.e., $(\pi_i S_p \times \pi_i S_p)(\alpha) \in \mathfrak{I}$, $i = 1, 2$ and $(\nabla_p \times \nabla_p)(\alpha) \in \mathfrak{I}_d$. We need to show that $\alpha = [x_i] \times [x_i]$ $(i = 1, 2)$ or $\alpha = [p] \times [p]$ or $\alpha = [\phi]$.

If $(\nabla_p \times \nabla_p)(\alpha) = [p] \times [p]$, then $\alpha = [p_i] \times [p_i]$ $(i = 1, 2)$ since $(\nabla_p)^{-1}\{p\} = \{p_i = (p, p)\}$ $(i = 1, 2)$.

If $(\nabla_p \times \nabla_p)(\alpha) = [\phi]$, then $\alpha = [\phi]$.

If $(\nabla_p \times \nabla_p)(\alpha) = [x] \times [x]$ for some $x \in B$ with $x \neq p$, then $\{x_1, x_2\} \times \{x_1, x_2\} \in \alpha$. Therefore α contains a finite set and so there is some $M_0 \in \alpha$ such that $\alpha = [M_0]$. Clearly, $M_0 \subseteq \{x_1, x_2\} \times \{x_1, x_2\}$ and if $i \neq j$, then by the first condition, it can be easily shown that $(x_i, x_j) \notin M_0$ and that $M_0 = \{(x_1, x_1), (x_2, x_2)\}$ is not possible by the second condition.

Hence, we must have $\alpha = [x_i] \times [x_i]$ $i = 1, 2$, and consequently, by Definition 3.1, 2.2, and 2.4, $(B, \mathfrak{I})$ is T_1 at p. □

Theorem 3.4. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space and $p \in B$. $\{p\}$ is closed in B iff for each $x \in B$ with $x \neq p$, the following conditions hold.*

(1) $[x] \times [p] \notin \mathfrak{I}$,

(2) $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$,

(3) $([x] \times [x]) \cap ([p] \times [p]) \cap ([p] \times [x]) \cap ([x] \times [p]) \notin \mathfrak{I}$,

Proof. Suppose that $\{p\}$ is closed in B. If $[x] \times [p] \in \mathfrak{I}$ for some $x \neq p$, then let $\alpha = [x_1] \times [x_2]$. Note that $(\pi_1 A_p^\infty \times \pi_1 A_p^\infty)(\alpha) = [x] \times [p] \in \mathfrak{I}$, $(\pi_2 A_p^\infty \times \pi_2 A_p^\infty)(\alpha) = [p] \times [x] \in \mathfrak{I}$, $(\pi_i A_p^\infty \times \pi_i A_p^\infty)(\alpha) = [p] \times [p] \in \mathfrak{I}$ for $i \geqslant 3$ and $(\nabla_p^\infty \times \nabla_p^\infty)(\alpha) = [x] \times [x] \in \mathfrak{I}_d$, a contradiction since $\{p\}$ is closed in B. Hence, $[x] \times [p] \notin \mathfrak{I}$ for all $x \neq p$.

If $([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$ for some $x \neq p$, then let $\alpha = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$. By Lemma 2.1(i), $(\pi_1 A_p^\infty \times \pi_1 A_p^\infty)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$, $(\pi_2 A_p^\infty \times \pi_2 A_p^\infty)(\alpha) = ([p] \times [p]) \cap ([x] \times [x]) \in \mathfrak{I}$ and for $i \geqslant 3$ $(\pi_i A_p^\infty \times \pi_i A_p^\infty)(\alpha) = ([p] \times [p]) \in \mathfrak{I}$ and $(\nabla_p^\infty \times \nabla_p^\infty)(\alpha) = [x] \times [x] \in \mathfrak{I}_d$, a contradiction since $\{p\}$ is closed in B. Hence, $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$ for all $x \neq p$.

If $([x] \times [x]) \cap ([p] \times [p]) \cap ([p] \times [x]) \cap ([x] \times [p]) \in \mathfrak{I}$ for some $x \neq p$, then let $\alpha = [\{x_1, x_2, ...\} \times \{x_1, x_2, ...\}]$. It follows that for all $i \in I$, $(\pi_i A_p^\infty \times \pi_i A_p^\infty)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \cap ([p] \times [x]) \cap ([x] \times [p]) \in \mathfrak{I}$ and $(\nabla_p^\infty \times \nabla_p^\infty)(\alpha) = [x] \times [x] \in \mathfrak{I}_d$, a contradiction since $\{p\}$ is closed in B.

Conversely, suppose that for each $x \in B$ with $x \neq p$, $[x] \times [p] \notin \mathfrak{I}$, $([x] \times [x]) \cap ([p] \times [p] \notin \mathfrak{I}$, and $([x] \times [x]) \cap ([p] \times [p]) \cap ([p] \times [x]) \cap ([x] \times [p]) \notin \mathfrak{I}$. We need to show that $\{p\}$ is closed in B, i.e., by Definition 3.1, 2.2, and 2.4, the semiuniform convergence structure $\mathfrak{I}_W^\infty$ on $\vee_p^\infty B$ induced by $A_p^\infty : \vee_p^\infty B \to U((B^\infty, \mathfrak{I}^\infty)) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \longrightarrow U((B, \mathfrak{I}_d)) = B$ is discrete, where $\mathfrak{I}^\infty$ and $\mathfrak{I}_d$ is the product semiuniform convergence structure on B^∞ and the discrete semiuniform convergence structure on B, respectively. Let α be any filter in $\mathfrak{I}_W^\infty$. Then $(\pi_i A_p^\infty \times \pi_i A_p^\infty)(\alpha) \in \mathfrak{I}$, for each $i \in I$ and $(\nabla_p^\infty \times \nabla_p^\infty)(\alpha) \in \mathfrak{I}_d$. By 2.4, we need to show that $\alpha = ([x_i] \times [x_i])(i \in I)$ or $\alpha = [(p,p,p...)] \times [(p,p,p,...)]$ or $\alpha = [\phi]$

If $(\nabla_p^\infty \times \nabla_p^\infty)(\alpha) = [p] \times [p]$, then $\alpha = [(p,p,p...)] \times [(p,p,p,...)]$ since $(\nabla_p^\infty)^{-1}\{p\} = \{p_i = (p,p,p,...)\}$.

