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On graded Ω-groups

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Abstract. In this paper we study the notion of a graded Ω-group $(X, +, \Omega)$, but graded in the sense of M. Krasner, i.e., we impose nothing on the grading set except that it is nonempty, since operations of Ω and the grading of $(X, +)$ induce operations (generally partial) on the grading set. We prove that graded Ω-groups in Krasner's sense are determined up to isomorphism by their homogeneous parts, which, with respect to induced operations, represent partial structures called Ω-*homogroupoids*, thus narrowing down the theory of graded Ω-groups to the theory of Ω-homogroupoids. This approach already proved to be useful in questions regarding A. V. Kelarev's S-graded rings inducing S, where S is a partial cancellative groupoid. Particularly, in this paper we prove that the homogeneous subring of a Jacobson S-graded ring inducing S is Jacobson under certain assumptions. We also discuss the theory of prime radicals for Ω-homogroupoids thus extending results of A. V. Mikhalev, I. N. Balaba and S. A. Pikhtilkov in a natural way. We study some classes of Ω-homogroupoids for which the lower and upper weakly solvable radicals coincide and also, study the question of the homogeneity of the prime radical of a graded ring.

1. Introduction

Recently, graded Ω-groups, graded by a group, were considered by A. V. Mikhalev, I. N. Balaba and S. A. Pikhtilkov in [27] in order to unify the theory of prime radicals for various structures such as groups, associative, conformal and vertex algebras, Lie algebras and graded algebras. Let $(X, +, \Omega)$ be a group-graded Ω-group with 0 as the neutral element of $(X, +)$. As we know (see e.g. [27]), a G-graded Ω-group $(X, +, \Omega)$, where G is a group, is an Ω-group $(X, +, \Omega)$, which is the direct sum (restricted)

$$X = \bigoplus_{g \in G} X_g, \tag{1}$$

of subgroups X_g of $(X, +)$, called *homogeneous components*, and

$$\omega(X_{g_1}, \dots, X_{g_n}) \subseteq X_{g_1 \dots g_n}, \tag{2}$$

for all $\omega \in \Omega$, and for all $g_1, \dots, g_n \in G$, where $n = n(\omega)$ is the arity of ω. Let us call elements of homogeneous components X_g *homogeneous*. Also, let us call $g \in G$ the *degree* of $0 \neq x \in X_g$ and denote it by $\delta(x)$, and call (1)

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the *additive grading* of X. We assume that an arbitrary operation $\omega \in \Omega$ is distributive with respect to "+". If we replace (2) with

$$(\exists h \in G)\ \omega(X_{g_1}, \ldots, X_{g_n}) \subseteq X_h, \tag{3}$$

for all $\omega \in \Omega$, $n = n(\omega)$, and for all $g_1, \ldots, g_n \in G$, we see that there is no need to assume anything on G apart from being a nonempty set, since then, (1) and (3) will imply operations (generally partial) on G. Indeed, if $\omega(X_{g_1}, \ldots, X_{g_n}) \neq 0$, then $h \in G$, for which $\omega(X_{g_1}, \ldots, X_{g_n}) \subseteq X_h$, is unique, and of course, is arbitrary if $\omega(X_{g_1}, \ldots, X_{g_n}) = 0$. For those $g_1, \ldots, g_n$ for which $\omega(X_{g_1}, \ldots, X_{g_n}) \neq 0$, we may define $\omega(g_1, \ldots, g_n) := h$. For those $g_1, \ldots, g_n$ for which $\omega(X_{g_1}, \ldots, X_{g_n}) = 0$, $\omega(g_1, \ldots, g_n)$ may be defined arbitrarily in order to make ω an operation defined everywhere on G. Condition (2) can also be interpreted as $\delta(\omega(x_1, \ldots, x_n)) = g_1 \ldots g_n = \delta(x_1) \ldots \delta(x_n)$ if $\omega(x_1, \ldots, x_n) \neq 0$, where $x_1 \in X_{g_1}, \ldots, x_n \in X_{g_n}$, which will also be satisfied, as a special case for some binary operation, if any, in our definition of a graded Ω-group, which we give and discuss thoroughly in the rest of the article, since it will be proved that the degree of a homogeneous element $\omega(x_1, \ldots, x_n) \neq 0$, where x_i are homogeneous, depends only on degrees of $x_1, \ldots, x_n$, and, hence we may write $\delta(\omega(x_1, \ldots, x_n)) = \omega(\delta(x_1), \ldots, \delta(x_n))$ if $\omega(X_{\delta(x_1)}, \ldots, X_{\delta(x_n)}) \neq 0$. To resume, (3) along with (1) and the structure (X, Ω) imply the whole set of operations (generally partial) on G, and so, (2) represents a very special case. M. Krasner in [20] made similar observations concerning the notions of graded rings and modules given in [1]. Let us notice that the associativity of a particular operation from Ω of a graded Ω-group does not have to imply the associativity of the obtained (partial) operation on the grading set (see [2, 20] for the case of graded rings). One of features of M. Krasner's research is studying partial structures obtained by inducing operations of a graded structure to its set of homogeneous elements (the *homogeneous part*): *homogroupoids* in the case of graded groups, *anneids* in the case of graded rings and *moduloids* in the case of graded modules (see [20]). The origin of studying homogeneous parts only, which are of special interest to those who are dealing with homogeneous elements and substructures only, goes back to [17], continued through [2, 7, 18–21, 33], and was motivated by the following observation. Let K be a division ring, $|\cdot|$ its valuation onto a totally ordered group T, and observe the family $\{A_\gamma\}_{\gamma\in T}$, where $A_\gamma = \{x \in K \mid |x| \leq \gamma\}$. Then $\{A_\gamma\}_{\gamma\in T}$ is a total filtration of $(K, +)$. If $\overline{A}_\gamma = \bigcup_{\gamma'<\gamma} A_{\gamma'}$, $K_\gamma = A_\gamma/\overline{A}_\gamma$ and $gr(K) = \bigoplus_{\gamma\in T} K_\gamma$, then, as we all know, $gr(K)$ can be made into a graded ring. Let $s(K) = \bigcup_{\gamma\in T} K_\gamma$ be its homogeneous part. Notice that $(s(K), +)$ is a partial structure, since the sum of two homogeneous elements does not have to be homogeneous; however, if that is the case, we say that those elements are *addible* and we denote that relation by #. $(s(K), +, \cdot)$ has the following properties:

i) $(s(K), \cdot)$ is a group with the biabsorbing element 0, i.e. $\bar{x}0 = 0\bar{x} = 0$, for all $\bar{x} \in s(K)$.

ii) $(\forall \bar{x}, \bar{y}, \bar{z} \in s(K))\ \bar{x}\#0;\ \bar{x}\#\bar{x};\ \bar{x}\#\bar{y} \wedge \bar{y}\#\bar{z} \wedge \bar{y} \neq 0 \Rightarrow \bar{x}\#\bar{z}$;

iii) for all $0 \neq \bar{a} \in s(K)$, $\{\bar{x} \in s(K) \mid \bar{a}\#\bar{x}\}$ is an additive Abelian group;

iv) $(\forall \bar{x}, \bar{y}, \bar{z} \in s(K))\ \bar{x}\#\bar{y} \Rightarrow \bar{z}\bar{x}\#\bar{z}\bar{y} \wedge \bar{x}\bar{z}\#\bar{y}\bar{z} \wedge \bar{z}(\bar{x} + \bar{y}) = \bar{z}\bar{x} + \bar{z}\bar{y} \wedge (\bar{x} + \bar{y})\bar{z} = \bar{x}\bar{z} + \bar{y}\bar{z}$.

M. Krasner named this kind of structure a *corpoid* [17], which, in the sense of the above story, represents the homogeneous part (with induced operations) of a graded division ring. The main aim of this paper is hence, not only to define graded Ω-groups in the sense of M. Krasner, but to characterize and study their homogeneous parts with induced operations, called Ω-*homogroupoids*. We will also observe corresponding substructures and homomorphisms. This is also related to [10]. In the language of categories, we thus obtain a wider category compared to the classical gradings, where all objects are graded by the same set and morphisms are degree-preserving, while here, the set of objects consists of all graded Ω-groups, regardless the grading set, and morphisms are homogeneity-preserving. One particular case of our graded Ω-groups – graded rings in Krasner's sense [2, 7, 20], has already proved to be useful for it so far represents the best known tool for dealing with some open problems concerning the so called S-graded rings inducing S, where S is a partial groupoid, see [11, 16]. A. V. Kelarev gave a list of such problems in [12] and an application of Krasner graded rings to these questions is presented in [8]. In this paper we also proved that the homogeneous subring of the Jacobson graded ring (graded in the forementioned sense) is Jacobson under certain assumptions, thus generalizing the same result which is independently obtained for $\mathbb{Z}$-graded rings by A. Smoktunowicz [32] and P.-H. Lee and E. R. Puczyłowski [25]. The study of the case of semigroup-graded rings can be found in [26]. We also give the theory of prime radicals of Ω-homogroupoids, which is analogous to that developed in [27], and which can be applied not only to group-graded Ω-groups, but to any graded Ω-group. We study some classes of Ω-homogroupoids for which the lower and upper weakly solvable radicals coincide. Particularly, we examine the case of anneids – homogeneous parts of graded rings, and among other, prove that the prime and the Levitzki radical of the anneid of an incidence ring of the finite group automation (see [14]) coincide. Also, we study the question of homogeneity of the prime radical of a graded ring.

