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Infinitesimal Bending Influence on the Willmore Energy of Curves

Marija S. Najdanović[a,b]

[a] *Faculty of Sciences and Mathematics, University of Niš, Serbia*
[b] *Preschool Teacher Training College, Kruševac, Serbia*

Abstract. In this paper we study the change of the Willmore energy of curves, as a special case of so-called Helfrich energy, under infinitesimal bending determined by the stationarity of arc length. We examine the variation of the unit tangent, principal normal and binormal vector fields, the curvature and the torsion of the curve. We obtain an explicit formula for calculating the variation of the Willmore energy, as well as the Euler-Lagrange equations describing equilibrium. We find an infinitesimal bending field for a helix and compute the variation of its Willmore energy under such infinitesimal bending.

1. Introduction

Let $C : \mathbf{r} = \mathbf{r}(s)$ ($\mathbf{r} : \mathcal{I} \mapsto \mathcal{R}^3$), $\mathcal{I} \subseteq \mathcal{R}$, be a regular curve of the class C^α, $\alpha \geq 2$. The Helfrich energy of the curve C is given by

$$\mathcal{H}_\lambda(C) = \frac{1}{2}\int_{\mathcal{I}} (k - c_0)^2\, ds + \lambda \mathcal{L}(C), \tag{1}$$

where $k = \mathbf{r}'' \cdot \mathbf{n}_1$ denotes the scalar curvature of the curve, $\mathbf{n}_1$ is the unit principal normal, s denotes the arc length and $\mathcal{L}(C) = \int_{\mathcal{I}} ds$ the length of C. The map $c_0 : \mathcal{I} \mapsto \mathcal{R}$ is called spontaneous curvature. The constant $\lambda \in \mathcal{R}$ is taken to be positive, so that the growth in length of a curve is penalized. The above functional is motivated by the modeling of cell membranes [5].

The special case where $c_0 = 0$ and $\lambda = 0$ is known as Willmore energy

$$\mathcal{W}(C) = \frac{1}{2}\int_{\mathcal{I}} k^2\, ds, \tag{2}$$

and it can also be historically motivated by the so-called Euler-Bernoulli model of elastic rods [12]. Willmore energy penalizes bending models the stiffness of a polymer, and it has been used to model the elastic properties of DNA [2].

The Helfrich and Willmore energies are mathematically very interesting and in particular the Willmore flow is nowadays considered to be one of the most important models in which fourth order PDEs appear. Both functionals have been extensively investigated analytically and numerically in recent years and the

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Email address: marijamath@yahoo.com (Marija S. Najdanović)

literature is by now rather vast. In [3] the authors study the long-time evolution of regular open curves in $\mathcal{R}^n$, $n \geq 2$, moving according to the L^2-gradient flow for a generalization of the Helfrich functional. In [2], the authors examine the equilibrium conditions of a curve in space when a local energy penalty is associated with its extrinsic geometrical state characterized by its curvature and torsion. To do this they tailor the theory of deformations to the Frenet-Serret frame of the curve. Many papers are related to the Willmore energy of surfaces (see [6], [19], [20], ...)

2. Infinitesimal bending of a curve in $\mathcal{R}^3$

We begin by studying infinitesimal bending of a curve. More information about infinitesimal bending of the curves and the surfaces one can get from [1, 4, 7, 13], [14]-[20]. The concept on infinitesimal deformations of curves in the spaces with linear connection is given in [21]. Infinitesimal rigidity and flexibility of a non-symmetric affine connection space is considered in [8], [9], [10], [18].

Definition 2.1. *Let us consider continuous regular curve*

$$C : \mathbf{r} = \mathbf{r}(u), \quad u \in J \subseteq \mathcal{R} \tag{3}$$

included in a family of the curves

$$C_\epsilon : \tilde{\mathbf{r}}(u, \epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \mathbf{z}(u), \quad u \in J, \quad \epsilon \in (-1, 1), \tag{4}$$

where u is a real parameter and we get C for $\epsilon = 0$ ($C = C_0$). Family of curves C_ϵ is **infinitesimal bending of a curve** *C if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon), \tag{5}$$

where $\mathbf{z} = \mathbf{z}(u)$, $\mathbf{z} \in C^1$ is **infinitesimal bending field** *of the curve C.*