If $(\nabla_p^\infty \times \nabla_p^\infty)(\alpha) = [\phi]$, then $\alpha = [\phi]$.

If $(\nabla_p^\infty \times \nabla_p^\infty)(\alpha) = [x] \times [x]$ for some point x in B with $x \neq p$, then α contains either a finite set of the form $W = \{x_{i_1}, x_{i_2}, ...x_{i_n}\} \times \{x_{i_1}, x_{i_2}, ...x_{i_n}\}$ or an infinite set of the form $W = \{x_1, x_2, ...\} \times \{x_1, x_2, ...\}$.

If $W = \{x_{i_1}, x_{i_2}, ...x_{i_n}\} \times \{x_{i_1}, x_{i_2}, ...x_{i_n}\} \in \alpha$, then α contains a finite set and so there is some $M_0 \in \alpha$ such that $\alpha = [M_0]$. Clearly, $M_0 \subseteq \{x_{i_1}, x_{i_2}, ...x_{i_n}\} \times \{x_{i_1}, x_{i_2}, ...x_{i_n}\}$ and if $i_k \neq i_l$ for $k, l = 1, 2, ..., n$ and $k \neq l$, then by the first condition, it can be easily shown that $(x_{i_k}, x_{i_l}) \notin M_0$ and that $M_0 = \{(x_{i_1}, x_{i_1}), (x_{i_2}, x_{i_2}), ..., (x_{i_n}, x_{i_n})\}$ is not possible by the second condition.

Hence, $\alpha = [M_0] = [x_i] \times [x_i]$, $(i \in I)$.

If $W = \{x_1, x_2, ...\} \times \{x_1, x_2, ...\} \in \alpha$, then α contains an infinite set M_0 such that $\alpha = [M_0]$. Note that $M_0 \subseteq W = \{x_1, x_2, ...\} \times \{x_1, x_2, ...\}$. If $M_0 = W = \{x_1, x_2, ...\} \times \{x_1, x_2, ...\}$, then for each $i \in I$, $(\pi_i A_p^\infty \times \pi_i A_p^\infty)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \cap ([p] \times [x]) \cap ([x] \times [p]) \in \mathfrak{I}$. Hence, by the third condition, $M_0 = W = \{x_1, x_2, ...\} \times \{x_1, x_2, ...\}$ is not possible. If $M_0 = \{x_k, x_{k+1}, ...\} \times \{x_m, x_{m+1}, ...\}$ for $k, m > 1$ and $m \leq k$, then, in particular, $(\pi_k A_p^\infty \times \pi_k A_p^\infty)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \cap ([p] \times [x]) \cap ([x] \times [p]) \in \mathfrak{I}$. Hence, by the third condition, $M_0 = \{x_k, x_{k+1}, ...\} \times \{x_m, x_{m+1}, ...\}$ is not possible. If $M_0 = \{(x_i, x_j), i, j \in I, i \neq j\}$, then for each $i \in I$, $(\pi_i A_p^\infty \times \pi_i A_p^\infty)(\alpha) = ([x] \times [p]) \in \mathfrak{I}$ or $([x] \times [p]) \in \mathfrak{I}$. Hence, by the first condition, $M_0 = \{(x_i, x_j), i, j \in I, i \neq j\}$ is not possible.

If $M_0 = \{(x_i, x_i), i \in I\}$, then for each $i \in I$, $(\pi_i A_p^\infty \times \pi_i A_p^\infty)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$. Hence, by the second condition, $M_0 = \{(x_i, x_i), i \in I\}$ is not possible. If $M_0 = \{(x_i, x_i), i \in I\} \cup \{(x_k, x_{k+1})\}$ or If $M_0 = \{(x_i, x_i), i \in I\} \cup \{(x_k, x_{k+1}), (x_{k+3}, x_k)\}$ or If $M_0 = \{(x_i, x_i), i \in I\} \cup \{(x_k, x_{k+1}), (x_{k+1}, x_k), (x_{k+3}, x_{k+10})\}$, k fixed, then $(\pi_{k-1} A_p^\infty \times \pi_{k-1} A_p^\infty)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$. Hence, by the second condition, these cases of M_0 are not possible.

Therefore, if $\alpha \in \mathfrak{I}_W^\infty$, then $\alpha = [x_i] \times [x_i]$ $(i \in I)$, $\alpha = [(p,p,p...)] \times [(p,p,p,...)]$ or $[\phi]$, i.e., by 2.4, the semiuniform convergence structure $\mathfrak{I}_W^\infty$ on $\vee_p^\infty B$ induced by $A_p^\infty : \vee_p^\infty B \to U((B^\infty, \mathfrak{I}^\infty)) = B^\infty$ and $\nabla_p^\infty : \vee_p^\infty B \longrightarrow U((B, \mathfrak{I}_d)) = B$ is discrete. Hence, by Definition 3.1, $\{p\}$ is closed in B. □

Lemma 3.5. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space, $\phi \neq M \subset B$, $\beta \in \mathfrak{I}$ and $q : \mathcal{U}(X) = B \to B/M = (B \backslash M) \cup \{*\}$ be the epi map that is the identity on $B \backslash M$ and identifying M with a point $*$. For all $a, b \in B$ with $a \notin M$ and $b \in M$*

(i) $\beta \subset [b] \times [a]$ or $\beta \cup ([M] \times [a])$ is proper iff $(q \times q)(\beta) \subset [*] \times [a]$.

(ii) $\beta \cap ([M] \times [M]) \subset ([a] \times [a]) \cap ([M] \times [M])$ and $\beta \cup ([M] \times [M])$ is proper iff $(q \times q)(\beta) \subset ([a] \times [a]) \cap ([*] \times [*])$.

(iii)$(q \times q)(\beta) \subset ([a] \times [a]) \cap ([*] \times [*]) \cap ([*] \times [a]) \cap ([a] \times [*])$ iff the following conditions hold.

(I) $\beta \subset [b] \times [a]$ or $\beta \cup ([M] \times [a])$ is proper,

(II) $\beta \subset [a] \times [b]$ or $\beta \cup ([a] \times [M])$ is proper,

(III) $\beta \cap ([M] \times [M]) \subset ([a] \times [a]) \cap ([M] \times [M])$ and $\beta \cup ([M] \times [M])$ is proper.

