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An Extension Of Two Fixed Point Theorems of Fisher to Partial Metric Spaces

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Abstract. In S.G. Matthews [Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 728, 1994, pp. 183-197], the author introduced and studied the concept of partial metric space, and obtained a Banach type fixed point theorem on complete partial metric spaces. The present paper forms part of the study of possible extensions of metric fixed point results to the context of partial metric spaces, in particular two theorems due to Fisher. The theory is illustrated by some examples.

1. Introduction

Amongst the many generalizations of the concept of metric spaces that can be found in literature, a relatively recently introduced one is that of partial metric spaces. The notion was introduced by Matthews ([9]) as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Ever since, fixed point results in partial metric spaces have been studied by a large number of other authors. References [1], [2] [8], [12], [18] are some works in this line of research. The existence of several connections between partial metrics and topological aspects of domain theory have been pointed in, e,g., [2], [8], [3], [10], [17], [19].

The purpose of present paper is to extend the results of [7] to the context of partial metric spaces. The theory is illustrated by some examples.

2. Preliminaries

Let us recall [9] that a mapping $p : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is said to be a *partial metric on* X if for any $x, y, z \in X$ the following four conditions hold true:

(P1) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$

(P2) $p(x, x) \le p(x, y)$

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(P3) $p(x, y) = p(y, x)$

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is then called a *partial metric space*. A sequence $\{x_m\}_{m=0}^{\infty}$ of elements of X is called *p-Cauchy* if the limit $\lim_{m,n} p(x_n, x_m)$ exists and is finite. The partial metric space (X, p) is called *complete* if for each p-Cauchy sequence $\{x_m\}_{m=0}^{\infty}$ there is some $z \in X$ such that

$$p(z, z) = \lim_{n} p(z, x_n) = \lim_{n,m} p(x_n, x_m). \tag{1}$$

If (X, p) is a partial metric space, then $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, $x, y \in X$, is a metric on X, $\{x_n\}_{n\geq 1}$ converges to $z \in X$ with respect to p^s if and only if (1) holds, and (X, p) is a complete partial metric space if and only if (X, p^s) is a complete metric space (see [9, 12]).

A sequence x_n in a partial metric space (X, p), is called 0-Cauchy ([18]) if $\lim_{m,n} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to p, to a point $x \in X$ such that $p(x, x) = 0$. Note that every 0-Cauchy sequence in (X, p) is Cauchy in (X, p^s), and that every complete partial metric space is 0-complete. A paradigm for partial metric spaces is the pair (X, p) where $X = \mathbb{Q} \cap [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \geq 0$, which provides an example of a 0-complete partial metric space which is not complete.

Bellow we give two more examples of partial metrics both of which are taken from [9].

Example 2.1. If $X := \{[a, b] |\ a, b \in \mathbb{R},\ a \leq b\}$ then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X. □

Example 2.2. Let $X := \mathbb{R}^{\mathbb{N}_0} \cup \bigcup_{n\geq 1} \mathbb{R}^{\{0,1,\ldots,n-1\}}$, where $\mathbb{N}_0$ is the set of nonnegative integers.

By $L(x)$ denote the set $\{0, 1, \ldots, n\}$ if $x \in \mathbb{R}^{\{0,1,\ldots,n-1\}}$ for some $n \in \mathbb{N}$, and the set $\mathbb{N}_0$ if $x \in \mathbb{R}^{\mathbb{N}_0}$. Then a partial metric is defined on X by

$$p(x, y) = \inf\{2^{-i} | i \in L(x) \cap L(y) \text{ and } \forall j \in \mathbb{N}_0\ (j < i \implies x(j) = y(j))\}. \quad \square$$

We proceed by recalling that quasi-contractions on metric spaces were introduced and studied by Ćirić [5] (see also [6], [16]) as one of the most general types of contractive maps. Given a metric space (X, ρ), a map $T : X \mapsto X$ such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$ there is the inequality

$$\rho(Tx, Ty) \leq \lambda \cdot \max\Big\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(y, Tx)\Big\} \tag{2}$$

is called a *quasi-contraction*. A well-known result of Ćirić says that every quasi-contraction on a complete metric space has a unique fixed point.

In 1979 Fisher [7] came up with a generalization of Ćirić's result as follows. He calls a quasi-contraction any mapping $T : X \to X$, of a metric space (X, ρ) to itself, for which there are $\alpha \in [0, 1)$ and positive integers l, q such that for each $x, y \in X$ the following condition holds

$$\rho(T^l x, T^q y) \leq \alpha \max\left\{\rho(T^i x, T^j y),\ \rho(T^i x, T^{i'} x),\ \rho(T^j y, T^{j'} y)\right\}.$$

for some $0 \leq i,\ i' \leq l$ and $0 \leq j,\ j' \leq q$. His results about such mappings then go as follows.