2. Preliminaries

In this section, we give definitions of a *graded group in Krasner's sense* and of a *homogroupoid* which are essential for the rest of the article.
Let G be a multiplicative group with the neutral element e and let Δ be a nonempty set. Although with multiplicative operation, the direct product (restricted) [22] will be denoted by $\bigoplus$.

Definition 2.1 ([20]). *Every mapping*

$$\gamma : \Delta \to \mathrm{Sg}(G),\ \gamma(\delta) = G_\delta\ (\delta \in \Delta), \tag{4}$$

such that $G = \bigoplus_{\delta\in\Delta} G_\delta$*, where* $\mathrm{Sg}(G)$ *is the set of all subgroups of* G*, is called a* grading *or* gradiation*. A group with a grading is called a* graded group*. A grading is called* strict *if* $G_\delta \neq \{e\}$*, for all* $\delta \in \Delta$*. If* $\Delta^* = \{\delta \in \Delta \mid G_\delta \neq \{e\}\}$*, then the mapping* $\gamma^* = \gamma|_{\Delta^*}$ *is a strict grading of* G *and is called the* strict kernel *of* (4)*. Elements* $\delta \in \Delta$ *are called* degrees *and the corresponding* G_δ *are called* homogeneous components*. The set* $H = \bigcup_{\delta\in\Delta} G_\delta$ *is called the* homogeneous part *of a graded group, and elements* $x \in H$ *are called* homogeneous*. For a homogeneous element* $x \neq e$*, the unique degree* $\xi \in \Delta^*$*, for which* $x \in G_\xi$*, is called the* degree *of* x *and is denoted by* $\delta(x)$*.*

Element e generally speaking does not have a degree and $\delta \in \Delta \setminus \Delta^*$ are called *empty degrees*. However, it is useful to associate a degree from $\Delta \setminus \Delta^*$ to e, which we denote by 0 and call it the *zero degree* [20]. If $\Delta = \Delta^* \cup 0$, and if we put $\delta(e) = 0$, the grading is called *proper* [20].
Let G be a graded group with grading (4) and let

$$\gamma' : \Delta' \to \mathrm{Sg}(G),\ \gamma'(\delta') = G_{\delta'}\ (\delta' \in \Delta') \tag{5}$$

be another grading of G. Then gradings (4) and (5) are called *equivalent* [20] if there exists a bijective mapping $\varphi : \Delta \to \Delta'$ such that $\gamma(\delta) = \gamma'(\varphi(\delta))$ $(\delta \in \Delta)$. Two gradings are called *weakly equivalent* [20] if their strict kernels are equivalent.
Obviously, a strict grading is determined up to equivalence by $\{G_\delta \mid \delta \in \Delta\}$, while a grading is determined up to weak equivalence by $\{G_\delta \mid \delta \in \Delta^*\}$.
The homogeneous part H of a graded group G, together with the group structure of G, determine the corresponding grading up to weak equivalence. Indeed, if $a \neq e$ and b are from H, and if $a \in G_\delta$, then $b \in G_\delta$ if and only if $ab \in H$. Hence, if for $e \neq a \in H$, we define $G(a) = \{x \in H \mid ax \in H\}$, then $\{G_\delta \mid \delta \in \Delta\}$ coincides with $\{G(a) \mid a \in H^* = H \setminus \{e\}\}$.
This observation led to an idea of defining a graded group as an ordered pair (G, H), where $H \subseteq G$ is the homogeneous part with respect to some grading of G in the previous sense. This however required the characterization of that homogeneous part.

Theorem 2.2 ([20]). *A nonempty subset* H *of a group* G *is the homogeneous part of* G *with respect to some grading of* G *if and only if the following conditions are satisfied:*

i) $e \in H$;

ii) $x \in H \Rightarrow x^{-1} \in H$;

iii) $x, y, z, xy, yz \in H \wedge y \neq e \Rightarrow xz \in H$;

iv) $x, y \in H \wedge xy \notin H \Rightarrow xy = yx$;

v) *H generates G;*

vi) *If $n \geq 2$ and if elements $x_1, \ldots, x_n \in H$ are such that for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, $x_i x_j \notin H$, then $x_1 \ldots x_n \neq e$.*

Let G be a graded group with the homogeneous part H. Multiplicative operation on G induces a partial operation on H, since if $x, y \in H$, then xy is defined in H if and only if $xy \in G$ is the element from H, and in that case the result is the same and we write it the same way. If this situation occurs, we say that elements x, y are *composable* or *multipliable* (*addible* in the case of an additive operation) and we write $x\#y$ [20]. Clearly, $x\#y$ if and only if x, y come from the same subgroup $G(a)$, $a \in H^*$.
In case when H with the induced operation from G is given, we may reconstruct G up to H-isomorphism. Indeed, if $a \in H^*$, then $G(a)$ may be defined as $G(a) = \{x \in H \mid a\#x\}$. G is then the direct sum of distinct subgroups $G(a)$ and H is obviously the homogeneous part of G. The group G which is obtained in this way is called the *linearization* of $(H, \cdot)$ and we denote it by $\overline{H}$ [20].
There is a natural idea now to define a graded group using the corresponding homogeneous part, at least up to isomorphism. In order to do that, we need to characterize the structure $(H, \cdot)$, which is the homogeneous part of some graded group, with the partial operation induced from the operation of that group.

Theorem 2.3 ([20]). *Let H be a nonempty set with a partial multiplicative operation "$\cdot$". Then $(H, \cdot)$ is the homogeneous part of some graded group G, with partial operation induced by that group, if and only if:*

i) $(\exists e \in H)(\forall x \in H)\ x\#e \wedge xe = x$;

ii) $(\forall x \in H)\ x\#x$;

iii) $(\forall x, y, z \in H)\ x\#y \wedge y\#z \wedge y \neq e \Rightarrow x\#z$;

iv) For all $e \neq a \in H$, $H(a) = \{x \in H \mid a\#x\}$ is a group with respect to "$\cdot$",

where $x\#y$ means that $x \cdot y$ exists.

Definition 2.4 ([20]). *A partial structure $(H, \cdot)$ which satisfies conditions of Theorem 2.3 is called a* homogroupoid*.*

3. Graded Ω-groups

Let $(X, +, \Omega)$ be an Ω-group, 0 the neutral element of $(X, +)$, and let Δ be a nonempty set. It is assumed that Ω is nonempty and that it contains at least one operation of arity $n \geq 2$. We also assume that an arbitrary $\omega \in \Omega$ is distributive with respect to "+".

Definition 3.1. *An Ω-group $(X, +, \Omega)$ is called a* graded Ω-group *if there exists a mapping γ, called* grading*, which maps elements $\delta \in \Delta$ to subgroups X_δ of $(X, +)$ such that $X = \bigoplus_{\delta \in \Delta} X_\delta$ and if for an arbitrary operation $\omega \in \Omega$ we have*

$$(\forall \delta_1, \ldots, \delta_{n(\omega)} \in \Delta)(\exists \xi \in \Delta)\ \omega(X_{\delta_1}, \ldots, X_{\delta_{n(\omega)}}) \subseteq X_\xi, \tag{6}$$

where $n = n(\omega)$ is the arity of ω. The set $H = \bigcup_{\delta \in \Delta} X_\delta$ is called the homogeneous part *of a graded Ω-group X. Elements h of H are called* homogeneous *and $\eta \in \Delta^*$ is called the* degree *of a homogeneous element $0 \neq h \in H$, and is denoted by $\delta(h)$, if $h \in X_\eta$.*

Remark 3.2. *Let us notice that $\xi \in \Delta$ in (6) is unique if $\omega(X_{\delta_1}, \ldots, X_{\delta_{n(\omega)}}) \neq 0$. Indeed, if it is not unique, there would exist another $\eta \in \Delta$ such that $\omega(X_{\delta_1}, \ldots, X_{\delta_{n(\omega)}}) \subseteq X_\eta$. Hence, $\omega(X_{\delta_1}, \ldots, X_{\delta_{n(\omega)}}) \subseteq X_\xi \cap X_\eta = 0$, a contradiction.*

Just like in the case of graded rings [20], we establish the following lemma.

