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Family of Simultaneous Methods with Corrections for Approximating Zeros of Analytic Functions

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Abstract. A family of accelerated iterative methods for the simultaneous approximation of complex zeros of a class of analytic functions is proposed. Considered analytic functions have only simple zeros inside a simple smooth closed contour in the complex plane. It is shown that the order of convergence of the basic family can be increased from four to five and six using Newton's and Halley's corrections, respectively. The improved convergence is achieved on the account of additional calculations of low computational cost, which significantly increases the computational efficiency of the accelerated methods. Numerical examples demonstrate a good convergence properties, fitting very well theoretical results.

1. Introduction

Finding all zeros of analytic functions is a very important problem in numerical analysis and applied scientific disciplines. In this paper we consider a class of analytic functions having only simple zeros inside a simple smooth closed contour in the complex plane. The aim of this paper is to present a new family of accelerated iterative methods for the simultaneous computation of zeros of an analytic function from the considered class. Also, the convergence properties of the presented family are studied. Good convergent properties are illustrated by two numerical examples.

The derivation of the basic family is proposed in [10]. It is based on suitable combination of appropriate analytic function with a cubically convergent iterative method for finding a single zero of an analytic function f, proposed by Gutiérez and Hernández [4],

$$\hat{x} = x - \frac{f(x)}{f'(x)}\Big(1 + \frac{1}{s(x) - \alpha}\Big). \tag{1}$$

Here α is a real parameter, x is a current approximation, $\hat{x}$ is a new approximation to the wanted zero and

$$s(x) = \frac{2f'(x)^2}{f(x)f''(x)}.$$

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The family (1) includes various methods with a cubic convergence when α is finite: Halley's method ($\alpha = 1$) and Chebyshev-Euler's method ($\alpha = 0$), for example. As a limiting case, when $|\alpha| \to \infty$, the family (1) behaves as quadratically convergent Newton's method

$$x = x - \frac{f(x)}{f'(x)}.$$

For this reason, it is preferable to avoid the choice of large parameter α in (1).

2. Derivation of the accelerated families

Let $z \mapsto \Phi(z)$ be an analytic function inside and on the simple smooth closed contour Γ, without zeros on Γ and with a known number n of simple zeros $\zeta_1, \ldots, \zeta_n$ inside Γ. We note that there exist several reliable approaches for computing the number of zeros of an analytic function inside a given smooth closed contour in the complex plane (see [2], [5], [7], [8], for instance). The number of zeros n of Φ iside Γ may be determined by the *argument principle* (see, e.g., [5])

$$n = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi'(w)}{\Phi(w)}\, dw = \frac{1}{2\pi}\Big[\arg \Phi(\omega)\Big]_{\omega \in \Gamma} = n\Big(\Phi(\Gamma), 0\Big)\,. \tag{2}$$

$\Phi(\Gamma)$ denotes the image of the curve Γ under the mapping Φ. The integer $n\Big(\Phi(\Gamma), 0\Big)$ is the so-called *winding number* of $\Phi(\Gamma)$ with respect to the origin and it is equal to the number of times that the curve $\Phi(\Gamma)$ "winds" itself around the origin.

Following Smirnov [12] and Iokimidis and Anastasselou [6] Φ can be represented in the form of product

$$\Phi(z) = \exp\Big(\Psi(z)\Big) \prod_{j=1}^{n} (z - \zeta_j), \tag{3}$$

where Ψ is an analytic function inside Γ given by

$$\Psi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\log\Big[(w-\eta)^{-n}\Phi(w)\Big]}{w - z}\, dw \tag{4}$$

and η is an arbitrary point inside Γ.