Proof. *(i) Suppose that $\beta \in \mathfrak{I}$, $\phi \neq M \subset B$, $\beta \subset [b] \times [a]$ or $\beta \cup ([M] \times [a])$ is proper for all $a, b \in B$ with $a \notin M$ and $b \in M$. We need to show that $(q \times q)(\beta) \subset [*] \times [a]$. If $\beta \subset [b] \times [a]$, then by Lemma 2.1 (ii), $(q \times q)(\beta) \subset (q \times q)([b] \times [a]) = [*] \times [a]$.*

If $\beta \cup ([M] \times [a])$ is proper, then for all $V \in \beta$, $V \cap (M \times \{a\}) \neq \phi$. Let $W \in (q \times q)(\beta)$. There exists $U \in \beta$ such that $W \supset (q \times q)(U)$. Since $\beta \cup ([M] \times [a])$ is proper, it follows that $U \cap (M \times \{a\}) \neq \phi$. Hence, $(b, a) \in U$ for some $b \in M$ and $(q \times q)((b, a)) = (*, a) \in (q \times q)(U) \subset W$, and consequently, $W \in [*] \times [a]$ and $(q \times q)(\beta) \subset [*] \times [a]$.

Conversely, suppose that $\beta \in \mathfrak{I}$, $a \notin M$ and $(q \times q)(\beta) \subset [*] \times [a]$. If $\beta \not\subset [b] \times [a]$ and $\beta \cup ([M] \times [a])$ is improper for some $b \in M$, then $U \cap (M \times \{a\}) = \phi$ for some U in β. It follows that $(c, a) \notin U$ for all $c \in M$ and

$(q\times q)((c,a)) = (*,a) \notin (q\times q)(U) \in (q\times q)(\beta)$. *Hence,* $(q\times q)(\beta) \not\subseteq [*]\times[a]$, *a contradiction. Hence,* $\beta \subset [b]\times[a]$ *or* $\beta \cup ([M]\times[a])$ *is proper for all* $a \notin M$ *and* $b \in M$.

(ii) Suppose that $\beta\cap([M]\times[M]) \subset ([a]\times[a])\cap([M]\times[M])$ *and* $\beta\cup([M]\times[M])$ *is proper. We need to show that* $(q\times q)(\beta) \subset ([a]\times[a])\cap([*]\times[*])$. *First, we will show that* $(q\times q)(\beta) \subset [*]\times[*]$. *If* $(q\times q)(\beta) \not\subseteq [*]\times[*]$, *then, there exists* $W \in (q\times q)(\beta)$ *such that* $(*,*) \notin W$. *Since* $W \in (q\times q)(\beta)$, *there exists* $V \in \beta$ *such that* $(q\times q)(V) \subset W$. $\beta \cup ([M]\times[M])$ *is proper implies that* $V\cap(M\times M) \neq \phi$ *and consequently,* $(q\times q)(V\cap(M\times M)) = (*,*) \in (q\times q)(V) \subset W$. *It follows that* $(*,*) \in W$, *a contradiction. Therefore, we must have* $(q\times q)(\beta) \subset [*]\times[*]$. *By Lemma 2.1 (i),* $(q\times q)(\beta\cap([M]\times[M])) = (q\times q)(\beta)\cap(q\times q)([M]\times[M]) = (q\times q)(\beta)\cap([*]\times[*]) = (q\times q)(\beta)$, *and consequently* $(q\times q)(\beta) = (q\times q)(\beta\cap([M]\times[M]) \subset (q\times q)(([a]\times[a])\cap([M]\times[M])) = ([a]\times[a])\cap([*]\times[*])$.

Conversely, suppose that $(q\times q)(\beta) \subset ([a]\times[a])\cap([*]\times[*])$. *We will show that* $\beta\cup([M]\times[M])$ *is proper. If* $\beta\cup([M]\times[M])$ *is improper, then there exists* $W \in \beta$ *such that* $W\cap(M\times M) = \phi$. $(q\times q)(W) \in (q\times q)(\beta) \subset [*]\times[*]$, *and consequently,* $(*,*) \in (q\times q)(W)$. *Hence, there exists* $(x,y) \in W$ *such that* $(q\times q)((x,y)) = (*,*)$. *It follows that* $(x,y) \in W\cap(M\times M)$, *a contradiction. Hence,* $\beta\cup([M]\times[M])$ *is proper. Now, we show that* $\beta\cap([M]\times[M]) \subset ([a]\times[a])\cap([M]\times[M])$. *If* $U \in \beta\cap([M]\times[M])$, *then* $M\times M \subset U \in \beta$ *and* $(q\times q)(U) \in (q\times q)(\beta) \subset ([a]\times[a])\cap([*]\times[*]) = (q\times q)(([a]\times[a])\cap([M]\times[M]))$. *It follows that there exists* $V \in ([a]\times[a])\cap([M]\times[M])$ *such that* $(q\times q)(V) \subset (q\times q)(U)$. *Since* $V \in ([a]\times[a])\cap([M]\times[M])$, $V\cap(M\times M) \neq \phi$ *and* $V \subset V\cup(M\times M) = (q\times q)^{-1}((q\times q)(V)) \subset (q\times q)^{-1}((q\times q)(U)) = U\cup(M\times M) = U$ *and consequently,* $U \in ([a]\times[a])\cap([M]\times[M])$. *Hence,* $\beta\cap([M]\times[M]) \subset ([a]\times[a])\cap([M]\times[M])$.