Theorem 2.2. *[4] Necessary and sufficient condition for $\mathbf{z}(u)$ to be an infinitesimal bending field of a curve C is to be*

$$d\mathbf{r} \cdot d\mathbf{z} = 0, \tag{6}$$

where $\cdot$ stands for the scalar product in $\mathcal{R}^3$. □

Theorem 2.3. *[15] Infinitesimal bending field for the curve C is*

$$\mathbf{z}(u) = \int [p(u)\mathbf{n}_1(u) + q(u)\mathbf{n}_2(u)]\, du, \tag{7}$$

where $p(u)$ and $q(u)$, are arbitrary integrable functions and vectors $\mathbf{n}_1(u)$ and $\mathbf{n}_2(u)$ are respectively unit principal normal and binormal vector fields of the curve C. □

Under infinitesimal bending of the surfaces each line element gets non-negative addition (see [13]). Let us prove an analogous theorem which holds for the curves.

Theorem 2.4. *Under infinitesimal bending of the curves each line element gets non-negative addition, which is the infinitesimal value of the order higher than the first with respect to ϵ, i. e.*

$$ds_\epsilon - ds = o(\varepsilon) \geq 0. \tag{8}$$

Proof. As

$$d\mathbf{r} = \dot{\mathbf{r}}(u)du, \quad d\mathbf{z} = \dot{\mathbf{z}}(u)du,$$

according to (6), for infinitesimal bending field of a curve C we have

$$\dot{\mathbf{r}}(u) \cdot \dot{\mathbf{z}}(u) = 0, \tag{9}$$

where dot denotes derivative with respect to u. Based on that we have

$$\begin{aligned} ds_\epsilon &= \|\dot{\mathbf{r}}_\epsilon(u)\| \, du = \|\dot{\mathbf{r}}(u) + \epsilon \dot{\mathbf{z}}(u)\| \, du = (\|\dot{\mathbf{r}}(u)\|^2 + \epsilon^2 \|\dot{\mathbf{z}}(u)\|^2)^{\frac{1}{2}} \, du \\ &= \|\dot{\mathbf{r}}(u)\| \Big(1 + \epsilon^2 \frac{\|\dot{\mathbf{z}}(u)\|^2}{\|\dot{\mathbf{r}}(u)\|^2}\Big)^{\frac{1}{2}} \, du = ds \Big(1 + \epsilon^2 \frac{\|\dot{\mathbf{z}}(u)\|^2}{\|\dot{\mathbf{r}}(u)\|^2}\Big)^{\frac{1}{2}} \end{aligned}$$

After using the Maclaurin formula we get

$$ds_\epsilon = ds \Big(1 + \epsilon^2 \frac{\|\dot{\mathbf{z}}(u)\|^2}{2\|\dot{\mathbf{r}}(u)\|^2} - \epsilon^4 \frac{\|\dot{\mathbf{z}}(u)\|^4}{8\|\dot{\mathbf{r}}(u)\|^4} + \dots \Big)$$

i.e.

$$ds_\epsilon - ds = \epsilon^2 \frac{\|\dot{\mathbf{z}}(u)\|^2}{2\|\dot{\mathbf{r}}(u)\|^2} - \dots,$$

which leads to (8). ∎

A curve parameterized by the arc length. Consider a regular curve

$$C : \mathbf{r} = \mathbf{r}(s) = \mathbf{r}[u(s)], \quad s \in \mathcal{I}, \tag{10}$$

of the class C^α, $\alpha \geq 3$, parameterized by the arc length s. The unit tangent to the curve is given by $\mathbf{t} = \mathbf{r}'$, where prime denotes a derivative with respect to arc length s. Clearly, $\mathbf{t}'$ is orthogonal to $\mathbf{t}$, but $\mathbf{t}''$ is not. The Frenet equations

$$\begin{aligned} \mathbf{t}' &= k\mathbf{n}_1, \\ \mathbf{n}_1' &= -k\mathbf{t} + \tau\mathbf{n}_2, \\ \mathbf{n}_2' &= -\tau\mathbf{n}_1, \end{aligned} \tag{11}$$

describe the construction of an orthonormal basis $\{\mathbf{t}, \mathbf{n_1}, \mathbf{n_2}\}$ along a curve, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are respectively unit principal normal and binormal vector fields of the curve. We choose an orientation with $\mathbf{n}_2 = \mathbf{t} \times \mathbf{n}_1$. k and τ are respectively the curvature and the torsion.