In order to increase computational efficiency of the basic method proposed in [10], we state some modifications. It is obvious that the zeros of analytic function Φ inside Γ coincide with the zeros of function

$$V_i(z) = \frac{\Phi(z)}{\exp\Big(\Psi(z)\Big) \prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (z - v_j)}, \tag{5}$$

where $v_1, \ldots, v_n$ are some approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of Φ. The computational cost could be increased if the approximations v_j use the already calculated values $\Phi(z_j), \Phi'(z_j)$ and $\Phi''(z_j)$ in a suitable way. We consider the following approximations $(v_1, \ldots, v_n) = (z_1^{(k)}, \ldots, z_n^{(k)})$, $k = 1, 2, 3$, where

$$\begin{aligned} z_j^{(1)} &= z_j \quad \text{(current approximation)},\\ z_j^{(2)} &= z_j - N_j \quad \text{(Newton's approximation)},\\ z_j^{(3)} &= z_j - H_j \quad \text{(Halley's approximation)}. \end{aligned} \tag{6}$$

Newton's and Halley's approximations occur in the classic iterative methods

$$\hat{z}_j = z_j - N_j = z_j - \frac{\Phi(z_j)}{\Phi'(z_j)} \quad \text{(Newton's method, order 2)},$$

$$\hat{z}_j = z_j - H_j = z_j - \left(\frac{\Phi'(z_j)}{\Phi(z_j)} - \frac{\Phi''(z_j)}{2\Phi'(z_j)}\right)^{-1} \quad \text{(Halley's method, order 3)}.$$

In the sequel, for $i \in I_n := \{1, \dots, n\}$, we will use the abbreviations

$$\delta_{q,i} = \frac{\Phi^{(q)}(z_i)}{\Phi(z_i)}, \qquad S_{q,i}^{(k)} = \sum_{j \in I_n \setminus \{i\}} \frac{1}{\left(z_i - z_j^{(k)}\right)^q}, \qquad \Sigma_{q,i} = \sum_{j \in I_n \setminus \{i\}} \frac{1}{\left(z_i - \zeta_j\right)^q}, \qquad (q = 1, 2,\ k = 1, 2, 3), \tag{7}$$

where $z_j^{(k)}$ is defined by (6).

For simplicity, we will write $\Psi_i' = \Psi'(z_i)$, $\Psi_i'' = \Psi''(z_i)$. If we use the logarithmic derivative in (5), according to introduced abbreviations (7), we find

$$\left.\frac{\left(V_i(z)\right)'}{V_i(z)}\right|_{z=z_i} = \delta_{1,i} - S_{1,i}^{(k)} - \Psi_i' = T_{i,k}, \tag{8}$$

$$\left.\frac{\left(V_i(z)\right)''}{\left(V_i(z)\right)'}\right|_{z=z_i} = \delta_{1,i} - S_{1,i}^{(k)} - \Psi_i' - \frac{\delta_{1,i}^2 - \delta_{2,i} - S_{2,i}^{(k)} + \Psi_i''}{\delta_{1,i} - S_{1,i}^{(k)} - \Psi_i'} = T_{i,k} - \frac{H_{i,k}}{T_{i,k}}, \tag{9}$$

where we set

$$T_{i,k} = \delta_{1,i} - S_{1,i}^{(k)} - \Psi_i', \qquad H_{i,k} = \delta_{1,i}^2 - \delta_{2,i} - S_{2,i}^{(k)} + \Psi_i''. \tag{10}$$

Taking the iterative formula (1) with the function V instead of Φ and substituting $V'(z)/V(z)$ and $V''(z)/V'(z)$ evaluated at the point $z = z_i$ and given by (8) and (9), we construct the following one parameter family of iterative methods for finding, simultaneously, simple zeros of the analytic function Φ inside the contour Γ,

$$\hat{z}_i = z_i - \frac{1}{T_{i,k}}\left(1 + \frac{\left(T_{i,k}\right)^2 - H_{i,k}}{2\left(T_{i,k}\right)^2 - \alpha\left(\left(T_{i,k}\right)^2 - H_{i,k}\right)}\right) \qquad (i \in I_n,\ k = 1, 2, 3). \tag{11}$$

Introducing the iteration index m and assuming that $T_{i,k}^{(m)}$ and $H_{i,k}^{(m)}$ involve quantities obtained in the m-th iteration, the last iterative formula gets the form

$$z_i^{(m+1)} = z_i^{(m)} - \frac{1}{T_{i,k}^{(m)}}\left(1 + \frac{\left(T_{i,k}^{(m)}\right)^2 - H_{i,k}^{(m)}}{2\left(T_{i,k}^{(m)}\right)^2 - \alpha\left(\left(T_{i,k}^{(m)}\right)^2 - H_{i,k}^{(m)}\right)}\right) \qquad (i \in I_n,\ k = 1, 2, 3,\ m = 0, 1, \dots). \tag{12}$$

For simplicity, we will often omit the iteration index in our analysis.