Lemma 3.3. *The condition* (6) *is satisfied if and only if the following axioms hold:*

i) *For an arbitrary operation $\omega \in \Omega$ of arity n, $\omega(h_1, \dots, h_n) \in H$, for all $h_1, \dots, h_n \in H$.*

ii) *For an arbitrary operation $\omega \in \Omega$ of arity n, if $\omega(h_1, \dots, h_n) \neq 0$, $\delta(\omega(h_1, \dots, h_n))$ depends only on $\delta(h_1), \dots, \delta(h_n)$, $h_1, \dots, h_n \in H$.*

Moreover, i) *implies ii*).

Proof. If axioms *i*) and *ii*) are satisfied, then it is obvious that (6) holds. Conversely, let us assume that (6) holds and let $\omega \in \Omega$ and $h_1, \dots, h_n \in H$ be arbitrary, where $n = n(\omega)$. Then there exist $\delta_1, \dots, \delta_n \in \Delta$ such that $h_j \in X_{\delta_j}$, $j = 1, \dots, n$. Hence, according to (6), there exists $\xi \in \Delta$ such that $\omega(h_1, \dots, h_n) \in X_\xi \subseteq H$, which proves that *i*) holds. However, *i*) implies *ii*). Indeed, let $h_1, \dots, h_j, h'_j, \dots, h_n \in H$, $h_j, h'_j \neq 0$, $\omega(h_1, \dots, h_j, \dots, h_n) \neq 0$, $\omega(h_1, \dots, h'_j, \dots, h_n) \neq 0$ and $\delta(h_j) = \delta(h'_j)$. Then $h_j + h'_j \in H$, hence $\omega(h_1, \dots, h_j, \dots, h_n) + \omega(h_1, \dots, h'_j, \dots, h_n) = \omega(h_1, \dots, h_j + h'_j, \dots, h_n) \in H$, according to *i*), which implies

$$\delta(\omega(h_1, \dots, h_j, \dots, h_n)) = \delta(\omega(h_1, \dots, h'_j, \dots, h_n)).$$

This, of course, holds for every $j = 1, \dots, n$. Now, let $h_1, h'_1, \dots, h_n, h'_n \in H$ be such that $\omega(h_1, \dots, h_n) \neq 0$, $\omega(h'_1, \dots, h'_n) \neq 0$ (hence, all of h_j, h'_j are distinct from 0) and $\delta(h_j) = \delta(h'_j)$, $j = 1, \dots, n$. If

$$\omega(h_1, \dots, h_{j-1}, h'_j, h_{j+1}, \dots, h_n) \neq 0,$$

then

$$\begin{aligned} \delta(\omega(h_1, \dots, h_j, \dots, h_n)) &= \delta(\omega(h_1, \dots, h_{j-1}, h'_j, h_{j+1}, \dots, h_n)) \\ &= \delta(\omega(h'_1, \dots, h'_j, \dots, h'_n)), \end{aligned}$$

$j = 1, \dots, n$. Analogously, if $\omega(h'_1, \dots, h'_{j-1}, h_j, h'_{j+1}, \dots, h'_n) \neq 0$, then

$$\delta(\omega(h_1, \dots, h_j, \dots, h_n)) = \delta(\omega(h'_1, \dots, h'_j, \dots, h'_n)), \; j = 1, \dots, n.$$

In case when

$$\omega(h_1, \dots, h_{j-1}, h'_j, h_{j+1}, \dots, h_n) = \omega(h'_1, \dots, h'_{k-1}, h_k, h'_{k+1}, \dots, h'_n) = 0,$$

$j, k = 1, \dots, n$, it is easy to see that we have

$$\omega(h_1, \dots, h_n) + \omega(h'_1, \dots, h'_n) = \omega(h_1 + h'_1, \dots, h_n + h'_n) \in H$$

and so, $\delta(\omega(h_1, \dots, h_j, \dots, h_n)) = \delta(\omega(h'_1, \dots, h'_j, \dots, h'_n))$. Hence, *ii*) is satisfied. □

According to Remark 3.2 and Lemma 3.3, for an arbitrary operation $\omega \in \Omega$ of arity n, if $\delta_1, \dots, \delta_n \in \Delta$ are such that $\omega(X_{\delta_1}, \dots, X_{\delta_n}) \neq 0$, then we may define $\omega(\delta_1, \dots, \delta_n) = \delta(\omega(x_1, \dots, x_n))$, where $x_1, \dots, x_n$ are homogeneous elements of X such that $\delta(x_1) = \delta_1, \dots, \delta(x_n) = \delta_n$ and $\omega(x_1, \dots, x_n) \neq 0$. For those $\delta_1, \dots, \delta_n \in \Delta$ for which $\omega(X_{\delta_1}, \dots, X_{\delta_n}) = 0$, $\omega(\delta_1, \dots, \delta_n)$ may be defined arbitrarily in order to make ω an operation defined everywhere on Δ. However, if the grading of $(X, +)$ is proper, i.e., if there is no other empty degree than 0 and $\delta(0) = 0$, then, for $\delta_1, \dots, \delta_n \in \Delta$ for which $\omega(X_{\delta_1}, \dots, X_{\delta_n}) = 0$, we may put $\omega(\delta_1, \dots, \delta_n) = 0$, and then, the grading of $(X, +, \Omega)$ is called *proper* as well. The grading of $(X, +, \Omega)$ is called *strict* if the grading of $(X, +)$ is strict, i.e., if there are no empty degrees, and $\omega(\delta_1, \dots, \delta_n)$ is not defined unless $\omega(X_{\delta_1}, \dots, X_{\delta_n}) \neq 0$.
Let us now characterize the homogeneous part of a graded Ω-group X in order to view graded Ω-groups as ordered pairs (X, H), where H is a nonempty subset of an Ω-group X, and where H is the homogeneous part with respect to some grading of X which determines that grading up to weak equivalence (of the graded group $(X, +)$).

Theorem 3.4. *Let X be an Ω-group and $\emptyset \neq H \subseteq X$. Then H is the homogeneous part of X with respect to some grading of X if and only if the following conditions are satisfied:*

i) $0 \in H$;

ii) $h \in H \Rightarrow -h \in H$;

iii) $h_1, h_2, h_3 \in H, h_1 + h_2, h_2 + h_3 \in H \wedge h_2 \neq 0 \Rightarrow h_1 + h_3 \in H$;

iv) $h_1, h_2 \in H \wedge h_1 + h_2 \notin H \Rightarrow h_1 + h_2 = h_2 + h_1$;

v) *H generates* $(X, +)$;

vi) *If $n \geq 2$ and if elements $h_1, \dots, h_n \in H$ are such that for all $j, k = 1, \dots, n$, $j \neq k$, $h_j + h_k \notin H$, then $h_1 + \cdots + h_n \neq 0$;*

vii) *For an arbitrary operation $\omega \in \Omega$ of arity n, $\omega(h_1, \dots, h_n) \in H$, for all $h_1, \dots, h_n \in H$.*

Proof. It follows from Theorem 2.2 and Lemma 3.3. □

4. Ω-homogroupoids

According to Theorem 3.4.**vii)**, operations $\omega \in \Omega$ of a graded Ω-group $(X, +, \Omega)$ induce operations ω_H on H. For the sake of simplicity, we will denote ω_H again by ω and the set of all induced operations Ω_H simply by Ω. Additive operation, on the other hand, induces a partial operation, i.e., $x +_H y$ exists and is equal to $x + y \in X$ if $x + y \in H$, and then, according to Definition 2.4, $(H, +)$ is a homogroupoid. Hence, the following notion is justified.

Definition 4.1. *Structures $(H, +, \Omega)$ which are homogeneous parts of graded Ω-groups $(X, +, \Omega)$ and whose operations are induced by those of X are called* Ω-homogroupoids.
The corresponding graded Ω-groups of Ω-homogroupoids $(H, +, \Omega)$ are denoted $\overline{(H, +, \Omega)}$ and are called linearizations *of Ω-homogroupoids $(H, +, \Omega)$.*

Remark 4.2. *Let us notice that, just like in the case of homogroupoids, linearizations of Ω-homogroupoids $(H, +, \Omega)$ are unique up to H-isomorphism.*

The following theorem is an analogue to the corresponding theorem for anneids presented in [20].