Consider an infinitesimal bending of the curve (10),

$$C_\epsilon : \tilde{\mathbf{r}}(s, \epsilon) = \mathbf{r}_\epsilon(s) = \mathbf{r}(s) + \epsilon \mathbf{z}(s). \tag{12}$$

As the vector field $\mathbf{z}$ is defined in the points of the curve (10), it can be presented in the form

$$\mathbf{z} = z\mathbf{t} + z_1\mathbf{n}_1 + z_2\mathbf{n}_2, \tag{13}$$

where $z\mathbf{t}$ is tangential and $z_1\mathbf{n}_1 + z_2\mathbf{n}_2$ is normal component, z, z_1, z_2 are the functions of s.

Theorem 2.5. *Necessary and sufficient condition for the field* $\mathbf{z}$*, (13), to be infinitesimal bending field of the curve* C*, (10), is*

$$z' - kz_1 = 0, \tag{14}$$

where k *is the curvature of* C*.*

Proof. According to (6), the necessary and sufficient condition for the field $\mathbf{z}$ to be infinitesimal bending field of the curve C is

$$\mathbf{r}' \cdot \mathbf{z}' = 0, \tag{15}$$

i. e. $\mathbf{t} \cdot \mathbf{z}' = 0$. Substituting Eq. (13) into the previous equation and using Frenet equations (11), we obtain (14). ∎

3. Change of geometric magnitudes under infinitesimal bending of curves

Geometric magnitudes are changing under infinitesimal bending and that changing is described by the variation of a geometric magnitude. We define the variations of the geometric magnitudes according to [13].

Definition 3.1. *Let $\mathcal{A} = \mathcal{A}(u)$ be the magnitude that characterizes a geometric property on the curve C and $\mathcal{A}_\epsilon = \mathcal{A}_\epsilon(u)$ the corresponding magnitude on the curve C_ϵ being infinitesimal bending of the curve C,*

$$\Delta\mathcal{A} = \mathcal{A}_\epsilon - \mathcal{A} = \epsilon\,\delta\mathcal{A} + \epsilon^2\,\delta^2\mathcal{A} + \ldots \epsilon^n\,\delta^n\mathcal{A} + \ldots \tag{16}$$

Coefficients $\delta\mathcal{A}, \delta^2\mathcal{A}, \ldots, \delta^n\mathcal{A}, \ldots$ are **the first, the second, ..., the nth variation** *of the geometric magnitude $\mathcal{A}$, respectively under infinitesimal bending C_ϵ of the curve C.*

In this paper we will consider the first variations under infinitesimal bending of the first order. For this reason, we can represent the magnitude $\mathcal{A}_\epsilon$ as

$$\mathcal{A}_\epsilon = \mathcal{A} + \epsilon\,\delta\mathcal{A},$$

by neglecting the terms of order higher than 1.

Obviously, for the first variation is effective

$$\delta\mathcal{A} = \frac{d}{d\epsilon}\mathcal{A}_\epsilon(u)\Big|_{\epsilon=0}, \tag{17}$$

i. e.

$$\delta\mathcal{A} = \lim_{\epsilon\to 0}\frac{\Delta\mathcal{A}}{\epsilon} = \lim_{\epsilon\to 0}\frac{\mathcal{A}_\epsilon(u) - \mathcal{A}(u)}{\epsilon}. \tag{18}$$

It is easy to prove that (see [17])

$$\mathbf{a})\quad \delta(\mathcal{A}\mathcal{B}) = \mathcal{A}\delta\mathcal{B} + \mathcal{B}\delta\mathcal{A}, \quad \mathbf{b})\quad \delta\Big(\frac{\partial\mathcal{A}}{\partial u}\Big) = \frac{\partial(\delta\mathcal{A})}{\partial u}, \quad \mathbf{c})\quad \delta(d\mathcal{A}) = d(\delta\mathcal{A}). \tag{19}$$

Let us describe the behavior of some geometric magnitudes under infinitesimal bending of a curve.

Lemma 3.2. *Under infinitesimal bending of the curve C, (10), a unit vector of the orthonormal basis and its variation are orthogonal.*

Proof. The condition that the unit tangent vector remains unit after bending $\mathbf{t}_\epsilon \cdot \mathbf{t}_\epsilon = 1$, i. e.

$$(\mathbf{t} + \epsilon\,\delta\mathbf{t}) \cdot (\mathbf{t} + \epsilon\,\delta\mathbf{t}) = 1$$

shows that $\mathbf{t} \cdot \delta\mathbf{t} = 0$, after neglecting the terms of order higher than 1. Similarly we show the statement for $\mathbf{n}_1$ and $\mathbf{n}_2$. ∎

Note that the above statement is true for an arbitrary unit vector.