In the particular case when $\alpha = 1$ from the iterative formula (11) we obtain Halley-like method

$$\hat{z}_i = z_i - \frac{2T_{i,k}}{\left(T_{i,k}\right)^2 + H_{i,k}} \qquad (i \in I_n)$$

considered for $k = 1$ (basic method) in [9]. If $\alpha \to +\infty$, the basic method in (11) reduces to the iterative method

$$\hat{z}_i = z_i - \frac{1}{\dfrac{\Phi'(z_i)}{\Phi(z_i)} - \Psi'(z_i) - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)^{-1}} \qquad (i \in I_n)$$

of the third order, considered by Iokimidis and Anastasselou in [6].

3. Convergence analysis

The subject of the following theorem is the order of convergence of the proposed family (12):

Theorem 3.1. *If the initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ are sufficiently close to the respective zeros $\zeta_1, \ldots, \zeta_n$ of Φ, then the family of iterative methods* (12) *has the order of convergence equal to* $k+3$ ($k = 1, 2, 3$).

Proof. We omit iteration index for short and use the iterative formula (11) instead of (12). For Newton's and Halley's corrections we have

$$N_j = \frac{\Phi(z_j)}{\Phi'(z_j)} = \frac{\varepsilon_j}{1 + \varepsilon_j(\Psi'_j + \Sigma_{1,j})}\,, \tag{13}$$

$$H_j = \left(\frac{\Phi'(z_j)}{\Phi(z_j)} - \frac{\Phi''(z_j)}{2\Phi'(z_j)}\right)^{-1} = \frac{2\varepsilon_j\Big(1 + \varepsilon_j(\Psi'_j + \Sigma_{1,j})\Big)}{2 + 2\varepsilon_j(\Psi'_j + \Sigma_{1,j}) + \varepsilon_j^2\Big((\Psi'_j + \Sigma_{1,j})^2 + \Sigma_{2,j} - \Psi''_j\Big)}\,. \tag{14}$$

According to (6), (13) and (14), we find the errors $z_j^{(k)} - \zeta_j$, for $k = 1, 2, 3$

$$\begin{aligned}
z_j^{(1)} - \zeta_j &= z_i - \zeta_j = \varepsilon_j = \omega_j^{(1)}\varepsilon_j,\\
z_j^{(2)} - \zeta_j &= z_j - N_j - \zeta_j = \frac{\varepsilon_j^2(\Psi'_j + \Sigma_{1,j})}{1 + \varepsilon_j(\Psi'_j + \Sigma_{1,j})} = \omega_j^{(2)}\varepsilon_j^2,\\
z_j^{(3)} - \zeta_j &= z_j - H_j - \zeta_j\\
&= \frac{\varepsilon_j^3\Big((\Psi'_j + \Sigma_{1,j})^2 + \Sigma_{2,j} - \Psi''_j\Big)}{2 + 2\varepsilon_j(\Psi'_j + \Sigma_{1,j}) + \varepsilon_j^2\Big((\Psi'_j + \Sigma_{1,j})^2 + \Sigma_{2,j} - \Psi''_j\Big)} = \omega_j^{(3)}\varepsilon_j^3.
\end{aligned}$$