Theorem 4.3. *Let H be a nonempty set with a partial additive operation "+" and with the set Ω of operations on H. Then $(H, +, \Omega)$ is an Ω-homogroupoid if and only if the following hold:*

a) *There exists $0 \in H$ such that for all j, $\omega(h_1, \dots, h_{j-1}, 0, h_{j+1}, \dots, h_n) = 0$ $(h_k \in H)$ for all $\omega \in \Omega$;*

b) *Every $h \in H$ is addible with 0, addibility is reflexive and it is almost transitive, i.e., if $h_1 \# h_2$, $h_2 \# h_3$ and $h_2 \neq 0$, then $h_1 \# h_3$;*

c) *If $0 \neq a \in H$, $H(a) = \{h \in H \mid a \# h\}$ is a group with respect to addition;*

d) *All operations $\omega \in \Omega$ are distributive with respect to $+$, i.e.*

$$\begin{aligned} \omega \quad & (h_1, \dots, h_{j-1}, h_j + h'_j, h_{j+1}, \dots, h_n) = \\ = \quad & \omega(h_1, \dots, h_{j-1}, h_j, h_{j+1}, \dots, h_n) + \omega(h_1, \dots, h_{j-1}, h'_j, h_{j+1}, \dots, h_n) \end{aligned}$$

if $h_j \# h'_j$, for all $\omega \in \Omega$.

Proof. The necessity is clear. Let us assume that **a)-d)** hold. Let us notice first that 0 is the neutral element of H since, by **b)** 0#0, and by **a)** and **d)**,

$$\begin{aligned} 0 &= \omega(h_1, \ldots, h_{j-1}, 0, h_{j+1}, \ldots, h_{k-1}, 0+0, h_{k+1}, \ldots, h_n) \\ &= \omega(h_1, \ldots, h_{j-1}, 0, h_{j+1}, \ldots, h_{k-1}, 0, h_{k+1}, \ldots, h_n) + \omega(h_1, \ldots, h_{j-1}, 0, h_{j+1}, \ldots, h_{k-1}, 0, h_{k+1}, \ldots, h_n) \\ &= 0+0, \end{aligned}$$

which implies that 0 is the neutral element of $H(a)$, for all $0 \neq a \in H$, since, by **c)**, $H(a)$ is a group with respect to addition. Hence, 0 is the neutral element of $(H, +)$. Hence, according to Theorem 2.3, $(H, +)$ is a homogroupoid. Let $(X, +) = \overline{(H, +)}$ and $\bar{h}^1, \ldots, \bar{h}^n \in \overline{H}$. Then $\bar{h}^j = \sum_{\delta^j \in \Delta} h^j_{\delta^j}$, $h^j_{\delta^j} \in X_{\delta^j}$ and $h^j_{\delta^j} = 0$ for all but finitely many $\delta^j \in \Delta$. Let us put

$$\omega(\bar{h}^1, \ldots, \bar{h}^n) = \sum_{\delta^1 \in \Delta} \cdots \sum_{\delta^n \in \Delta} \omega(h^1_{\delta^1}, \ldots, h^n_{\delta^n}),$$

where $\omega \in \Omega$ is arbitrary. It is not hard to prove that this operation is distributive with respect to addition. Now, according to Lemma 3.3, $(\overline{H}, +, \Omega)$ is a graded Ω-group. $\square$

Remark 4.4. *It is obvious that every Ω-group is an Ω-homogroupoid, and as we have just seen, every graded Ω-group is determined by its Ω-homogroupoid. Hence, all results obtained in the theory of Ω-homogroupoids remain true in the theory of Ω-groups and graded Ω-groups.*

5. Homogeneous substructures of graded Ω-groups. Homomorphisms

Naturally, M. Krasner [20] defined a *homogeneous subgroup* K of a graded group G to be a subgroup of G generated by $K \cap H$, where H is the homogeneous part of G. We define a *homogeneous Ω-subgroup* of a graded Ω-group analogously.

Definition 5.1. *An Ω-subgroup $(Y, +, \Omega)$ of a graded Ω-group $(X, +, \Omega)$ is called a* homogeneous Ω-subgroup *if Y is generated by $Y \cap H$, where H is the homogeneous part of X.*

Next we define the notion of a *homogeneous ideal* of a graded Ω-group.

Definition 5.2. *Let $(X, +, \Omega)$ be a graded Ω-group. A nonempty substructure $(Y, +, \Omega)$ of X is called a* homogeneous ideal *of X if:*

a) *$(Y, +)$ is a normal homogeneous subgroup of $(X, +)$;*

b) $(\forall \omega \in \Omega)(\forall x_1, \ldots, x_n \in X)((\exists i)\, x_i \in Y \Rightarrow \omega(x_1, \ldots, x_n) \in Y)$.

Since graded Ω-groups are up to H-isomorphism determined by their homogeneous parts H, we intend to observe the homogeneous counterparts of above definitions.

Definition 5.3. *A subset K of an Ω-homogroupoid $(H, +, \Omega)$ is called an* Ω-subhomogroupoid *if K is the homogeneous part of a homogeneous Ω-subgroup of $(\overline{H}, +, \Omega)$.*

The following Lemma is obvious.

Lemma 5.4. *Let $(H, +, \Omega)$ be an Ω-homogroupoid. Then $K \subseteq H$ is an Ω-subhomogroupoid of H if and only if:*

a) *$(K, +)$ is a subhomogroupoid of $(H, +)$;*

b) *K is closed with respect to Ω.*

Definition 5.5. *An Ω-subhomogroupoid K is called an* ideal *of an Ω-homogroupoid $(H, +, \Omega)$ if:*

a) *$(K, +)$ is a normal subhomogroupoid of $(H, +)$;*

b) $(\forall\omega \in \Omega)(\forall h_1, \dots, h_n \in H)((\exists j)\, h_j \in K \Rightarrow \omega(h_1, \dots, h_n) \in K)$.

Let $(X, +, \Omega)$ be a graded Ω-group with the homogeneous part H and $(X', +', \Omega')$ a graded Ω'-group with the homogeneous part H'. We also assume that X and X' are structures of the same type, i.e., that there exists a bijective mapping $\omega \to \omega'$ from Ω onto Ω' such that $n(\omega) = n(\omega')$ $(\omega \in \Omega)$. For the sake of simplicity, additive operation "$+'$" will also be denoted by "$+$". However, we will denote the neutral element of $(X', +)$ by $0'$.

Definition 5.6. *A homomorphism* $f : (X, +, \Omega) \to (X', +, \Omega')$ *is called a* quasihomogeneous homomorphism *if* $f(H) \subseteq H'$.

It follows that if $h, k \in H$ are such that $f(h), f(k) \neq 0'$, then $\delta(h) = \delta(k) \Rightarrow \delta(f(h)) = \delta(f(k))$.

Definition 5.7. *A quasihomogeneous homomorphism* $f : (X, +, \Omega) \to (X', +, \Omega')$ *is called* homogeneous *if* $\delta(f(h)) = \delta(f(k)) \Rightarrow \delta(h) = \delta(k)$ *for all* $h, k \in H$ *such that* $f(h), f(k) \neq 0'$.

Now, let us observe restrictions of quasihomogeneous resp. homogeneous homomomorphisms to the corresponding homogroupoids. We then come to the following notions.

Definition 5.8. *The mapping* $f : (H, +, \Omega) \to (H', +, \Omega')$ *from an* Ω*-homogroupoid* H *to an* Ω'*-homogroupoid* H' *is called a* quasimorphism *if:*

a) $(\forall\omega \in \Omega)(\forall h_1, \dots, h_n \in H)\; f(\omega(h_1, \dots, h_n)) = \omega'(f(h_1), \dots, f(h_n))$;

b) $(\forall h_1, h_2 \in H)\; h_1\#h_2 \Rightarrow f(h_1)\#f(h_2) \wedge f(h_1 + h_2) = f(h_1) + f(h_2)$.

If moreover $0' \neq f(h_1)\#f(h_2) \neq 0'$ *implies* $h_1\#h_2$, f *is called a* homomorphism.

Remark 5.9. *The kernel and the image are defined as usual. It is important to notice that if f is a quasimorphism, then* $\ker f = f^{-1}(0')$ *is an ideal of* $(H, +, \Omega)$*, but* $f(H)$ *does not have to be an ideal of* $(H', +, \Omega')$*. The image* $f(H)$ *is an ideal, however, if f is a homomorphism. This is purely the consequence of the properties of the mapping* $(H, +) \to (H', +)$ *of homogroupoids [2, 7, 20].*

Theorem 5.10. *Let* $(K, +, \Omega)$ *be an ideal of an* Ω*-homogroupoid* $(H, +, \Omega)$*. If* ρ *is defined by* $h_1\rho h_2$ *if and only if* h_1, $h_2 \in K$ *or* $h_1\#h_2$ *and* $h_1 - h_2 \in K$ $(h_1, h_2 \in H)$*, then* ρ *is a congruence relation on* H *and the factor set* H/ρ*, denoted by* H/K*, is an* Ω*-homogroupoid, called the* factor Ω-homogroupoid.