The proof of (iii) follows easily from the Parts (i) and (ii). □

Theorem 3.6. *Let* $(B,\mathfrak{I})$ *be a semiuniform convergence space.* $\phi \neq M \subset B$ *is closed iff for all* $a,b \in B$ *with* $a \notin M$, $b \in M$ *and* $\beta \in \mathfrak{I}$, *the conditions* (*i*), (*ii*), *and* (*iii*) *hold.*

(i) $\beta \not\subseteq [a]\times[b]$ *and* $\beta\cup([a]\times[M])$ *is improper.*

(ii) $\beta\cap([M]\times[M]) \not\subseteq ([a]\times[a])\cap([M]\times[M])$ *or* $\beta\cup([M]\times[M])$ *is improper.*

(iii) $\beta \not\subseteq [b]\times[a]$ *and* $\beta\cup([M]\times[a])$ *is improper, or* $\beta \not\subseteq [a]\times[b]$ *and* $\beta\cup([a]\times[M])$ *is proper, or* $\beta\cap([M]\times[M]) \not\subseteq ([a]\times[a])\cap([M]\times[M])$ *or* $\beta\cup([M]\times[M])$ *is improper.*

Proof. *M is closed iff, by Definition 3.1,* $\{*\}$ *is closed in* B/M *iff, by Theorem 3.4, for each* $a \neq *$ *in* B/M, $[a]\times[*] \notin \mathfrak{I}'$, $([a]\times[a])\cap([*]\times[*]) \notin \mathfrak{I}'$, *and* $([a]\times[a])\cap([*]\times[*])\cap([a]\times[*])\cap([*]\times[a]) \notin \mathfrak{I}'$, *where* $\mathfrak{I}'$ *is the quotient semiuniform structure on* B/M *that is induced by the map* $q: B \to B/M$ *iff, by definition of* $\mathfrak{I}'$ *and 2.3, for each* $\beta \in \mathfrak{I}$ *and* $a \notin M$, $(q\times q)(\beta) \not\subseteq ([a]\times[*])$, $(q\times q)(\beta) \not\subseteq ([a]\times[a])\cap([a]\times[*])$, *and* $(q\times q)(\beta) \not\subseteq ([a]\times[a])\cap([*]\times[*])\cap([*]\times[a])\cap([a]\times[*])$ *iff, by Lemma 3.5, for all* $a,b \in B$ *and* $\beta \in \mathfrak{I}$ *with* $a \notin M$, $b \in M$, *the conditions* (*i*), (*ii*), *and* (*iii*) *hold.* □

Theorem 3.7. *Let* $(B,\mathfrak{I})$ *be a semiuniform convergence space.* $\phi \neq M \subset B$ *is strongly closed iff for all* $a,b \in B$ *with* $a \notin M$, $b \in M$ *and* $\beta \in \mathfrak{I}$, *the following conditions hold.*

(*i*) $\beta \not\subseteq [a]\times[b]$ *and* $\beta\cup([a]\times[M])$ *is improper.*

(*ii*) $\beta\cap([M]\times[M]) \not\subseteq ([a]\times[a])\cap([M]\times[M])$ *or* $\beta\cup([M]\times[M])$ *is improper.*

Proof. *M is strongly closed iff, by Definition 3.1,* X/M *is* T_1 *at* $*$ *iff, by Theorem 3.3, for each* $a \neq *$ *in* B/M, $[a]\times[*] \notin \mathfrak{I}'$ *and* $([a]\times[a])\cap([*]\times[*]) \notin \mathfrak{I}'$ *iff, by definition of* $\mathfrak{I}'$ *and 2.3, for each* $\beta \in \mathfrak{I}$ *and* $a \notin M$, $(q\times q)(\beta) \not\subseteq [a]\times[*]$ *and* $(q\times q)(\beta) \not\subseteq ([a]\times[a])\cap([*]\times[*])$ *iff, by Lemma 3.5, for all* $a,b \in B$ *and* $\beta \in \mathfrak{I}$ *with* $a \notin M$, $b \in M$, *the conditions hold.* □

Theorem 3.8. *Let* $(B,\mathfrak{I})$ *be a semiuniform convergence space and* $p \in B$. *Then, the followings are equivalent.*

(*i*) $(B,\mathfrak{I})$ *is* T_1 *at* p.

(*ii*) $\{p\}$ *is strongly closed.*

(*iii*) *For each* $x \neq p$, $[x]\times[p] \notin \mathfrak{I}$ *and* $([x]\times[x])\cap([p]\times[p]) \notin \mathfrak{I}$.

Proof. *It follows from Theorem 3.3, Theorem 3.7, and Definition 3.1.* □

Note that the full subcategory ***SConv*** of ***SUConv*** whose object class consists of all convergence spaces is isomorphic to full subcategory $\boldsymbol{Conv_S}$ of ***Conv*** (= the category of Kent convergence spaces) whose object class consists of all symmetric Kent convergence spaces (Theorem 3.2 of [31] p. 473). A functor $G : \boldsymbol{Conv_S} \to \boldsymbol{SConv}$ is defined by $G((B, q)) = (B, \mathfrak{I}_q)$, where $\mathfrak{I}_q = \{\beta \in F(B \times B) : \exists x \in B, \exists \alpha \in F(B)$ with $(\alpha, x) \in q$ such that $\beta \supset (\alpha \times \alpha)\}$

Let (B, q) be a Kent(local filter) convergence space and $M \subset B$. Define $K(M)$ =$\{x \in B :$ there exists $(\alpha, x) \in q$ such that $\alpha \cup [M]$ is proper$\}$ [19] p.198. If $K(M) = M$, then M is said to be closed (we call it as closedness in the usual sense). $M \subset B$ is closed in a semiuniform convergence space $(B, \mathfrak{I})$, if it is closed in its underlying Kent convergence space $(B, q_{\mathfrak{I}})$, where $q_{\mathfrak{I}} = \{(\alpha, x) : \alpha \cap [x] \in \gamma_{\mathfrak{I}}\}$, where $\gamma_{\mathfrak{I}} = \{\beta \in F(B) : \beta \times \beta \in \mathfrak{I}\}$ [32] or [30].

Let (B, q) be a Kent convergence space and $M \subset B$. Note that $M \subset B$ is closed in the usual sense iff for all $a \in B$ if $a \notin M$, then $\alpha \cup [M]$ is improper for all $(\alpha, x) \in q$ ([9], 2.7(1)).

Theorem 3.9. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space. $\phi \neq M \subset B$ is strongly closed iff M is closed in the usual sense.*

Proof. Suppose that M is strongly closed. We show that M is closed in the usual sense. Suppose that for all $a \in B$ with $a \notin M$, and for all $(\alpha, a) \in q$. Note that $(\alpha \times \alpha) \in \mathfrak{I}_q \subset \mathfrak{I}$, and in particular, by Theorem 3.7 (i), $(\alpha \times \alpha) \cup ([a] \times [M]$ is improper, and consequently, $\alpha \cup [M]$ is improper. Hence, M is closed in the usual sense.