Theorem 2.3. ([7]) *Suppose a mapping T is given on a complete metric space. If T is either a continuous quasi-contraction or one for which $q = 1$, then it necessarily has a unique fixed point and for each point $x \in X$ the sequence $(T^n x\ :\ n \in \mathbb{N})$ converges to it.*

He also provides an example showing that in general, the continuity assumption cannot be removed from the result.

In the next section we will partially generalize Theorem 2.3 to the context of partial metric spaces, and to do that we need suitable modifications of the notions of both a quasi-contraction and of continuity.

Definition 2.4. *Let (X, p) be a partial metric space and $T : X \to X$ a given mapping. If there are $\alpha \in [0, 1)$ and positive integers l, q such that for each $x, y \in X$ the following condition holds*

$$p(T^l x, T^q y) \leq \alpha p(T^i x, T^j y) \tag{3}$$

for some $0 \leq i \leq l$, $0 \leq j \leq q$, then we shall say that T is a Fisher quasicontraction. If $q = 1$, in the preceding definition, then we will call T a strict Fisher quasicontraction.

Definition 2.5. *A mapping $T : X_1 \to X_2$ is said to be continuous if it is a continuous mapping from the metric space $\left(X_1, p_1^s\right)$ to the metric space $\left(X_2, p_2^s\right)$, which amounts to saying that*

$$p(a, a) = \lim_n p(x_n, a) = \lim_{n,m} p(x_n, x_m)$$

implies

$$p(Ta, Ta) = \lim_n p(Tx_n, Ta) = \lim_{n,m} p(Tx_n, Tx_m)\,.$$

It is said to be p-metrically continuous if $p_1(x_n, a) \to p(a, a)$ implies $p_2(Tx_n, Ta) \to p(Ta, Ta)$ (i.e. if it is continuous with respect to the topologies that the two partial metrics induce – see [9]). □

The following example shows that neither of the notions of (metric) continuity and p-metric continuity implies the other.

Example Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$. It is easy to see that every p-metrically continuous mapping must be monotonically nondecreasing. Thus any continuous mapping $T : X \to X$ that fails to be monotonically nondecreasing is automatically not p-metrically continuous (take e.g. $Tx = 1 - x$). The mapping $S : X \to X$ defined by

$$Sx = \begin{cases} x/2 & \text{, if } 0 \leq x < 1/2 \\ x & \text{, if } 1/2 \leq x \leq 1 \end{cases}$$

is p-metrically continuous but not continuous. □

3. The results

Lemma 3.1. *If T is a Fisher quasicontraction and $x \in X$, then* $\sup\left\{p(T^i x, T^j x) \,:\, i, j \geq 0\right\} < \infty$.

Proof. Set $r := \max\{l, q\}$. Replacing α by $\max\{1/2, \alpha\}$ if needed, we may assume without loss of generality that $1/2 \leq \alpha$, so that $1 \leq \dfrac{\alpha}{1-\alpha}$. To prove the assertion it suffices to show that for each nonnegative integer i there is the inequality

$$p(T^r x, T^i x) \leq \frac{\alpha}{1-\alpha} \max\left\{p(T^s x, T^r x) \,:\, 0 \leq s \leq r\right\}. \tag{4}$$

The proof of this is by induction on i. Set $M_x := \max\left\{p(T^s x, T^r x) \,:\, 0 \leq s \leq r\right\}$. For $0 \leq i \leq r$ the inequality (4) follows from the fact that $1 > \alpha \geq 1/2$. Suppose now that (4) holds for all $0 \leq i < k$ for some $k > r$ and suppose, to the contrary, that the inequality fails for $i = k$. Then we must have $p(T^r x, T^k x) > \dfrac{\alpha}{1-\alpha} M_x \geq p(T^r x, T^i x)$ for all $0 \leq i < k$.

Suppose that $0 \le i, j \le k$ are such that $p(T^r x, T^k x) \le \alpha p(T^j x, T^i x)$. Assume that $j \le r$. Then $p(T^r x, T^k x) \le \alpha p(T^j x, T^r x) + \alpha p(T^r x, T^i x) \le \alpha M_x + \alpha p(T^r x, T^k x)$ i.e. $p(T^r x, T^k x) \le \frac{\alpha}{1-\alpha} M_x$, which is not possible.