Proof. It is clear that ρ is a congruence on H and $[h]_\rho = h + K$. In order to prove that H/K is an Ω-homogroupoid, we will use the canonical mapping $f : H \to H/K$ (see [2, 7, 10]). For an arbitrary $\omega \in \Omega$ and $[h_i]_\rho \in H/K$, $i = 1, \dots, n = n(\omega)$, define $\omega([h_1]_\rho, \dots, [h_n]_\rho) = [\omega(h_1, \dots, h_n)]_\rho$, where h_i, $i = 1, \dots, n$, are such that $[h_i]_\rho = f(h_i)$, $i = 1, \dots, n$. We say that $[h_1]_\rho$ is addible to $[h_2]_\rho$, and write $[h_1]_\rho\#[h_2]_\rho$ if there exist elements h_1 and h_2 from H such that $[h_1]_\rho = f(h_1)$, $[h_2]_\rho = f(h_2)$, and $h_1\#h_2$; in that case we put $[h_1]_\rho + [h_2]_\rho = f(h_1 + h_2)$. A routine check verifies that these operations are well defined, and that $(H/K, +, \Omega)$ is an Ω-homogroupoid, i.e., that it satisfies all axioms of Theorem 4.3. □

Let us make a convention in denoting the neutral element of the factor Ω-homogroupoid, with respect to partial addition, by $\bar{0}$, since there is no danger of confusing it with the process of linearization (zero is the same in an Ω-homogroupoid and in its linearization).

Remark 5.11. *If K is an ideal of an* Ω*-homogroupoid* $(H, +, \Omega)$*, then, if for* $h \in H$, $[h]_\rho = f(h) \neq \bar{0}$*, then it is easily verified that* $(H/K)([h]_\rho) \cong H(h)/(H(h) \cap K)$*, just like in the case of homogroupoids [20], where* $f : H \to H/K$ *is the canonical mapping, which is, obviously, a homomorphism of* Ω*-homogroupoids. Using homomorphism* $H \to H/K$*, and a homogeneous homomorphism* $\overline{H} \to \overline{H}/\overline{K}$*, we easily arrive at* $\overline{H/K} = \overline{H}/\overline{K}$*, which represents the graded version of the previous theorem.*

Remark 5.12. *If we keep in mind what is stated in Remark 5.9, it is easy to formulate and prove isomorphism theorems for graded Ω-groups and Ω-homogroupoids, i.e., we need to assume that the mappings under consideration are homogeneous in the case of graded Ω-groups, and homomorphisms in the case of Ω-homogroupoids.*

If I and J are ideals of an Ω-homogroupoid H, let us see what their sum is. Let

$$I + J = \{a + b \mid a \in I,\ b \in J,\ a\#b\}.$$

It is easy to prove that $I + J$ is a normal subhomogroupoid of a homogroupoid $(H, +)$ [20]. Let $\omega \in \Omega$ be an arbitrary operation of arity n, $c_1, \ldots, c_n \in H$ and let $c_i \in I + J$. Then there exist $a_i \in I$ and $b_i \in J$ such that $a_i\#b_i$ and $c_i = a_i + b_i$. Since $a_i\#b_i$, we have

$$\omega(c_1, \ldots, c_{i-1}, a_i, c_{i+1}, \ldots, c_n)\#\omega(c_1, \ldots, c_{i-1}, b_i, c_{i+1}, \ldots, c_n)$$

and

$$\begin{aligned} \omega(c_1, \ldots, c_i, \ldots, c_n) &= \omega(c_1, \ldots, c_{i-1}, a_i + b_i, c_{i+1}, \ldots, c_n) \\ &= \omega(c_1, \ldots, c_{i-1}, a_i, c_{i+1}, \ldots, c_n) + \omega(c_1, \ldots, c_{i-1}, b_i, c_{i+1}, \ldots, c_n). \end{aligned}$$

Since I and J are ideals,

$$\omega(c_1, \ldots, c_{i-1}, a_i, c_{i+1}, \ldots, c_n) \in I,\ \omega(c_1, \ldots, c_{i-1}, b_i, c_{i+1}, \ldots, c_n) \in J,$$

hence, $I + J$ is an ideal of an Ω-homogroupoid H.

6. About the direct sum

It is, of course, of interest to know if we may derive a graded Ω-group by the means of the *direct sum of graded Ω-groups*. Once the direct sum is defined, other related notions will be clear enough. Since graded Ω-groups are determined by their homogeneous parts, we are actually interested in the notion of the *direct sum of Ω-homogroupoids*. The main difficulty is in defining the direct sum of homogroupoids, which is already done [7, 20], and once we recall what the direct sum of homogroupoids is, the rest will follow easily if we apply techniques known for Ω-groups (see e.g. [4, 31]). First we need to recall what the direct sum of the family of graded groups is [20]. Let $G = \bigoplus_{\delta\in\Delta} G_\delta$ be a graded group and let $\{G^\alpha \mid \alpha \in I\}$ be a family of homogeneous subgroups of G. We say that G is the direct sum $\bigoplus_{\alpha\in I} G^\alpha$ of homogeneous subgroups G^α if it is as an abstract group. It is immediate that this is the case if and only if $G_\delta = \bigoplus_{\alpha\in I} G^\alpha_\delta$, where $G^\alpha_\delta = G^\alpha \cap G_\delta$, $\delta \in \Delta$. This allows us to define the direct sum of the family of graded groups $\{G^\alpha \mid \alpha \in I\}$ whose sets of grades are assumed to be mutually disjoint. If $G^\alpha = \bigoplus_{d\in D_\alpha} G^\alpha_d$, put $D = \bigcup_{\alpha\in I} D_\alpha$. Sets D_α form a partition of D, and let ϵ_I be the corresponding equivalence relation. Take η to be an orthogonal equivalence relation to ϵ_I, i.e., a relation for which the intersection of an arbitrary class of ϵ_I and of an arbitrary class of η is an empty set or a singleton. Let $\Delta = D/\eta$. If $\delta \in \Delta$ and $\alpha \in I$, then put $G^\alpha_\delta = G^\alpha_d$ if $\delta \cap D_\alpha = \{d\}$ and $G^\alpha_\delta = \{e^\alpha\}$ if $\delta \cap D_\alpha = \emptyset$, where e^α is the neutral element of G^α. Then $\bigoplus^{(\eta)}_{\alpha\in I} G^\alpha = \bigoplus_{\delta\in\Delta} G_\delta$ is the direct sum of graded groups G^α, where $G_\delta = \bigoplus_{\alpha\in I} G^\alpha_\delta$. It is clear how to translate these notions to the case of homogroupoids. Now, we briefly turn our attention to the case of the family of graded Ω-homogroupoids. There, one should take the orthogonal equivalence relation η to be such that $[d^i_\alpha]_\eta = [d^i_{\alpha'}]_\eta$ implies $[\omega(d^1_\alpha, \ldots, d^n_\alpha)]_\eta = [\omega(d^1_{\alpha'}, \ldots, d^n_{\alpha'})]_\eta$, if $\omega(d^1_\alpha, \ldots, d^n_\alpha)$ and $\omega(d^1_{\alpha'}, \ldots, d^n_{\alpha'})$ exist, where $d^i_\alpha \in D_\alpha$, $d^i_{\alpha'} \in D_{\alpha'}$, $i = 1, \ldots, n = n(\omega)$, and then, it is clear enough how to define the direct sum of Ω-homogroupoids.

7. Graded Ω-group vs. Ω-homogroupoid and some questions regarding the Jacobson radical of a graded ring

We have seen that graded Ω-groups are determined by Ω-homogroupoids, but the question is which one to choose in a particular situation, graded Ω-groups or Ω-homogroupoids. Well, if one is interested in homogeneous elements and substructures only, then, the environment of Ω-homogroupoids is certainly the right one. If one observes elements both homogeneous and nonhomogeneous equally, then graded Ω-groups are, of course, sufficient. Nevertheless, when proving some properties concerning homogeneity, in this, nonhomogeneous setting, i.e., when observing graded Ω-groups, it is not always clear which nonhomogeneous elements would play the right part. This is however clear in homogeneous setting, i.e., when observing Ω-homogroupoids. Also, as M. Krasner pointed out in [20] while favorising homogeneous over nonhomogeneous aspect of graded rings, it is practically much simpler to say element than homogeneous element and to write $a\#b$ instead of $ab \in H$. We can use the similar argument to put Ω-homogroupoids in front of graded Ω-groups. Let us look at one particular case of a graded Ω-group with grading set Δ, a graded ring with grading set Δ [2, 7, 20]. We call such rings *Krasner graded rings* or *Δ-graded rings inducing* Δ, according to [11, 16], or just *graded rings*. The corresponding Ω-homogroupoid will be called an *anneid*, according to M. Krasner, see [2]. We could also call these as *ringoids*, but the original French term seems to be more appropriate, since the term "ringoid" is already reserved for a set with two binary operations, while in our case, we have one operation and one partial operation. Also, "anneid" carries a root of the Latin word for ring, "annulus". In [16], A. V. Kelarev and A. Plant considered the question of homogeneity of the Jacobson radical of Δ-graded rings inducing Δ. The homogeneous aspect, i.e., the theory of anneids proved to be useful in providing some answers to following questions.