These errors can be written in the unique form

$$z_j^{(k)} - \zeta_j = \omega_j^{(k)}\varepsilon_j^k \qquad (k = 1, 2, 3).$$

Let us introduce the notation

$$A_{i,k} = S_{1,i}^{(k)} - \Sigma_{1,i} = \sum_{j \in I_n \setminus \{i\}} \frac{\varepsilon_j^k \omega_j^{(k)}}{(z_i - z_j^{(k)})(z_i - \zeta_j)}, \tag{15}$$

and

$$B_{i,k} = S_{2,i}^{(k)} - \Sigma_{2,i} = \sum_{j \in I_n \setminus \{i\}} \frac{\varepsilon_j^k (2z_i - z_j^{(k)} - \zeta_j)\omega_j^{(k)}}{(z_i - z_j^{(k)})^2 (z_i - \zeta_j)^2}. \tag{16}$$

Using the logarithmic derivative in (3) and the introduced notation (7) we obtain

$$\begin{aligned}
\delta_{1,i} &= \frac{\Phi'(z_i)}{\Phi(z_i)} = \frac{1}{\varepsilon_i} + \Sigma_{1,i} + \Psi'_i,\\
\delta_{1,i}^2 - \delta_{2,i} &= -\frac{d}{dz}\frac{\Phi'(z)}{\Phi(z)}\Big|_{z=z_i} = \frac{\Phi'(z_i)^2 - \Phi''(z_i)\Phi(z_i)}{\Phi(z_i)^2} = \frac{1}{\varepsilon_i^2} + \Sigma_{2,i} - \Psi''_i.
\end{aligned}$$

Regarding (10) we estimate

$$T_{i,k} = \delta_{1,i} - S_{1,i}^{(k)} - \Psi_i' = \frac{1}{\varepsilon_i} + \Sigma_{1,i} - S_{1,i}^{(k)} = \frac{1}{\varepsilon_i} - A_{i,k}, \tag{17}$$

$$H_{i,k} = \delta_{1,i}^2 - \delta_{2,i} - S_{2,i}^{(k)} + \Psi_i'' = \frac{1}{\varepsilon_i^2} + \Sigma_{2,i} - S_{2,i}^{(k)} = \frac{1}{\varepsilon_i^2} - B_{i,k}. \tag{18}$$

Starting from the family of iterative methods (11), from (17) and (18) we have

$$\begin{aligned}
\hat{\varepsilon}_i &= \hat{z}_i - \zeta_i = \varepsilon_i - \frac{1}{T_{i,k}}\left(1 + \frac{(T_{i,k})^2 - H_{i,k}}{2(T_{i,k})^2 - \alpha\Big((T_{i,k})^2 - H_{i,k}\Big)}\right) \\
&= \varepsilon_i - \frac{\varepsilon_i}{1 - \varepsilon_i A_{i,k}}\left(1 + \frac{\Big(1/\varepsilon_i - A_{i,k}\Big)^2 + B_{i,k} - 1/\varepsilon_i^2}{2\Big(1/\varepsilon_i - A_{i,k}\Big)^2 - \alpha\Big(\Big(1/\varepsilon_i - A_{i,k}\Big)^2 + B_{i,k} - 1/\varepsilon_i^2\Big)}\right) \\
&= \varepsilon_i - \frac{\varepsilon_i}{1 - \varepsilon_i A_i^{(k)}}\left(1 + \frac{\varepsilon_i^2\Big((A_i^{(k)})^2 + B_i^{(k)}\Big) - 2\varepsilon_i A_i^{(k)}}{2 + 2(\alpha - 2)\varepsilon_i A_i^{(k)} - \varepsilon_i^2\Big((\alpha - 2)(A_i^{(k)})^2 + \alpha B_i^{(k)}\Big)}\right) \\
&= \frac{\varepsilon_i^3\Big((A_{i,k})^2(3 - 2\alpha) - B_{i,k} + \varepsilon_i\Big((\alpha - 2)(A_{i,k})^3 + \alpha A_{i,k} B_{i,k}\Big)\Big)}{2 + 2(\alpha - 2)\varepsilon_i A_{i,k} - \varepsilon_i^2\Big((\alpha - 2)(A_{i,k})^2 + \alpha B_{i,k}\Big)}.
\end{aligned} \tag{19}$$