Thus if $0 \le i, j \le k$ are such that $p(T^r x, T^k x) \le \alpha p(T^j x, T^i x)$, then we must $i, j > r$. Using this and (3) repeatedly we can find sequences of positive integers $(j_n : n \in \mathbb{N})$ and $(i_n : n \in \mathbb{N})$ with $r < i_n \le k$ and $r < j_n \le k$ such that $p(T^r x, T^k x) \le \alpha^n p(T^{j_n} x, T^{i_n} x) \le \alpha^n \max\left\{p(T^s x, T^l x) : 0 \le s, l \le k\right\}$, leading to $p(T^r x, T^k x) = 0$ – a contradiction. □

Now we can state and prove our first result.

Theorem 3.2. *Suppose (X, p) is a 0-complete partial metric space and $T : X \to X$ is either a continuous or a p-metrically continuous Fisher quasicontraction. Then there is a unique fixed point $a \in X$ of the mapping T. Furthermore $p(a, a) = 0$ and for each $x \in X$ the sequence $(T^n x : n \in \mathbb{N})$ converges to the point a with respect to the metric p^s.*

Proof. By Lemma 3.1 we can choose some $S > 0$ such that $p(T^n x, T^m x) \le S$ for all $n, m \in \mathbb{N}$. We prove that $\lim_{n,m} p(T^n x, T^m x) = 0$.

Given $\epsilon > 0$ let $k \in \mathbb{N}$ be such that $\alpha^k S < \epsilon$. Set $r := \max\{p, q\}$ and let $n, m > kr$ be arbitrary. Applying (3) consecutively k many times we get that $p(T^n x, T^m x) \le \alpha^k p(T^i x, T^j x)$, for some $i \ge n - kr$ and $j \ge m - kr$, hence $p(T^n x, T^m x) \le \alpha^k S < \epsilon$.

Now, (X, p) is 0-complete, so there must be some $a \in X$ with

$$p(a, a) = \lim_n p(T^n x, a) = \lim_{n,m} p(T^n x, T^m x) = 0\,. \tag{5}$$

If T is continuous then from $\lim_n p^s(T^n x, a) = 0$ it follows $\lim_n p^s(T^n x, Ta) = 0$ and we immediately have $a = Ta$.

Suppose now that T is p-metrically continuous. From (5) it follows that $\lim_n p(T^n x, Ta) = p(Ta, Ta)$. Proceeding inductively, we conclude that

$$\lim_n p(T^n x, T^i a) = p(T^i a, T^i a) \quad \text{for all } i \ge 0\,. \tag{6}$$

For any $i \ge 0$ we have $p(a, T^i a) \le p(a, T^n x) + p(T^n x, T^i a)$, whence using (5) and (6) we obtain

$$p(a, T^i a) = p(T^i a, T^i a) \quad \text{for all } i \ge 0\,. \tag{7}$$

Fix an $i \ge 0$. (7) means that $\lim_n p(y_n, T^i a) = p(T^i a, T^i a)$ where $y_n = a$ for every $n \in \mathbb{N}$. By p-metric continuity of T we must therefore have $\lim_n p(Ty_n, T^{i+1} a) = p(T^{i+1} a, T^{i+1} a)$, i.e.

$$p(Ta, T^{i+1} a) = p(T^{i+1} a, T^{i+1} a) \tag{8}$$

Since (8) holds for any $i \ge 0$, we can now use (6) and the fact that $\lim_{n,m} p(T^n a, T^m a) = 0$ (note that when proving (5) we did not impose any particular assumptions on the point $x \in X$) to conclude $p(Ta, Ta) = 0$. But by (7) this means that $p(a, Ta) = 0$, i.e. $a = Ta$.

That a is the unique fixed point of T follows in the usual way: if $Tb = b$ then $p(a, b) = p(T^l a, T^q b) \le \alpha p(T^i a, T^j b) = \alpha p(a, b)$ for some $i \le l$, and $j \le q$, so $p(a, b) = 0$ and $a = b$. □

As shown in the example below, the preceding theorem fails to hold even for complete partial metric spaces, if the assumption of p-metric continuity is weakened to that of 0-partial continuity, where $T : X \to X$ is said to be 0-partially continuous if whenever it happens that $\lim_n p(x_n, a) = p(a, a) = 0$, then it must also be that $\lim_n p(Tx_n, Ta) = p(Ta, Ta)$.