Lemma 3.3. *Under infinitesimal bending of the curve C, (10), variation of the line element ds is equal to zero, i. e.*

$$\delta(ds) = 0. \tag{20}$$

Proof. Let us consider the curve (10). We conclude that

$$d\mathbf{r} = \mathbf{r}'(s)\, ds = \mathbf{t}\, ds,$$

i. e. after scalar product of previous equation with $\mathbf{t}$,

$$\mathbf{t} \cdot d\mathbf{r} = ds.$$

Applying Leibniz's law and the fact that differential and variation are commutative (19) and Lemma 3.2, we have

$$\delta(ds) = \delta\mathbf{t} \cdot d\mathbf{r} + \mathbf{t} \cdot \delta(d\mathbf{r}) = \delta\mathbf{t} \cdot \mathbf{r}'\, ds + \mathbf{t} \cdot d(\delta\mathbf{r}) = \delta\mathbf{t} \cdot \mathbf{t}\, ds + \mathbf{t} \cdot d\mathbf{z} = \mathbf{t} \cdot (dz\, \mathbf{t} + z\, d\mathbf{t} + dz_1\, \mathbf{n}_1 + z_1\, d\mathbf{n}_1 + dz_2\, \mathbf{n}_2 + z_2\, d\mathbf{n}_2)$$

Using Frenet equations we obtain

$$\delta(ds) = (z' - kz_1)\, ds, \tag{21}$$

which, due to (14), gives (20). ■

Lemma 3.4. *Under infinitesimal bending of the curve C, (10), variation of the unit tangent vector is*

$$\delta\mathbf{t} = (z_1' - \tau z_2 + kz)\, \mathbf{n}_1 + (z_2' + \tau z_1)\, \mathbf{n}_2. \tag{22}$$

Proof. As it is $\delta\mathbf{t} = \delta\mathbf{r}' = (\delta\mathbf{r})' = \mathbf{z}'$, using (13), (14) and Frenet equations we obtain (22). ■

Lemma 3.5. *Under infinitesimal bending of the curve C, (10), variations of the unit principal normal and binormal vectors are respectively*

$$\delta\mathbf{n}_1 = -(kz + z_1' - \tau z_2)\mathbf{t} + \frac{1}{k}(k\tau z + z_2'' - \tau^2 z_2 + 2\tau z_1' + \tau' z_1)\mathbf{n}_2, \tag{23}$$

$$\delta\mathbf{n}_2 = -(z_2' + \tau z_1)\mathbf{t} - \frac{1}{k}(k\tau z + z_2'' - \tau^2 z_2 + 2\tau z_1' + \tau' z_1)\mathbf{n}_1. \tag{24}$$

Proof. The unit normal vector remains unit after infinitesimal bending, which means $\mathbf{n}_{1\epsilon} \cdot \mathbf{n}_{1\epsilon} = 1$, i. e. $(\mathbf{n}_1 + \epsilon\delta\mathbf{n}_1) \cdot (\mathbf{n}_1 + \epsilon\delta\mathbf{n}_1) = 1$ and gives

$$\mathbf{n}_1 \cdot \delta\mathbf{n}_1 = 0. \tag{25}$$

Also, the unit normal vector remains perpendicular to the unit tangent vector, $\mathbf{n}_{1\epsilon} \cdot \mathbf{t}_\epsilon = 0$, i. e. $(\mathbf{n}_1 + \epsilon\delta\mathbf{n}_1) \cdot (\mathbf{t} + \epsilon\delta\mathbf{t}) = 0$, wherefrom we have

$$\mathbf{t} \cdot \delta\mathbf{n}_1 = -\mathbf{n}_1 \cdot \delta\mathbf{t} = -(z_1' - \tau z_2 + kz). \tag{26}$$