Question 7.1 ([12]). *Is it true that, if S is a finite cancellative partial groupoid with idempotent e and $R = \bigoplus_{s\in S} R_s$ is an S-graded ring inducing S, then $J(R_e) = R_e \cap J(R)$?*

Question 7.2 ([12]). *Is it true that, for every groupoid S and every S-graded ring R satisfying a polynomial identity, if the radicals of subrings R_e are nil for all idempotents e in S, then the radical of R is nil?*

In [8], theory of anneids (especially [6, 7]) was used in order to answer affirmatively to Question 7.1 in case when the *Jacobson radical* and the *large Jacobson radical* of an anneid coincide. Here, the Jacobson radical of an anneid A is defined as the intersection of annihilators of all regular irreducible A-moduloids and the large Jacobson radical as the intersection of annihilators of all irreducible A-moduloids [6, 7]. Moduloids are, as already mentioned, homogeneous parts of graded modules, and a commutative graded group $M = \bigoplus_{d\in D} M_d$ is a graded right (left) R-module, where R is Δ-graded ring inducing Δ, if for all $\delta \in \Delta$ and $d \in D$ there exists $t \in D$ such that $M_d R_\delta \subseteq M_t$ ($R_\delta M_d \subseteq M_t$) [2, 7, 20]. As for the Question 7.2, in [8] it is proved that the answer is positive under certain assumptions which include the regularity (cancellativity) of a graded PI-ring with the finite support and the assumption that the Jacobson radical and the large Jacobson radical coincide. While proving this, one more open problem which can be found in [12] was dealt, which asks whether a certain power of the Jacobson radical is contained in the homogeneous Jacobson radical of an S-graded ring inducing S, where S is a finite cancellative partial groupoid.

However, there are cases when one might find the nonhomogeneous aspect more useful. For instance, let the grading set be totally ordered. In that case, the degree may also be associated to a nonhomogeneous element of a graded Ω-group as it can be done in the case of graded rings [20]. Namely, let $(X, +, \Omega)$ be a graded Ω-group, whose grading is proper, with the totally ordered grading set $(\Delta, \leq)$, where the order is compatible with Ω, i.e., if $\omega \in \Omega$, $n = n(\omega)$, $\delta_1, \dots, \delta_n, \delta'_1, \dots, \delta'_n \in \Delta$, then

$$\delta_1 \leq \delta'_1 \wedge \cdots \wedge \delta_n \leq \delta'_n \Rightarrow \omega(\delta_1, \dots, \delta_n) \leq \omega(\delta'_1, \dots, \delta'_n).$$

Suppose that 0 is the least element of Δ. Let $x = \sum_{\delta\in\Delta} x_\delta \in X$, where $x_\delta \in X_\delta$ if $\delta \neq 0$ and $x_0 = 0$. Then we may define the degree $\delta(x)$ of x by $\delta(x) = \max_{\delta\in\Delta} \delta(x_\delta)$.

We will now focus on the special case of regular proper graded rings. Recently, A. Smoktunowicz [32] and P.-H Lee and E. R. Puczyłowski [25] proved independently that the homogeneous subring of a Jacobson $\mathbb{Z}$-graded ring is also Jacobson, where $\mathbb{Z}$ denotes the set of integers. We prove that the same result holds for Δ-graded rings inducing Δ under certain assumptions. For results on semigroup-graded rings, see [26].

Let R be a regular proper graded ring with the totally ordered grading set Δ, with 0 as the least element. Here, by regularity we mean that the grading set Δ is cancellative with respect to induced operation. Also, assume that the order is compatible with the operation induced on Δ, i.e., if $\delta_1, \delta_2, \delta_1', \delta_2' \in \Delta$ are such that $\delta_1 \leq \delta_1'$ and $\delta_2 \leq \delta_2'$, then $\delta_1\delta_2 \leq \delta_1'\delta_2'$. Let us observe the set

$$S = \{\bar{x} = \sum_{\delta\in\Delta} x_\delta \mid x_\delta \in R_\delta,\ \mathrm{supp}(\bar{x}) \text{ is well ordered}\}, \tag{7}$$

where $\mathrm{supp}(\bar{x})$ denotes the *support of* $\bar{x}$, i.e., the set $\{\delta \in \Delta \mid x_\delta \neq 0\}$ [20].
For $\bar{x} = \sum_{\delta\in\Delta} x_\delta$, $\bar{y} = \sum_{\delta\in\Delta} y_\delta \in S$, define

$$\bar{x}\bar{y} := \sum_{\xi\in\Delta}\sum_{\delta\delta'=\xi} x_\delta y_{\delta'}, \tag{8}$$

and let $D(\xi) := \{(\delta, \delta') \in \Delta \times \Delta \mid \delta\delta' = \xi \text{ and } x_\delta y_{\delta'} \neq 0\}$ [20].
The following lemma is known, but we include its proof for convenience.

Lemma 7.3 ([20]). *Element* (8) *belongs to* S.

Proof. If $\delta_1\delta_1' = \delta_2\delta_2'$ and $\delta_1 < \delta_2$, then $\delta_1' > \delta_2'$. Indeed, if $\delta_1' \leq \delta_2'$, then $\delta_1\delta_1' \leq \delta_2\delta_1' \leq \delta_2\delta_2'$. Hence, $\delta_1\delta_1' = \delta_2\delta_1' = \delta_2\delta_2'$, and since R is regular, it follows that $\delta_1 = \delta_2$. Therefore, the set $\{\delta \in \Delta \mid (\delta, \delta') \in D(\xi)\}$ is well ordered, since for those δ, $x_\delta y_{\delta'} \neq 0$, hence $x_\delta \neq 0$ and $\delta \in \mathrm{supp}(\bar{x})$. Also, the set $\{\delta' \in \Delta \mid (\delta, \delta') \in D(\xi)\}$ is well ordered with respect to reverse order, and since it is contained in $\mathrm{supp}(\bar{y})$, it is well ordered and finite. This means that $D(\xi)$ is finite. Obviously, the set $\{\delta\delta' \mid (\delta, \delta') \in \mathrm{supp}(\bar{x}) \times \mathrm{supp}(\bar{y})\}$ is well ordered, and hence, element (8) belongs to S. $\square$

The set S, with addition defined component-wise and with the multiplication (8), is a ring, called the *Hahn completion of* R [20].

Theorem 7.4. *Let R be a regular proper graded ring with the totally ordered grading set Δ such that the order is compatible with the induced operation on Δ and such that for all $\delta \in \Delta$, for which $\delta^2 \neq 0$, we have $\delta^2 > \delta$. Then, if T is a homogeneous subring of a graded ring R, then $J(R) \cap T \subseteq J(T)$. In particular, if R is Jacobson, then T is also Jacobson.*

Proof. According to assumptions, we may observe the ring S from (7) with addition defined component-wise and with the multiplication (8). Observe also that, under our assumptions, if $\xi\eta \neq 0$, then $\xi\eta > \xi$ and $\xi\eta > \eta$, where $\xi, \eta \in \Delta$. The ring S is Jacobson. Indeed, let $\bar{x} \in S$. Then its quasi-inverse is $\bar{x}' = \sum_{i=1}^{\infty}(-1)^i\bar{x}^i$, which is, under our assumptions, a well defined element in S according to [29], Theorem 4.7.
Let $x \in J(R) \cap T$ and $x = \sum_{i=1}^{k} r_{\delta_i}$, $r_{\delta_i} \in R_{\delta_i}$. Since T is homogeneous, $r_{\delta_i} \in T$. Let $x' = \sum_{j=1}^{l} r_{\lambda_j}$ be the quasi-inverse of x in R. If x'' is a quasi-inverse of x in S, then, since $x' \in S$, and there can be only one quasi-inverse in S, $x' = x''$, i.e., $\sum_{j=1}^{l} r_{\lambda_j} = \sum_{i=1}^{\infty}(-1)^i x^i$. This means that homogeneous elements from the left-hand side belong to the corresponding homogeneous components of homogeneous elements from the right-hand side, i.e., they belong to T. Hence, x' is a quasi-inverse of x which belongs to T, and the claim follows. $\square$

The following result is of interest as well.