Suppose that M is closed in the usual sense. We show that M is strongly closed. Suppose that for all $a, b \in B$ with $a \notin M$, $b \in M$ and $\beta \in \mathfrak{I}$. We show that the conditions (i) and (ii) of Theorem 3.7 hold. $\beta \in \mathfrak{I}$ implies that there exists $(\alpha, a) \in q$ such that $\beta \supset (\alpha \times \alpha)$. We show that $\beta \not\subseteq [a] \times [b]$. If $\beta \subset [a] \times [b]$, then $\alpha \subset [a]$ and $\alpha \subset [b]$ since $\beta \supset (\alpha \times \alpha)$. It follows that $\alpha \cup [M]$ is proper since $b \in M$ and $[M] \subset [b]$, a contradiction to M is being closed in the usual sense. Hence, $\beta \not\subseteq [a] \times [b]$. Now, we show that $(\beta) \cup ([a] \times [M]$ is improper. Suppose that it is proper. It follows that $(\alpha \times \alpha) \cup ([a] \times [M]$ is proper (since $\beta \supset (\alpha \times \alpha)\}$), and consequently, $\alpha \cup [M]$ is proper, a contradiction. The condition (ii) of Theorem 3.4 follows easily. Hence, M is strongly closed. □

Remark 3.10. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space and $\phi \neq M \subset B$. Then, by Theorem 3.6, Theorem 3.7, and Theorem 3.9, if M is closed, then M is both strongly closed and closed in the usual sense.*

4. Closure Operators

Let $\mathcal{E}$ be a set based topological category.

Recall, (cf. [19] p.132 or [21] p.25), that a closure operator C of $\mathcal{E}$ is an assignment to each subset M of (the underlying set of) any object X of a subset CM of X such that:

a) $M \subset CM$;

b) $CN \subset CM$ whenever $N \subset M$;

c) (continuity condition). For each $f : X \to Y$ in $\mathcal{E}$ and M subset of Y, $C(f^{-1}(M)) \subset f^{-1}(CM)$, or equivalently, $f(CM) \subset C(f(M))$.

$M \subset X$ is called C-closed (C-dense) in X if $CM = M$ ($CM = X$), and an $\mathcal{E}$-morphism $f : X \to Y$ is called C-closed if $f(M)$ is C-closed in Y for each C-closed M in X [21].

A closure operator C is called idempotent if $C(CM) = CM$, and it is weakly hereditary if every subobject of any object in $\mathcal{E}$ is C- dense in its C-closure [19] or [21].

The discrete closure operator δ is defined by setting $\delta(M) = M$ for each $X \in \mathcal{E}$ and $M \subset X$. The trivial closure operator ∂ is defined by setting ∂(M) = X for each $X \in \mathcal{E}$ and $M \subset X$ [21] p. 39 or [20] p.21.

The closure operators of $\mathcal{E}$ form a large complete lattice with δ and ∂ as bottom and top elements. For closure operators C and D the meet $C \bigwedge D$ and the join $C \bigvee D$ are defined by $(C \bigwedge D)(M) = C(M) \cap D(M)$ and ($C \bigvee D)(M) = C(M) \cup D(M)$ for each $X \in \mathcal{E}$ and $M \subset X$. The idempotent hull of a closure operator C is denoted by $\hat{C}$ (this is the least idempotent closure operator above C) [19] p.134. A closure operator C is said to be additive if for each $X \in \mathcal{E}$ and M and $N \subset X$, $C(M \bigvee N) = C(M) \cup C(N) = C(M) \bigvee C(N)$ [19] or [21], and it is called hereditary if for each $X \in \mathcal{E}$ and M and $N \subset X$ with $M \subset N$, $C(M_N) = N \bigwedge C(M)$, where

$C(M_N)$ denotes C-closure of M in N [19] or [21].
More on closure operators can be found in [19], [20] and [21].

We, now, show that the notions of closedness and strong closedness form appropriate closure operators $cl^{\mathcal{E}}$ and $scl^{\mathcal{E}}$ of $\mathcal{E}$, where $\mathcal{E}$ is the category ***SUConv*** of semiuniform convergence spaces and investigate which of the properties idempotency, (weak) hereditariness, additivity and productivity are enjoyed by each of them.

Definition 4.1. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space and $M \subset B$. The (strong) closure of M is the intersection of all (strongly) closed subsets of B containing M and it is denoted by $cl^{\mathcal{E}}(M)$ (resp. $scl^{\mathcal{E}}(M)$), where $\mathcal{E}$ is the category* ***SUConv***.

Lemma 4.2. *Let $f : (A, \mathfrak{I}') \to (B, \mathfrak{I})$ be in* ***SUConv***.
(1) If $D \subset B$ is strongly closed, so also is $f^{-1}(D)$.
(2) Let $(B, \mathfrak{I})$ be a semiuniform convergence space. If $N \subset B$ is strongly closed and $M \subset N$ is strongly closed, so also is $M \subset B$.

Proof. *(1). Suppose $D \subset B$ is strongly closed and for all $a, b \in A$ with $a \notin f^{-1}(D)$, $b \in f^{-1}(D)$ and $\beta \in \mathfrak{I}'$, we need to show that*
(i) $\beta \not\subseteq [a] \times [b]$ and $\beta \cup \left([a] \times \left[f^{-1}(D)\right]\right)$ is improper.
(ii) $\beta \cap \left(\left[f^{-1}(D)\right] \times \left[f^{-1}(D)\right]\right) \not\subseteq ([a] \times [a]) \cap \left(\left[f^{-1}(D)\right] \times \left[f^{-1}(D)\right]\right)$ or $\beta \cup \left(\left[f^{-1}(D)\right] \times \left[f^{-1}(D)\right]\right)$ is improper.
Note that $f(a), f(b) \in B$ with $f(a) \notin D$, $f(b) \in D$, and $(f \times f)(\beta) \in \mathfrak{I}$. Since D is strongly closed, by Theorem 3.7, we have
(i) $(f \times f)(\beta) \not\subseteq [f(a)] \times [f(b)]$ and $(f \times f)(\beta) \cup ([f(a)] \times [D])$ is improper.
(ii) $(f \times f)(\beta) \cap ([D] \times [D]) \not\subseteq ([f(a)] \times [f(a)]) \cap ([D] \times [D])$ or $(f \times f)(\beta) \cup ([D] \times [D])$ is improper.
If $(f \times f)(\beta) \not\subseteq [f(a)] \times [f(b)]$, then clearly we get $\beta \not\subseteq [a] \times [b]$. We show that $\beta \cup \left([a] \times \left[f^{-1}(D)\right]\right)$ is improper. Suppose that it is proper. By Lemma 2.1, we have $(f \times f)(\beta) \cup ([f(a)] \times [D]) \subset (f \times f)(\beta) \cup \left([f(a)] \times \left[ff^{-1}(D)\right]\right) \subset (f \times f)(\beta \cup ([a] \times [f^{-1}(D)])$, and consequently $(f \times f)(\beta) \cup ([f(a)] \times [D])$ is proper, a contradiction. Hence, $\beta \cup ([a] \times [f^{-1}(D)])$ is improper.