Example Let $X = [0,1] \cup \{2\}$ and let p be given by

$$p(x,y) = \begin{cases} |x-y|, & \text{if } 2 \notin \{x,y\} \\ 1, & \text{if } 2 \in \{x,y\}. \end{cases}$$

Then (X,p) is a partial metric space as can easily be verified. To see that it is a complete one suppose that $\lim_{n,m} p(x_n, x_m) = r$. As $p(x_n, x_m) \in [0,1]$ for all n, m, we have $r \in [0,1]$.

If $r < 1$ then there is some n_0 such that $p(x_n, x_n) < 1$ for all $n \geq n_0$. Thus $x_n \in [0,1]$ for $n \geq n_0$. Consequently $p(x_n, x_m) = |x_n - x_m|$ for $n, m \geq n_0$ and thus $r = \lim_{n,m} p(x_n, x_n) = 0$, so the sequence is a Cauchy one with respect to the usual Euclidean metric of the segment $[0,1]$. This in turn implies that there is some $a \in [0,1]$ such that $0 = \lim_n |x_n - a| = \lim_n p(x_n, a)$.

If on the other hand $r = 1$, then there is some n_0 such that $p(x_n, x_n) > 0$ for all $n \geq n_0$. So $x_n = 2$ for all $n \geq n_0$ and thus $r = 1$ and also $\lim_n p(x_n, 2) = 1 = p(2,2)$.

Now we define a mapping $T : X \to X$ by

$$Tx = \begin{cases} x/2, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0. \end{cases}$$

We see that $T^i x \in (0,1]$ for all $x \in X$ and all $i \geq 2$. So $T^{i+1}x = \frac{1}{2}T^i x$ and $p(T^i x, T^i y) = |T^i x - T^i y|$ if $i \geq 2$ and hence

$$p(T^3x, T^3y) = |T^3x - T^3y| = \frac{1}{2}|T^2x - T^2y| = \frac{1}{2}p(T^2x, T^2y).$$

So T is a Fisher quasicontraction with no fixed points.

However, T is 0-partially continuous with respect to p. To see this let $\lim_n p(x_n, a) = p(a,a) = 0$. Clearly $a \neq 2$. We distinguish two cases:

Case 1 $a = 0$. Here we have $\lim_n p(Tx_n, Ta) = 1 = p(Ta, Ta)$.

Case 2 $a \in (0,1]$. There is some n_0 such that $p(x_n, a) < 1$ for all $n \geq n_0$. So $x_n \in [0,1]$ for $n \geq n_0$ and thus $p(x_n, a) = |x_n - a|$ for such n. It follows that $\lim_n |x_n - a| = 0$. As $a \in (0,1]$ this means that there is some n_1 such that $x_n \in (0,1]$ and thus $Tx_n = \frac{1}{2}x_n \in [0,1]$ for all $n \geq n_1$. Also $Ta = \frac{a}{2}$ and hence $\lim_n p(Tx_n, Ta) = \lim_n |Tx_n - Ta| = \frac{1}{2}\lim_n |x_n - a| = 0 = p(Ta, Ta)$.

Of course, T is not p-metrically continuous since it has no fixed points (or simply look at the sequence $x_n = 1/n$ and the point $a = 2$). □

Theorem 3.3. *Suppose (X,p) is a 0-complete partial metric space and $T : X \to X$ is a strict Fisher quasicontraction. Then there is a unique fixed point $a \in T$ of the mapping T. Furthermore $p(a,a) = 0$ and for each $x \in X$ the sequence $(T^n x : n \in \mathbb{N})$ converges to the point a with respect to the metric p^s.*

Proof. Given $x \in X$ we can prove, the same way we did in Theorem 3.2, that there is a point $a \in X$ for which (5) holds true. By Lemma 3.1 there is some $S > 0$ such that $S \geq p(Ta, T^n x)$ for all $n \in \mathbb{N}$. Fix a positive integer $m \in \mathbb{N}$ with $\alpha^m S < \epsilon/2$ and then one $k \geq ml$ such that $p(a, T^n x) < \epsilon/2$ for all $n \geq k - ml$. We have $p(Ta, a) \leq p(Ta, T^k x) + p(T^k x, a)$. Applying (3) with $q = 1$ at most m many times starting with the pair $p(T^k x, Ta)$, we obtain that for some $i \geq k - ml$ it must be

$$p(Ta, T^k x) \leq \max\left\{\alpha^m p(Ta, T^i x),\ \alpha p(a, T^i x)\right\}$$

and thus $p(Ta, a) \leq p(T^k x, a) + \max\{\alpha^m S, p(a, T^i x)\} < \epsilon$. This proves that $p(Ta, a) = 0$, i.e. $a = Ta$. As before, the fixed point of T is readily seen to be unique. □

The potentially nonzero self-distance, that is built into Matthew's definition of partial metrics, was taken into account in [13] in an essential way by a rather mild variation of the classical Banach's contractive condition, and in [14] further considerations in this direction were carried out which were in turn generalized by Chi et al. in [4]. As an example of that type of extensions of metric fixed point results to partial metric spaces we state the following theorem.