Theorem 7.5. *Let R be a regular proper graded ring with the totally ordered grading set Δ such that the order is compatible with the induced operation on Δ and such that for all $\delta \in \Delta$, for which $\delta^2 \neq 0$, we have $\delta^2 > \delta$. If R is a Jacobson graded ring, then R is a graded-nil ring, i.e., every homogeneous element is nilpotent.*

Proof. Again, we observe the ring S from (7) with addition defined component-wise and with the multiplication (8). Embed R into S. For a homogeneous element $x \in R$, if x' is its quasi-inverse in R and x'' its quasi-inverse in S, we have $x' = x''$ in S, which implies

$$\sum_{j=1}^{l} r_{\lambda_j} = \sum_{i=1}^{\infty}(-1)^i x^i,$$

and therefore, there exists a natural number n such that $x^n = 0$, q.e.d. $\square$

Example 7.6. *If the grading set has nonzero idempotents, then the statements of previous theorems do not have to hold. Indeed, let* $\mathbb{Q}_1 = \{\frac{2m}{2n+1} \mid m, n \in \mathbb{Z},\ (2m, 2n+1) = 1\}$. *Then* $R = \begin{pmatrix} \mathbb{Q}_1 & \mathbb{Q}_1 \\ 0 & \mathbb{Q}_1 \end{pmatrix} = \begin{pmatrix} \mathbb{Q}_1 & 0 \\ 0 & \mathbb{Q}_1 \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathbb{Q}_1 \\ 0 & 0 \end{pmatrix}$ *is a Jacobson regular Krasner graded ring which has a nonzero idempotent degree corresponding to* $\begin{pmatrix} \mathbb{Q}_1 & 0 \\ 0 & \mathbb{Q}_1 \end{pmatrix}$. *However,* $\begin{pmatrix} \mathbb{Q}_1 & 0 \\ 0 & \mathbb{Q}_1 \end{pmatrix}$ *is a homogeneous subring which is Jacobson but not nil since* $\mathbb{Q}_1$ *is a Jacobson but not a nil ring.*

8. The theory of prime radicals

We show that the theory of prime radicals can also be developed for Ω-homogroupoids as it is developed for group-graded Ω-groups in [27]. As in [27], we will follow the usual construction of the theory of prime radicals (see [27] and references therein). Proofs of results in [27] were included for the sake of convenience in reading and because the graded case is considered, and here we do the same for results on Ω-homogroupoids. All statements are easily translated to graded Ω-groups and may be applied to any graded Ω-group, in the spirit of Remark 4.4. Particularly, at the end of this section, we observe prime radicals of anneids.

Definition 8.1. *Let* I, J *be* Ω*-subhomogroupoids of an* Ω*-homogroupoid* H. *The* commutator $[I, J]$ *of* I *and* J *is an* Ω*-subhomogroupoid of an* Ω*-homogroupoid* H *which is generated by the set of elements* $\omega(c_1, \dots, c_n)$, $n \geq 2$, $\omega \in \Omega$, *where* c_i *come from* I *or* J *but some elements from both* I *and* J *are among* c_i. *An* Ω*-subhomogroupoid* $[I, I]$ *is denoted* I^2.

Definition 8.2. *An element* a *of an* Ω*-homogroupoid* H *is called* strictly Engel *if for an arbitrary sequence* $a_0, a_1, \dots$ *of elements of* H, *the condition* $a_0 = a$, $a_{i+1} \in (a_i)^2$, $i = 0, 1, \dots$, *implies that all elements are* 0 *starting from some* i.

Definition 8.3. *An* Ω*-homogroupoid* H *is called* prime *if for every two ideals* I, J *of* H, $[I, J] = 0$ *implies* $I = 0$ *or* $J = 0$. *We say that* H *is* semiprime *if* $I^2 = 0$ *implies* $I = 0$.

Definition 8.4. *An ideal* P *of an* Ω*-homogroupoid* H *is called* prime *if* H/P *is a prime* Ω*-homogroupoid.*

Definition 8.5. *The* prime radical $P(H)$ *of an* Ω*-homogroupoid* H *is the intersection of all prime ideals of* H.

Using the same methods as in [27], one can prove the following theorems.

Theorem 8.6. *The prime radical of an* Ω*-homogroupoid* H *is the set of all strictly Engel elements of* H.

Proof. Let $a \in H$ be an element that does not belong to the prime radical, i.e., $a \notin P(H)$. Then there exists a prime ideal I of H such that $a \notin I$. This implies that the image of $(a)^2$ in the factor Ω-homogroupoid H/I under the canonical mapping is distinct from $\bar{0}$. Hence, $(a)^2$ is not contained in I. Let $a_0 = a$ and let $a_1 \in (a)^2 \setminus I$ be arbitrary. Then $a_1 \notin P(H)$. We continue this process, and construct a sequence of elements $a_0 = a, a_1, \dots,$ where $a_{i+1} \in (a_i)^2$ $(i = 0, 1, \dots)$ none of which belongs to $P(H)$. Hence, a is not strictly Engel.
Conversely, let $a \in H$ be an element which is not strictly Engel. Then there exists a sequence of elements $a_0 = a, a_1, \dots$ such that $a_{i+1} \in (a_i)^2$ $(i = 0, 1, \dots)$ all of which are distinct from 0. According to Zorn's Lemma, there exists a maximal ideal I of H which does not contain the sequence $a_0, a_1, \dots$. We claim that I is a prime ideal. Let M and N be two ideals such that $I \subsetneq M$ and $I \subsetneq N$. Then there exist natural numbers k and l such that $a_k \in M$ and $a_l \in N$. If $m = \max\{k, l\}$, then all elements of sequence a_i, where $i \geq m$, starting with the number $m + 1$ lie in $M \cap N$ and in $[M, N]$. Let $f : H \to H/I$ be the canonical homomorphism (in the sense of Definition 5.8). The commutant $[f(M), f(N)]$ of the images $f(M)$ and $f(N)$ of the ideals M and N in H/I is not equal to $\bar{0}$. Therefore, I is a prime ideal of an Ω-homogroupoid H, and so, $a \notin P(H)$. $\square$

Theorem 8.7. *An* Ω*-homogroupoid* H *is semiprime if and only if* $P(H) = 0$.

Proof. 1. Let us first prove that an Ω-homogroupoid H is semiprime if and only if for all $a \in H$ the equality $(a)^2 = 0$ implies $a = 0$. Well, let H be a semiprime Ω-homogroupoid and assume that for some $a \in H$ we have $(a)^2 = 0$. Since H is semiprime, it follows that $(a) = 0$, and hence, $a = 0$. Conversely, assume that for all $a \in H$ the equality $(a)^2 = 0$ implies $a = 0$. If I is an ideal of H for which we have $I^2 = 0$ and $a \in I$, then $(a) \subseteq I$, which implies that $(a)^2 = 0$. Therefore, $a = 0$, and since a was arbitrarily chosen, $I = 0$, hence, H is semiprime.
2. Let I be an ideal of H such that the factor Ω-homogroupoid H/I is semiprime and $b \notin I$. We claim that there exists a prime ideal K such that $K \supseteq I$ and $b \notin K$. Let $b_0 = b$. Since H/I is semiprime, $(b_0)^2$ is not contained in I. Let $b_1 \in (b_0)^2 \setminus I$. Then $(b_1)^2$ is not contained in I and then choose $b_2 \in (b_1)^2 \setminus I$ and so on. We obtain a sequence of elements $b_0 = b, b_1, \dots,$ such that $b_{i+1} \in (b_i)^2$ $(i = 0, 1, \dots)$ and none of them belongs to I. According to Zorn's Lemma, there exists a maximal ideal K which contains I but does not contain any of the elements $b_0, b_1, \dots$. Now, mimicking the second part of the proof of the previous theorem, one may show that K is a prime ideal, and, moreover, $b \notin K$.
3. Assume now that the prime radical $P(H)$ of H is 0. Let $a \in H$ and $a \neq 0$. By the previous theorem, a is not a strictly Engel element. That means that there exists a sequence $a_0 = a, a_1, \dots,$ such that $a_{i+1} \in (a_i)^2$ $(i = 0, 1, \dots)$, and all a_i are distinct from 0. Hence, $(a)^2 \neq 0$. According to 1., H is semiprime.
Conversely, let H be semiprime, $a \in H$, and $a \neq 0$. According to 2., there exists a prime ideal $K \subseteq H$ such that $a \notin K$. Therefore, $a \notin P(H)$, and so, $P(H) = 0$. □

In Ω-homogroupoid setting, it is natural to observe solvability rather than nilpotency, since in the special case of Lie algebras, solvability and nilpotency do not coincide. V. A. Parfenov (Parfyonov) introduced *weakly solvable radicals* in the class of Lie algebras [30], since, at the time, it was not known whether the sum of locally solvable ideals is locally solvable. This question, known as the Amayo–Stewart problem, is later resolved in negative in [24] thus proving that the theory of locally solvable radicals of Lie algebras cannot be developed. Therefore, in [27], weakly solvable group-graded Ω-groups were introduced, and here we do similar for Ω-homogroupoids.