If $(f \times f)(\beta) \cap ([D] \times [D]) \not\subseteq ([f(a)] \times [f(a)]) \cap ([D] \times [D])$, then we get $\beta \cap \left(\left[f^{-1}(D)\right] \times \left[f^{-1}(D)\right]\right) \not\subseteq ([a] \times [a]) \cap \left(\left[f^{-1}(D)\right] \times \left[f^{-1}(D)\right]\right)$. It remains to show that $\beta \cup \left(\left[f^{-1}(D)\right] \times \left[f^{-1}(D)\right]\right)$ is improper. Suppose that it is proper. By Lemma 2.1, we have $(f \times f)(\beta) \cup ([D] \times [D]) \subset (f \times f)(\beta) \cup \left(\left[ff^{-1}(D)\right] \times \left[ff^{-1}(D)\right]\right) \subset (f \times f)(\beta \cup ([f^{-1}(D)] \times [f^{-1}(D)])$, and consequently $(f \times f)(\beta) \cup ([D] \times [D])$ is proper, a contradiction. Hence, $\beta \cup ([f^{-1}(D)] \times [f^{-1}(D)])$ is improper.
(2). Suppose $N \subset B$ and $M \subset N$ are (strongly) closed, for all $a, b \in B$ with $a \notin M$, $b \in M$, and $\beta \in \mathfrak{I}$. By Theorem 3.7, we need to show that
(i) $\beta \not\subseteq [a] \times [b]$ and $\beta \cup ([a] \times [M])$ is improper.
(ii) $\beta \cap ([M] \times [M]) \not\subseteq ([a] \times [a]) \cap ([M] \times [M])$ or $\beta \cup ([M] \times [M])$ is improper.
Suppose $a \notin N$. Since $N \subset B$ is strongly closed, by Theorem 3.7, we have
(i) $\beta \not\subseteq [a] \times [b]$ and $\beta \cup ([a] \times [N])$ is improper.
(ii) $\beta \cap ([N] \times [N]) \not\subseteq ([a] \times [a]) \cap ([N] \times [N])$ or $\beta \cup ([N] \times [N])$ is improper.
Consequently, we get (i) $\beta \not\subseteq [a] \times [b]$ and $\beta \cup ([a] \times [M])$ is improper.
(ii) $\beta \cap ([M] \times [M]) \not\subseteq ([a] \times [a]) \cap ([M] \times [M])$ or $\beta \cup ([M] \times [M])$ is improper (since $M \subset N$).
Suppose that $a \in N$. Since the inclusion map $i : (N, \mathfrak{I}') \to (B, \mathfrak{I})$ is initial lift and $\beta \in \mathfrak{I}$, it follows from 2.2 that $(i \times i)^{-1}(\beta) \in \mathfrak{I}'$ (note that $(i \times i)^{-1}(\beta) = \beta \cup [N] \times [N]$ and $\beta \subset (i \times i)((i \times i)^{-1}(\beta))$. Since $M \subset N$ is strongly closed and $a, b \in N$, $a \notin M, b \in M$, by Theorem 3.7 ,
(i) $(i \times i)^{-1}(\beta) \not\subseteq [a] \times [b]$ (and consequently, $\beta \not\subseteq [a] \times [b]$ and $(i \times i)^{-1}(\beta) \cup ([a] \times [M]) = \beta \cup ([a] \times [M])$ is improper.
(ii) $(i \times i)^{-1}(\beta) \cap ([M] \times [M]) = \beta \cap ([M] \times [M]) \not\subseteq ([a] \times [a]) \cap ([M] \times [M])$ or $(i \times i)^{-1}(\beta) \cup ([M] \times [M]) = \beta \cup ([M] \times [M])$ is improper. Hence, by Theorem 3.7, $M \subset B$ is strongly closed (since $M \subset N$). □

Lemma 4.3. *Let $f : (A, \mathfrak{I}') \to (B, \mathfrak{I})$ be in* ***SUConv***.
(1) If $D \subset B$ is closed, so also is $f^{-1}(D)$.

(2) Let $(B, \mathfrak{I})$ be a semiuniform convergence space. If $N \subset B$ is closed and $M \subset N$ is closed, so also is $M \subset B$.

Proof. It is similar to the proof of the Lemma 4.2 by using Theorem 3.6 instead of Theorem 3.7. $\square$

Theorem 4.4. *Let $\mathcal{E}$ be **SUConv**. Both $scl^{\mathcal{E}}$ and $cl^{\mathcal{E}}$ are idempotent, weakly hereditary, productive, and hereditary closure operators of $\mathcal{E}$.*

Proof. It follows from Definition 4.1, Lemma 4.2, Lemma 4.3, Exercise 2.D, Theorems 2.3 and 2.4, and Proposition 2.5 of [21]. $\square$

Let $\mathcal{E}$ be a set based topological category and C be a closure operator of $\mathcal{E}$.
$\mathcal{E}_{0C} = \{ X \in \mathcal{E} : x \in C(\{y\})$ and $y \in C(\{x\})$ implies $x = y\}$.
$\mathcal{E}_{1C} = \{X \in \mathcal{E} : \forall x \in X, C(\{x\}) = \{x\}\}$.
For $\mathcal{E} = \boldsymbol{Top}$, the category of topological spaces and $C = K$, the ordinary closure, we obtain the class of T_0-spaces and T_1-spaces, respectively.