Further, we take a variation of the first of (11),

$$\delta\mathbf{t}' = \mathbf{n}_1 \delta k + k\delta\mathbf{n}_1. \tag{27}$$

Dotting with $\mathbf{n}_2$ we obtain $\mathbf{n}_2 \cdot \delta\mathbf{n}_1 = \frac{1}{k}\mathbf{n}_2 \cdot \delta\mathbf{t}'$. To evaluate $\delta\mathbf{t}'$ we apply commutativity of the variation and the derivative and obtain $\delta\mathbf{t}' = (\delta\mathbf{t})'$. Based on (22), Frenet equations and $z' = kz_1$ (due to (14)), one obtains

$$\delta\mathbf{t}' = -k(kz + z_1' - \tau z_2)\, \mathbf{t} + (k'z + z_1'' + (k^2 - \tau^2)z_1 - 2\tau z_2' - \tau' z_2)\, \mathbf{n}_1 + (k\tau z + 2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2)\, \mathbf{n}_2. \tag{28}$$

Now we have

$$\mathbf{n}_2 \cdot \delta \mathbf{n}_1 = \frac{1}{k}(k\tau z + 2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2) \tag{29}$$

Comparing (25), (26) and (29) we obtain (23). Similarly, from the conditions $\mathbf{n}_{2\epsilon} \cdot \mathbf{n}_{2\epsilon} = 1$, $\mathbf{n}_{2\epsilon} \cdot \mathbf{t}_\epsilon = 0$ and $\mathbf{n}_{2\epsilon} \cdot \mathbf{n}_{1\epsilon} = 0$ we obtain (24). ∎

Lemma 3.6. *Under infinitesimal bending of the curve C, (10), variation of the curvature is*

$$\delta k = k' z + z_1'' + (k^2 - \tau^2) z_1 - 2\tau z_2' - \tau' z_2. \tag{30}$$

Proof. Dotting Eq. (27) with $\mathbf{n}_1$ and using Lemma 3.2, we obtain $\delta k = \mathbf{n}_1 \cdot \delta \mathbf{t}'$. This leads to (30) after using (28). ∎

Corollary 3.7. *Under infinitesimal bending of a plane curve, variation of the curvature is*

$$\delta k = k' z + z_1'' + k^2 z_1. \tag{31}$$

Lemma 3.8. *Under infinitesimal bending of the curve C, (10), variation of the torsion is*

$$\delta \tau = z\tau' + k(z_2' + 2\tau z_1) + \Big\{\frac{1}{k}\Big[2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2\Big]\Big\}'. \tag{32}$$

Proof. Let us take a variation of the Frenet equation for $\mathbf{n}_1'$ and dot with $\mathbf{n}_2$. We have

$$\delta \tau = k \mathbf{n}_2 \cdot \delta \mathbf{t} + \mathbf{n}_2 \cdot \delta \mathbf{n}_1'. \tag{33}$$

We now rewrite the second term on the right hand side as

$$\mathbf{n}_2 \cdot \delta \mathbf{n}_1' = (\mathbf{n}_2 \cdot \delta \mathbf{n}_1)' - \mathbf{n}_2' \cdot \delta \mathbf{n}_1 = (\mathbf{n}_2 \cdot \delta \mathbf{n}_1)', \tag{34}$$

after using the third Frenet equation and the Lema 3.2. As it is $\mathbf{r}'' = k\mathbf{n}_1$, we have $\mathbf{t}' = k\mathbf{n}_1$, i. e. $\mathbf{n}_1' = \frac{1}{k}\,\mathbf{t}'$. Farther, $\delta \mathbf{n}_1 = \delta(\frac{1}{k})\,\mathbf{t}' + \frac{1}{k}\,\delta \mathbf{t}'$,

$$\mathbf{n}_2 \cdot \delta \mathbf{n}_1 = \mathbf{n}_2 \cdot \Big[\delta(\frac{1}{k})\, k\mathbf{n}_1 + \frac{1}{k}\,\delta \mathbf{t}'\Big] = \frac{1}{k}\mathbf{n}_2 \cdot \delta \mathbf{t}'. \tag{35}$$

From (33), (34) and (35) we obtain

$$\delta \tau = k\mathbf{n}_2 \cdot \delta \mathbf{t} + \Big(\frac{1}{k}\mathbf{n}_2 \cdot \delta \mathbf{t}'\Big)' \tag{36}$$