According to the assumptions of the theorem, the approximations $z_1, \ldots, z_n$ are good enough, so the quantity $\varepsilon = \max_{1 \le j \le n} |\varepsilon_j|$ is sufficiently small. The denominator of the last expression is bounded and tends to 2 when $|\varepsilon| \to 0$. From (15) and (16) we have the estimations

$$A_{i,k} = O(\varepsilon^k), \quad B_{i,k} = O(\varepsilon^k). \tag{20}$$

Taking the relations (19) and (20) we find that

$$\hat{\varepsilon} = O(\varepsilon^{k+3}),$$

which completes the proof. □

To implement the iterative formula (11) it is necessary to calculate the derivatives $\Psi'(z)$ and $\Psi''(z)$ at the point z_i, $i \in I_n$. Starting from (4) we find

$$\Psi'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\log\left[(w - \eta)^{-n}\Phi(w)\right]}{(w - z)^2}\, dw. \tag{21}$$

Without loss of generality, we can take $\eta = 0$ in (21). Applying an integration by parts, from (21) we obtain

$$\Psi'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi'(w)}{\Phi(w)}\frac{dw}{w - z}, \tag{22}$$

(see [12]), wherefrom

$$\Psi''(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi'(w)}{\Phi(w)}\frac{dw}{(w - z)^2}. \tag{23}$$

The integral in (2) (the number of zeros) and the integrals in (22) and (23) have the similar form and we can calculate them by applying a numerical integration. We can use a convenient sufficiently accurate quadrature rule for contours of the form

$$\frac{1}{2\pi i}\int_\Gamma f(w)dw \cong \sum_{k=1}^{p} A_{kp} f(w_{kp})$$

based on the trapezoidal rule or on the orthogonal polynomials. A_{kp} are the weights and w_{kp} are the corresponding nodes of quadrature formula. The same values $\Phi(w_{kp})$ and $\Phi'(w_{kp})$ may be used in all three integrals. We refer the reader to [1], [2], [3], [6] and the references cited therein for details.

In [11] we discussed the influence of the error of numerical integration on the accuracy of the produced approximations. The influence of the increased number of iterative steps is very small. The same discussion can be performed for the iterative methods (11). The conclusion is that even rough approximations to $\Psi'(z_i)$ and $\Psi''(z_i)$ can provide a good approximations to the wanted zeros of Φ.

It should be noted that all zero-finding methods applied to the considered class of analytic functions are not effective if the sought zeros are very close to the contour Γ.

4. Numerical examples

To demonstrate the convergence speed of the proposed simultaneous methods (12), we tested a lot of analytic functions. In this section we give some selected examples chosen among many numerical experiments. The proposed simultaneous methods with Newton's and Halley's corrections use the already calculated values Φ, Φ', Φ'' at the points $z_1, \ldots, z_n$ so that the convergence speed of the implemented iterative methods is accelerated with the negligible number of additional operations. In this manner a very high computational efficiency of the proposed methods is provided, which is the main advantage of the presented methods. In order to save all significant digits of the obtained approximations, we implemented the considered methods using the programming package *Mathematica* 8 with multiple precision arithmetic.

The performed numerical experiments demonstrated very fast convergence of the modified methods for finding simple zeros of analytic function Φ. For illustration we present two numerical examples. As a measure of closeness of approximations with regard to the exact zeros, we have calculated Euclid's norm

$$e_m := \left(\sum_{i=1}^{n} \left| z_i^{(m)} - \zeta_i \right|^2 \right)^{1/2}. \tag{24}$$

Example 1. We applied the proposed family of iterative methods (12) obtained for $\alpha = 0$, $\alpha = 1$, $\alpha = -1$ and $\alpha = 100$, for the simultaneous approximation to the simple zeros of the analytic function

$$\Phi(z) = z(z-1)(z-2)(z-3)(z-4) + \cos z - 1$$

inside the contour $\Gamma = \{z \in \mathbb{C} : |z| = 5\}$.