Theorem 4.5. *Let $\mathcal{E}$ be **SUConv** and $(B, \mathfrak{I}) \in \mathcal{E}$. Then, the followings are equivalent.*
(1) $(B, \mathfrak{I}) \in \mathcal{E}_{1scl}$.
(3) $(B, \mathfrak{I})$ is $\overline{T_1}$.
(4) $(B, \mathfrak{I})$ is $\overline{T_0}$.
(5) for any distinct pair of points x and y in B, $[x] \times [y] \notin \mathfrak{I}$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \mathfrak{I}$.

Proof. It follows from Theorem 3.8 and Theorem 4.4, Theorem 4.6, and Remark 4.7(1) of [13]. $\square$

Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be a topological functor. Recall that a full and isomorphism-closed subcategory $\mathcal{S}$ of $\mathcal{E}$ is a) epireflective in $\mathcal{E}$ iff it is closed under the formation of products and extremal subobjects (i.e., subspaces); b) quotient-reflective in $\mathcal{E}$ iff it is epireflective and is closed under finer structures (i.e., if $X \in \mathcal{S}$, $Y \in \mathcal{E}$, $\mathcal{U}(X) = \mathcal{U}(Y)$, and $id : Y \to X$ is a $\mathcal{E}$-morphism, then $Y \in \mathcal{S}$).

Theorem 4.6. *Let $\mathcal{E} = \boldsymbol{SUConv}$. The subcategory $\mathcal{E}_{1scl}$ is quotient-reflective in $\mathcal{E}$.*

Proof. It is easy to see that this subcategory is full, isomorphism-closed, closed under formation of subspaces, and closed under finer structures. It remains to show thatit is closed under formation of products. Let $(B_i, \mathfrak{I}_i)$ and $B = \prod_{i \in I} B_i$. Suppose $(B_i, \mathfrak{I}_i) \in \mathcal{E}_{1scl}$. We show that $(B, \mathfrak{I}) \in \mathcal{E}_{1scl}$, where $\mathfrak{I}$ is the product structure on B. Suppose there exist $x \neq y$ in B such that $[x] \times [y] \in \mathfrak{I}$ or $([x] \times [x]) \cap ([y] \times [y]) \in \mathfrak{I}$. It follows that there exists $m \in I$ such that $x_m \neq y_m$ in B_m and $(\pi_m \times \pi_m)([x] \times [y]) = [x_m] \times [y_m] \in \mathfrak{I}_m$ or, by Lemma 2.1, $(\pi_m \times \pi_m)([x] \times [x]) \cap ([y] \times [y])) = ([x_m] \times [x_m]) \cap ([y_m] \times [y_m]) \in \mathfrak{I}_m$, a contradiction. Hence, for any $x \neq y$ in B, $[x] \times [y] \notin \mathfrak{I}$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \mathfrak{I}$, i.e., $(B, \mathfrak{I}) \in \mathcal{E}_{1scl}$.
Hence, the subcategory $\mathcal{E}_{1scl}$ is a quotient-reflective subcategory of $\mathcal{E}$. $\square$

Let (B, q) be a generalized (filter) convergence space and $M \subset B$. Define $K(M) = \{x \in B :$ there exists $(\alpha, x) \in q$ such that $\alpha \cup [M]$ is proper$\}$ [19] p.198 and $K^*(M) = \{x \in B : K(\{x\}) \cap M \neq \emptyset\} = \{x \in B : (\exists c \in M)$ and $([x], c) \in q\}$ [20] p.21. Note that K, the ordinary Kuratowski operator, and its opposite K^* are closure operators. If $K(M) = M$, then M is said to be closed (we call it as closedness in the usual sense). $M \subset B$ is closed in a semiuniform convergence space $(B, \mathfrak{I})$, if it is closed in its underlying Kent convergence space $(B, q_{\mathfrak{I}})$, where $q_{\mathfrak{I}} = \{(\alpha, x) : \alpha \cap [x] \in \gamma_{\mathfrak{I}}\}$, where $\gamma_{\mathfrak{I}} = \{\beta \in F(B) : \beta \times \beta \in \mathfrak{I}\}$ [32] or [30].

Remark 4.7. *(1) Let (B, q) be a Kent (local filter) convergence space and $M \subset B$.*
(i) By ([9], 2.6(4)), M is strongly closed iff M is $K \bigvee K^*$- closed (the join of K and K^*) .
(ii) By ([9], 2.7(1)), M is closed (in the usual sense) iff for $x \in B$ if there exists $(\alpha, x) \in q$ such that $\alpha \cup [M]$ is proper, then $x \in M$.
(iii) By ([9], 2.5(1)), M is closed (in our sense) iff M is $K \bigwedge K^*$- closed (the meet of K and K^*).
(iv) By (i), (ii), and (iii), if M is strongly closed, then M is closed (in both senses).
(v)By (ii) and (iii), if M is closed (in the usual sense), then M is closed (in our sense).

*(2) Let $\mathcal{U} : \mathcal{E} \to$ **Set** be a topological functor, X an object in $\mathcal{E}$ and $p \in \mathcal{U}(X)$ be a retract of X, i.e., the initial lift $h : \bar{1} \to X$ of the $\mathcal{U}-$source $p : 1 \to \mathcal{U}(X)$ is a retract ([4], Theorem 2.6). Then if X is T_i, then X is T_i at p, $i = 0, 1$ but the converse of implication is not true, in general.*
(3) Let (B, q) be a Kent (local filter) convergence space and $M \subset B$.
By ([9], Theorem 2.5(2)), if (B, q) is T_1, then all subsets of B are closed (in our sense) and M is strongly closed iff M is closed (in the usual sense).
(4) By 3.4 of [10], for $\mathcal{E} = \textbf{Top}$ and $C = scl^{\mathcal{E}}, K, K^*$ or $C = cl^{\mathcal{E}}$, $\mathcal{E}_{1C}$ is the class of T_1-spaces.
(5) By 2.2.6 of [2], $cl^{=\textbf{Top}} = K$ and $scl^{=\textbf{Top}} = \widehat{K \bigvee K^*}$, the idempotent hull of $K \bigvee K^*$. Furthermore, if a topological space X is T_1, then $cl^{=\textbf{Top}} = K = scl^{=\textbf{Top}}$.
(6) For $\mathcal{E} = \textbf{LFCO}$, by Theorem 2.5 of [9], $cl^{\mathcal{E}} = \widehat{K \bigwedge K^*}$, the idempotent hull of $K \bigwedge K^*$, and $scl^{\mathcal{E}} = \widehat{K \bigvee K^*}$. In particular, if X is T_1, then $cl^{\mathcal{E}} = \delta$, the discrete closure operator and $scl^{\mathcal{E}} = K$.
(7) For an arbitrary set based topological category $\mathcal{E}$, it is not known, in general, whether $cl^{\mathcal{E}}$ and $scl^{\mathcal{E}}$ are closure operators in the sense of [19] or not.