Theorem 3.4. *([13]) Suppose (X, p) is a complete partial metric space and $T : X \to X$ is a mapping such that there is such that for some $\alpha \in [0, 1)$ there is inequality*

$$p(Tx, Ty) \leq \max\{\alpha p(x, y), p(x, x), p(y, y)\} \tag{9}$$

for all $x, y \in X$, then T must have a fixed point.

We end the paper with a simple example showing that one possible attempt of a generalization of this sort of Theorem 2.3 is doomed to fail.

Example Let $X = \{1, 2, 3\}$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$. Then (X, p) is a complete partial metric space. Define a mapping $T : X \to X$ by setting $T1 = 2$, $T2 = 3$ and $T3 = 1$. It is trivial to see that the inequality

$$p(T^3x, T^3y) \leq \max\{\alpha p(x, y), p(x, x), p(y, y)\}$$

holds true for all $x, y \in X$. T is a continuous mapping without fixed points. □

However, the mapping T in the preceding example was not p-metrically continuous. This leaves us with the following two questions.

Suppose that (X, p) is a complete partial metric space and $T : X \to X$ a given mapping. Suppose further that there are $\alpha \in [0, 1)$ and positive integers l, q such that for each $x, y \in X$ the following condition holds

$$p(T^l x, T^q y) \leq \max\left\{\alpha p(T^i x, T^j y),\ p(x, x),\ p(y, y)\right\}$$

for some $0 \leq i \leq l$, $0 \leq j \leq q$. If T is p-metrically continuous, must there be a fixed point of T? The same question can be asked in the case when $q = 1$ regardless of any assumptions on continuity (we note that this indeed is the case if also $l = 1$, as shown in [15]).

References

[1] I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology Appl. 157:18 (2010), 2778-2785.

[2] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, Partial metric spaces, American Mathematical Monthly 116 (2009), 708-718.

[3] M. A. Bukatin, S. Yu. Shorina, Partial metrics and co-continuous valuations, in: Foundations of software science and computation structure, Lecture Notes in Computer Science 1378, eds. M. Nivat et.al., pp. 125-139, Springer, 1998.

[4] K.P. Chi, E. Karapinar, T.D. Thanh, A generalized contraction principle in partial metric spaces, Mathematical and Computer Modelling 55 (2012) 1673-1681.

[5] Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974) 267-273.

[6] Lj. B. Ćirić, Some recent results in metrical fixed point theory, University of Belgrade, Beograd, 2003.

[7] B. Fisher, Quasi-contrtactions on metric spaces, Proc. Amer. Math. Soc. 25:2 (1979) 321-325.

[8] A. F. Rabarison, Partial metrics, Supervised by Hans-Peter A. Künzi, African Institute for Mathematical Sciences, 2007.

[9] S.G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 728, 1994, pp. 183-197.

[10] S.G. Matthews, An extensional treatment of lazy data flow deadlock, Theoret. Comput. Sci. 151 (1995) 195-205.

[11] S. B. Nadler Jr., Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475-487.

[12] S. Oltra, O. Valero, Banach's fixed theorem for partial metric spaces, Rend. Istit. Mat. Univ. Trieste 36 (2004) 17-26.

[13] D. Ilić, V. Pavlović, V. Rakočević, Some new extensions of Banach's contraction principle to partial metric space, Applied Mathematics Letters 24 (2011) 1326-1330.

[14] D. Ilić, V. Pavlović, V. Rakočević, Extensions of the Zamfirescu theorem to partial metric spaces, Mathematical and Computer Modelling 55 (3-4) (2012) 801-809.

[15] D. Ilić, V. Pavlović, V. Rakočević, Three extensions of Ćirić quasicontraction on partial metric spaces, Fixed Point Theory and Applications 2013, 2013:303

[16] B. E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977) 257-290.
[17] S. Romaguera, M. Schellekens, Duality and quasi-normability for complexity spaces, Appl. General Topology 3 (2002) 91-112.
[18] S. Romaguera, A Kirk Type Characterization of Completeness for Partial Metric Spaces, Fixed Point Theory and Applications Volume 2010, Article ID 493298, 6 pages doi:10.1155/2010/493298.
[19] M. P. Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci. 315 (2004) 135-149.