The number n of zeros of Φ inside Γ and the values of $\Psi'(z_i)$ and $\Psi''(z_i)$ were calculated by the trapezoidal quadrature rule along the circle $\Gamma = \{z \ : \ |z| = 5\}$. The details of such kind of approximate integration may be found in [2], [6] and [9]. We found by this approach that the number of zeros in the disk $\{z \ : \ |z| < 5\}$ is $n = 5$. The number of zeros can be also determined by the winding number $n(\Phi(\Gamma), 0)$ (see Fig.1).

The following initial approximations

$$z_1^{(0)} = 0.3 - 0.3\,i,\ \ z_2^{(0)} = 1. + 0.1\,i,\ \ z_3^{(0)} = 2.4 + 0.4\,i,\ \ z_4^{(0)} = 2.4 - 0.4\,i,\ \ z_5^{(0)} = 4. - 0.6\,i,$$

were employed in the realization of the iterative methods (12). In each iteration we calculated the accuracy of approximations $z_i^{(m)}$ by the Euclidean norm (24). For the above initial approximations we have $e_0 = 0.790$.

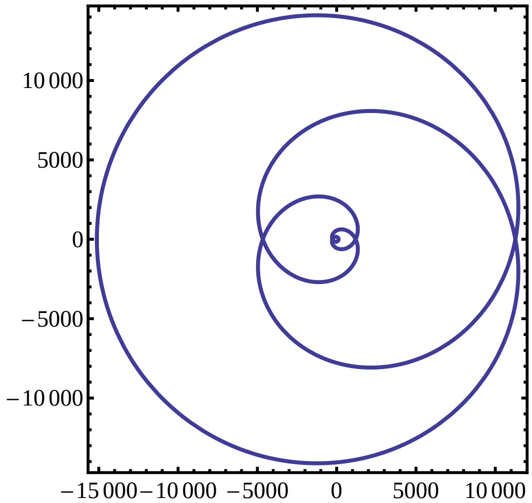

FIG. 1 The curve $\Phi(\Gamma)$ where $\Phi(z) = z(z-1)(z-2)(z-3)(z-4) + \cos z - 1$ and $\Gamma = \{z \in \mathbb{C} : |z| = 5\}$

The results of the first three iterations are given in Table 1, where $A(-h)$ means $A \times 10^{-h}$.

Methods		$k = 1$ (basic method)	$k = 2$ (Newton's corr.)	$k = 3$ (Halley's corr.)
$\alpha = 0$	e_1	3.26(−2)	4.46(−3)	4.50(−3)
	e_2	2.84(−8)	5.28(−14)	4.29(−17)
	e_3	5.48(−33)	2.75(−68)	3.76(−100)
$\alpha = 1$	e_1	2.90(−2)	4.82(−3)	3.72(−3)
	e_2	1.74(−8)	8.33(−14)	1.38(−17)
	e_3	7.40(−34)	2.93(−67)	1.55(−103)
$\alpha = -1$	e_1	3.63(−2)	4.25(−3)	5.42(−3)
	e_2	5.67(−8)	5.44(−14)	2.05(−16)
	e_3	6.30(−32)	5.14(−68)	2.81(−95)
$\alpha = 1000$	e_1	5.33(−2)	1.69(−2)	2.65(−2)
	e_2	1.60(−5)	2.95(−10)	7.15(−12)
	e_3	1.21(−19)	1.81(−49)	1.50(−68)

TABLE 1 The error e_m for the first three iterations

From Table 1 we observe that the basic method and the both accelerated methods from (11) for small α in magnitude possess very fast convergence and almost the same accuracy among approximations for a fixed k, in spite of the rough initial approximations. When α takes too large values, the family (11) reduces to the iterative methods

$$\hat{z} = z_i - \frac{1}{\dfrac{\Phi'(z_i)}{\Phi(z_i)} - \Psi'(z_i) - \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(z_i - z_j^{(k)}\right)^{-1}}, \tag{25}$$

and for $k = 1, 2, 3$ possess the order of convergence $k + 2$. For $k = 1$ the iterative method (25) was considered in [6].

Example 2. We applied the proposed family of methods (12) for the simultaneous approximation to the zeros of the analytic function

$$\Phi(z) = (z^2 - 4)(e^{2z} \cos z + z^3 - 1 - \sin z)$$

inside the contour $\Gamma = \{z \in \mathbb{C} \ : \ |z| = 3\}$.