Substituting (22) and (28) into (36) and using (14) we obtain (32). ∎

Corollary 3.9. *Under infinitesimal bending of a plane curve, variation of the torsion is*

$$\delta \tau = k z_2' + \Big(\frac{1}{k} z_2''\Big)'. \tag{37}$$

Based on the lemmas that precede, corresponding geometric magnitudes of deformed curves under infinitesimal bending are:

$$\begin{aligned}
\tilde{\mathbf{t}} &= \mathbf{t}_\epsilon = \mathbf{t} + \epsilon\Big[(z_1' - \tau z_2 + kz)\,\mathbf{n}_1 + (z_2' + \tau z_1)\,\mathbf{n}_2\Big],\\
\tilde{\mathbf{n}}_1 &= \mathbf{n}_{1\epsilon} = \mathbf{n}_1 + \epsilon\Big[-(kz + z_1' - \tau z_2)\mathbf{t} + \frac{1}{k}(k\tau z + z_2'' - \tau^2 z_2 + 2\tau z_1' + \tau' z_1)\mathbf{n}_2\Big],\\
\tilde{\mathbf{n}}_2 &= \mathbf{n}_{2\epsilon} = \mathbf{n}_2 + \epsilon\Big[-(z_2' + \tau z_1)\mathbf{t} - \frac{1}{k}(k\tau z + z_2'' - \tau^2 z_2 + 2\tau z_1' + \tau' z_1)\mathbf{n}_1\Big],\\
\tilde{k} &= k_\epsilon = k + \epsilon[k' z + z_1'' + (k^2 - \tau^2) z_1 - 2\tau z_2' - \tau' z_2],\\
\tilde{\tau} &= \tau_\epsilon = \tau + \epsilon\Big\{z\tau' + k(z_2' + 2\tau z_1) + \Big[\frac{1}{k}\Big(2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2\Big)\Big]'\Big\}.
\end{aligned} \tag{38}$$

after neglecting the terms of order higher than 1.

4. The change of the Willmore energy under infinitesimal bending

Let a regular curve of the class C^α, $\alpha \geq 3$, be given with

$$C : \mathbf{r} = \mathbf{r}(s), \quad s \in \mathcal{I}, \quad (\mathbf{r} : \mathcal{I} \mapsto \mathcal{R}^3), \tag{39}$$

The Willmore energy of the curve C is given with the following equation

$$\mathcal{W} = \frac{1}{2}\int_{\mathcal{I}} k^2\, ds. \tag{40}$$

The next theorem is related to determination of the Willmore energy of curve under infinitesimal bending.

Theorem 4.1. *Under infinitesimal bending of the curve C, (39), variation of its Willmore energy is*

$$\delta\mathcal{W} = \int_{\mathcal{I}} ds\left[(k'' + \frac{1}{2}k^3 - k\tau^2)z_1 + (2k'\tau + k\tau')z_2\right] + \int_{\mathcal{I}} ds\left[\frac{1}{2}k^2 z - k'z_1 + kz_1' - 2k\tau z_2\right]'. \tag{41}$$

Proof. The Willmore energy of deformed curve will be

$$\mathcal{W}_\epsilon = \frac{1}{2}\int_{\mathcal{I}} k_\epsilon^2\, ds_\epsilon = \frac{1}{2}\int_{\mathcal{I}} (k + \epsilon\,\delta k)^2[ds + \epsilon\,\delta(ds)], \tag{42}$$

i. e.

$$\mathcal{W}_\epsilon = \mathcal{W} + \epsilon\Big[\int_{\mathcal{I}} k\,\delta k\, ds + \frac{1}{2}\int_{\mathcal{I}} k^2\,\delta(ds)\Big]. \tag{43}$$

According to Lemma 3.3 we obtain that

$$\mathcal{W}_\epsilon = \mathcal{W} + \epsilon\int_{\mathcal{I}} k\,\delta k\, ds, \tag{44}$$

i. e.

$$\delta\mathcal{W} = \int_{\mathcal{I}} k\,\delta k\, ds, \tag{45}$$

Applying (30) we get

$$\delta\mathcal{W} = \int_{\mathcal{I}} k[k'z + z_1'' + (k^2 - \tau^2)z_1 - 2\tau z_2' - \tau' z_2]\, ds. \tag{46}$$

As it is $(kz_1')' = k'z_1' + kz_1''$, we have

$$kz_1'' = (kz_1')' - k'z_1'. \tag{47}$$

Also it is $(k'z_1)' = k''z_1 + k'z_1'$, i. e.