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References

[1] J. Adámek, H. Herrlich, and G.E. Strecker, *Abstract and Concrete Categories*, Wiley, New York, 1990.
[2] M. Baran, *Separation properties*, Indian J. Pure Appl. Math., **23** (1992), 333-341.
[3] M. Baran, *The notion of closedness in topological categories*, Comment. Math. Univ. Carolinae, **34** (1993), 383-395.
[4] M. Baran, *Generalized Local Separation Properties*, Indian J. pure appl. **25** (6) (1994), 615-620.
[5] M. Baran, and H. Altındiş, *T_2 -objects in topological categories*, Acta Math. Hungar., **71** (1996), no. 1-2, 41-48.
[6] M. Baran, *A notion of compactness in topological categories*, Publ. Math. Debrecen, **50** (1997), no. 3-4, 221-234.
[7] M. Baran, *Completely Regular Objects and Normal Objects in topological categories*, Acta Math. Hungar., **80** (1998), 211-224.
[8] M. Baran, *T_3 and T_4-objects in topological categories*, Indian J. Pure Appl. Math., **29** (1998), 59-69.
[9] M. Baran, *Closure operators in convergence spaces*, Acta Math. Hungar., **87** (2000), 33-45.
[10] M. Baran, *Compactness, perfectness, separation, minimality and closedness with respect to closure operators*, Applied Categorical Structures, **10** (2002), 403–415.
[11] M. Baran and M. Kula, *A note on connectedness*, Publ. Math. Debrecen, **68** (2006), no. 3-4, 489-501.
[12] M. Baran and J. Al-Safar, *Quotient-Reflective and Bireflective Subcategories of the category of Preordered Sets*, Topology and its Appl.,**158** (2011), 2076-2084, DOI 10.1016/j.topol.2011.06.043.
[13] M. Baran, S. Kula, and A. Erciyes, *T_0 and T_1semiuniform convergence spaces*, Filomat, **27** (2013), no. 4, 537546, DOI 10.2298/FIL1304537B.
[14] A. Behling *Einbettung Uniformer Räume in Topologische Universen*, Thesis, Freie Universität Berlin, 1992.
[15] R. Beattie, and H.P. Butzmann, *Convergence Structers and Applications to Functional Analysis*, Kluwer Academic Publishers (2002).
[16] C.H. Cook and H.R. Fisher, *On Equicontinuity and Continuous Convergence* , Math. Ann., **159** (1965), 94-104.
[17] A. Császár, *Foundations of General Topology* , Macmillan, New York, (1963).
[18] P. Fletscher and W.F. Lindgren, *Quasi-Uniform Spaces* , Marcel Dekker, New York, 1982.
[19] D. Dikranjan and E. Giuli, *Closure operators I*, Topology Appl., **27** (1987), 129-143.
[20] D. Dikranjan, E. Giuli and W. Tholen, *Closure Operators II*, Proceedings of the International Conference on Categorical Topology, (Prague,1988) (World Scientific Publ., Singapore,1989) 297-335.
[21] D. Dikranjan and W. Tholen, *Categorical Structure of Closure Operators*, Kluwer Academic Publishers, Dordrecht 1995.
[22] D.B.Doitchinov, *A Unified Teory of Topological Spaces, Proximity Spaces and Uniform Spaces*, Soviet Math. Dokl., **5** (1964), 595-598.
[23] H. Herrlich, *Topological Structures*, In: Math. Centre Tracts, Math. Centrum, Amsterdam, **52** (1974), 59-122.
[24] H. Herrlich, *Limit-Operators and Topological coreflection*, Trans. Amer. Math. Soc., **146**,(1969), 203-210.
[25] H.H. Hong, *Limit Operators and Reflective Subcategories*, Topo 72 II Pittsburg Internat. Confer., Lecture Notes Math. **378**(Springer, Berlin, 1973).
[26] M. Katétov, *On Continuity Structures and Spaces of Mappings*, Comment. Math. Univ. Carolinae, **6** (1965), 257-278.
[27] D. C. Kent, *Convergence Functions and Their Related Topologies*, Fund. Math., **54** (1964), 125-133.
[28] L. D. Nel , *Initially Structured Categories and Cartesian Closedness*, Canad. Journal of Math., **27** (1975), 1361-1377.
[29] G. Preuss, *Theory of Topological Structures. An Approach to Topological Categories*, D. Reidel Publ. Co., Dordrecht, 1988.
[30] G. Preuss, *Cauchy Spaces and Generalizations*, Math. Japonica, **38** No. 5(1993), 803-812.
[31] G. Preuss, *Semiuniform Convergence Spaces*, Math. Japonica, **41** No.3(1995), 465-491.
[32] G., Preuss, *Foundations of topology, An approach to Convenient topology*, Kluwer Academic Publishers, Dordrecht, (2002).
[33] G., Preuss, *Non-Symmetric Convenient Topology And its Relations to Convenient Topology*, Topology Proc., **29** (2005), 595-611.
[34] S. Salbany, *Reflective Subcategories and Closure Operators*, in in Proc. First Categorical Topology Symposium, Lecture Notes in Math., **540** (Springer, Berlin, 1976), 548-565.
[35] F. Schwarz, *Connections Between Convergence and Nearness*, Lecture Notes in Math., Springer-Verlag **719** (1978), 345-354.
[36] A. Weil, *Sur les Espaces á Structures et sur la topologie Générale*, Hermann, Paris, 1937.