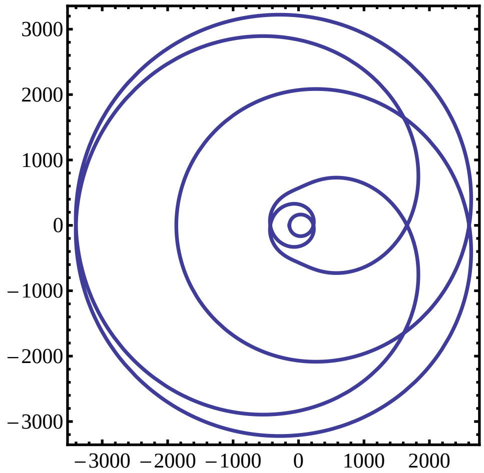

FIG. 2 The curve $\Phi(\Gamma)$ where $\Phi(z) = (z^2 - 4)(e^{2z}\cos z + z^3 - 1 - \sin z)$ and $\Gamma = \{z \in \mathbb{C} : |z| = 3\}$

The number n of zeros of Φ inside Γ and the values of $\Psi'(z_i)$ and $\Psi''(z_i)$ were calculated by the trapezoidal quadrature rule along the circle $\Gamma = \{z \ : \ |z| = 3\}$. We found by this approach that the number of zeros in the disk $\{z \ : \ |z| < 3\}$ is $n = 6$. Also, from Fig. 2 we observe that the winding number of Φ is six.

In the realization of the iterative methods (12) we use the following initial approximations

$$z_1^{(0)} = -0.6 + 0.7\,i, \quad z_2^{(0)} = -0.6 - 0.7\,i, \quad z_3^{(0)} = 0.2 - 0.1\,i,$$
$$z_4^{(0)} = 2.2 + 0.1\,i, \quad z_5^{(0)} = -2.2 + 0.1\,i, \quad z_6^{(0)} = 1.6 - 0.2\,i.$$

In each iteration we controlled the accuracy of approximations $z_i^{(m)}$ by the Euclidean norm (24). For the above initial approximations we have $e_0 = 0.494$. The results of the first three iterations are given in Table 2.

Methods		$k = 1$ (basic method)	$k = 2$ (Newton's corr.)	$k = 3$ (Halley's corr.)
$\alpha = 0$	e_1	1.97(−2)	9.61(−3)	4.76(−3)
	e_2	1.50(−6)	9.94(−10)	6.54(−14)
	e_3	4.56(−23)	1.64(−46)	6.13(−79)
$\alpha = 1$	e_1	1.75(−2)	8.97(−3)	4.57(−3)
	e_2	9.52(−7)	7.54(−10)	5.85(−14)
	e_3	7.53(−24)	4.19(−47)	3.15(−79)
$\alpha = -1$	e_1	2.16(−2)	1.02(−2)	4.94(−3)
	e_2	2.15(−6)	1.27(−9)	7.21(−14)
	e_3	1.91(−22)	5.34(−46)	1.10(−78)
$\alpha = 1000$	e_1	4.43(−2)	2.01(−2)	9.99(−3)
	e_2	2.86(−4)	8.50(−8)	6.84(−12)
	e_3	7.24(−14)	1.00(−36)	7.66(−67)

TABLE 2 The error e_m for the first three iterations

The tested methods from the family (12) showed good convergence behavior which coincides with the theoretical results. The order of convergence $k + 3$ ($k = 1, 2, 3$) is concerned for small α in magnitude. The presented examples and a number of numerical experiments did not yield a optimal value of α for all Φ. Also, an extensive numerical experimentation has shown wide domain of convergence of the proposed family (12).

The presented approach allows us to the following conclusions:

- The accelerated families of iterative methods possess very fast convergence and a wide domain of convergence.
- The convergence rate is accelerated at the price of a negligible number of additional operations because the Newton's and Halley's corrections use the already calculated values Φ, Φ' and Φ'' at the points $z_1, \ldots, z_n$.

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