$$-k'z_1' = k''z_1 - (k'z_1)'. \tag{48}$$

From (47) and (48) follows the equation

$$kz_1'' = (kz_1')' + k''z_1 - (k'z_1)'. \tag{49}$$

Further, $(-2\tau z_2 k)' = -2\tau' z_2 k - 2\tau z_2' k - 2\tau z_2 k'$, or

$$-2\tau z_2' k = (-2\tau z_2 k)' + (2\tau k)' z_2. \tag{50}$$

Also, using (14), we obtain

$$kk'z = (\frac{1}{2}k^2 z)' - \frac{1}{2}k^2 z' = (\frac{1}{2}k^2 z)' - \frac{1}{2}k^3 z_1. \tag{51}$$

Substituting (49), (50) and (51) into (46) we have (41). ■

Corollary 4.2. *The Willmore energy of deformed curve under infinitesimal bending is*

$$\mathcal{W}_\epsilon = \mathcal{W} + \epsilon\Big\{\int_I ds\Big[(k'' + \frac{1}{2}k^3 - k\tau^2)z_1 + (2k'\tau + k\tau')z_2\Big] + \int_I ds\Big[\frac{1}{2}k^2 z - k'z_1 + kz_1' - 2k\tau z_2\Big]'\Big\}. \tag{52}$$

Corollary 4.3. *Under infinitesimal bending of a plane curve, variation of its Willmore energy is*

$$\delta\mathcal{W} = \int_I ds\,(k'' + \frac{1}{2}k^3)z_1 + \int_I ds\Big[\frac{1}{2}k^2 z - k'z_1 + kz_1'\Big]'. \tag{53}$$

From the equation (41) we can see that the second integral is an integral of a total derivative. Also, we get the Euler-Lagrange equations

$$k'' + \frac{1}{2}k^3 - k\tau^2 = 0, \tag{54}$$

$$2k'\tau + k\tau' = 0. \tag{55}$$

These are the governing equations of the curve that minimize the integral (41). Integrating the second one gives

$$\tau k^2 = const, \tag{56}$$

which determines τ as a function of k. We substitute τ from (56) into (54) and get a second order differential equation for k. It is clear that the Euler-Lagrange equations (54) and (55) are integrable, τ is given as a function of k, k is determined as a quadrature.

Example 4.1. Let us examine infinitesimal bending of a helix parameterized by the arc length

$$C : \mathbf{r} = (\sqrt{2}\cos\frac{s}{2}, \sqrt{2}\sin\frac{s}{2}, \frac{\sqrt{2}s}{s}), \quad s \in \mathcal{I} = [a, b]. \tag{57}$$

The vector fields of the orthonormal basis are

$$\begin{aligned}
\mathbf{t} =& \mathbf{r}' = \Big(-\frac{\sqrt{2}}{2}\sin\frac{s}{2}, \frac{\sqrt{2}}{2}\cos\frac{s}{2}, \frac{\sqrt{2}}{2}\Big),\\
\mathbf{n}_1 =& \frac{\mathbf{r}''}{\|\mathbf{r}''\|} = \Big(-\cos\frac{s}{2}, -\sin\frac{s}{2}, 0\Big),\\
\mathbf{n}_2 =& \mathbf{t}\times\mathbf{n}_1 = \Big(\frac{\sqrt{2}}{2}\sin\frac{s}{2}, -\frac{\sqrt{2}}{2}\cos\frac{s}{2}, \frac{\sqrt{2}}{2}\Big).
\end{aligned} \tag{58}$$

The curvature and the torsion are respectively

$$k = \|\mathbf{r}''\| = \frac{\sqrt{2}}{4}, \quad \tau = \frac{\sqrt{2}}{4}. \tag{59}$$

According to Theorem 2.3, for $p = q = 1$ we obtain infinitesimal bending field for C in the form

$$\mathbf{z} = \Big(-2\sin\frac{s}{2} - \sqrt{2}\cos\frac{s}{2}, 2\cos\frac{s}{2} - \sqrt{2}\sin\frac{s}{2}, \frac{\sqrt{2}s}{2}\Big).$$

Let us decompose vector field $\mathbf{z}$ through the orthonormal basis, as in the Eq. (13). We obtain

$$\mathbf{z} = \frac{s + 2\sqrt{2}}{2}\mathbf{t} + \sqrt{2}\mathbf{n}_1 + \frac{s - 2\sqrt{2}}{2}\mathbf{n}_2. \tag{60}$$

Applying Theorem 4.1 we obtain that $\delta\mathcal{W} = \frac{1}{8}(a - b)$.

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