Definition 8.8. *An Ω-homogroupoid H is called* solvable of degree k *if there exists a natural number k such that $H_k = 0$, $H_{k-1} \neq 0$, where $H_0 = H$, $H_1 = [H_0, H_0], \dots, H_{i+1} = [H_i, H_i], \dots$.*

Definition 8.9. *An Ω-homogroupoid H is called* locally solvable *if every finitely generated subhomogroupoid is solvable.*

If H is an Ω-homogroupoid and $K \subseteq H$, then we define $K^{(0)} = K$, $K^{(k+1)} = \{\omega(a_1, \dots, a_n) \mid n \geq 2,\ a_1, \dots, a_n \in K^{(k)},\ \omega \in \Omega\}$, $k = 0, 1, \dots$.

Definition 8.10. *An Ω-homogroupoid H is called* weakly solvable *if for every finite subset K of H there exists a natural number k such that $K^{(k)} = 0$.*

We say that an Ω-homogroupoid H is with the *finiteness condition* if for every finite subset $K \subseteq H$ and for all $n \geq 2, k_1, \dots, k_n \in K, \omega \in \Omega$, we have that the set of elements $\omega(k_1, \dots, k_n)$ is finite. If H is an Ω-homogroupoid with the finiteness condition, then it is easy to prove that H is a weakly solvable Ω-homogroupoid if I and H/I are weakly solvable Ω-homogroupoids, where I is an ideal of H. It is also easy to prove that the sum of weakly solvable ideals of H is again a weakly solvable ideal. Now, one can prove that a class of weakly solvable Ω-homogroupoids with the finiteness condition is a radical class. Analogously to the case of group-graded Ω-groups in [27], we define the *upper weakly solvable radical* $T(H)$ of an Ω-homogroupoid H as the largest weakly solvable ideal of H, and we similarly construct the *lower weakly solvable radical* of an Ω-homogroupoid H with the finiteness condition. Namely, let $\rho(H)$ denote the sum of all ideals I of H such that $I^2 = 0$. The sum of two solvable ideals is solvable, and hence, $\rho(H)$ is locally solvable. For all ordinal numbers α, we define an ideal $\rho(\alpha)$, using transfinite induction in the following way:

1. $\rho(0) = 0$;
2. let $\rho(\alpha)$ be defined for all $\alpha < \beta$. Then:
   a) if $\beta = \lambda + 1$ is not a limit ordinal, then $\rho(\beta)$ is defined as an ideal of H such that $\rho(\beta)/\rho(\lambda) = \rho(H/\rho(\lambda))$;

b) if β is a limit ordinal, then we define $\rho(\beta) = \bigcup_{\lambda<\beta} \rho(\lambda)$.

There exists an ordinal β such that $\rho(\beta) = \rho(\beta + 1)$, and $\rho(\beta)$ is called the *lower weakly solvable radical* of an Ω-homogroupoid H with the finiteness condition. Next we give the relation between the prime radical and the lower weakly solvable radical which is an Ω-homogroupoid analogue of Theorem 3. from [27].

Theorem 8.11. *The prime radical of an Ω-homogroupoid H with the finiteness condition coincides with the lower weakly solvable radical.*

Proof. According to Theorem 8.7, the Ω-homogroupoid $H/P(H)$ is semiprime, and therefore, $\rho(H/P(H)) = \bar{0}$. This implies $\rho(H) \subseteq P(H)$. If we observe $H/\rho(H)$, we will obtain $\rho(2) \subseteq P(H)$. Continuing this process and using transfinite induction, we conclude that the lower weakly solvable radical is contained in $P(H)$.
The converse inclusion follows from the second part of the proof of Theorem 8.7. □

Corollary 8.12. *The prime radical $P(H)$ of an Ω-homogroupoid H with the finiteness condition is weakly solvable.*

The question of coincidence of lower and upper weakly solvable radical is of interest for particular classes of Ω-homogroupoids. At the end of this section, we will observe it in the case of anneids. Answering this question in the case of other particular classes of Ω-homogroupoids is left for future research (e.g. *Lie algebroids, Lie superalgebroids, special and generalized special Lie algebroids and superalgebroids* are yet to be defined and explored). Also, as in [27], we ask whether a prime radical is locally solvable.
We have the following theorem which gives us the connection between the upper weakly solvable radical and some prime ideals.

Theorem 8.13. *Let H be an Ω-homogroupoid with the finiteness condition and let $\{P_\alpha\}$ be the family of all prime ideals of H for which we have $T(H/P_\alpha) = \bar{0}$. Then $T(H)$ equals the intersection of all such prime ideals.*

Proof. We follow the arguments given for the proof of Theorem 4. in [27]. Let $P = \bigcap_{P_\alpha \in \{P_\alpha\}} P_\alpha$. Clearly, $T(H) \subseteq P$, since $T(H) \subseteq P_\alpha$, for all P_α. Let us now assume that $T(H) \subsetneq P$. Then there exists an element $b \in P$ such that $b \notin T(H)$. Hence, there exists a finite subset $X \subseteq (b)$ such that $X^{(k)} \nsubseteq T(H)$ $(k = 0, 1, \dots)$. Let us observe the set $\{Q_\beta\}$ of ideals of H such that

1. $T(H) \subseteq Q_\beta$;
2. $X^{(k)} \nsubseteq Q_\beta$ $(k = 0, 1, \dots)$.

Clearly, $T(H) \in \{Q_\beta\}$. Assume that there exists an infinite chain of ideals from $\{Q_\beta\}$ such that $Q_1 \subseteq Q_2 \subseteq \dots$. Then $Q = \bigcup_{i=1}^{\infty} Q_i$ does not contain $X^{(k)}$ $(k = 0, 1, \dots)$, since $X^{(k)}$ is finite for all $k = 0, 1, \dots$. According to Zorn's Lemma, there exists a maximal element $P_0 \in \{Q_\beta\}$. It is easily verified that H/P_0 is a prime Ω-homogroupoid. We claim that $T(H/P_0) = \bar{0}$. If $T(H/P_0) \neq \bar{0}$, then $P_0 \subsetneq f^{-1}(T(H/P_0))$, where f is the canonical homomorphism (in the sense of Definition 5.8). By the maximality of P_0, it follows that $X^{(l)} \subseteq f^{-1}(T(H/P_0))$. Hence, $f(X)^{(l)} \subseteq T(H/P_0)$. It follows that for some natural number $m > l$, we have $f(X)^{(m)} = \bar{0}$, i.e., $X^{(m)} \subseteq P_0$, a contradiction. We have constructed an ideal $P_0 \in \{Q_\beta\}$ which does not contain P, a contradiction. Therefore, $P = T(H)$. □

Theorem 8.14. *The prime radical of an Ω-homogroupoid H with the maximum condition on ideals is solvable.*

Proof. Let K be a maximal solvable ideal of H. It is the greatest solvable ideal since the sum of two solvable ideals is solvable. Also, an extension of a solvable Ω-homogroupoid by a solvable one is solvable. Hence, H/K contains no ideals I with $I^2 = 0$, and so, it is semiprime. Now, according to Theorem 8.7, $P(H) \subseteq K$. □

8.1. On prime radicals of associative anneids

Here we present some results regarding associative anneids, as special cases of Ω-homogroupoids.
An anneid A is said to be *Levitzki* or *locally nilpotent* if any finitely generated subanneid of A is nilpotent. It can be proved easily that the class of all locally nilpotent anneids is a radical class, called the *Levitzki radical*, in a universal class of all anneids. If we look at the notion of the weak solvability of an Ω-homogroupoid

in the case of an anneid, it comes clear that the weak solvability coincides with the local nilpotency, and thus, the upper weakly solvable radical and the Levitzki radical of an anneid coincide. Hence, according to Corollary 8.12, we have the following proposition, which gives a positive answer to a question whether a prime radical is locally solvable.

Proposition 8.15. *The prime radical of an anneid A is contained in the Levitzki radical of A.*

We may come to the same conclusion from the following results which give connection between the prime radical of an annied A and the prime radical of the linearization $\overline{A}$ thus generalizing results obtained for group-graded rings in [3] (see also [28]).

Lemma 8.16. *Let A be an anneid and $\overline{A}$ its linearization.*

i) *If I is a prime ideal of $\overline{A}$, then $I \cap A$ is a prime ideal of A.*

ii) *If J is a graded prime ideal of $\overline{A}$, then there exists a prime ideal I of $\overline{A}$ such that J is generated by $I \cap A$.*

iii) *The prime radical of A coincides with $P(\overline{A}) \cap A$, where $P(\overline{A})$ denotes the prime radical of $\overline{A}$ regarded as a ring.*

Proof. *i*) Let M, N be ideals of A such that $MN \subseteq I \cap A$. Then $\overline{M}\,\overline{N} \subseteq I$, and so, $\overline{M} \subseteq I$ or $\overline{N} \subseteq I$, i.e., $M \subseteq I \cap A$ or $N \subseteq I \cap A$.
ii) Let I be a maximal ideal with respect to the property that J is generated by $I \cap A$. That ideal exists according to Zorn's Lemma. It is enough now to prove that I is a prime ideal. As usual, let M, N be ideals of $\overline{A}$ such that $I \subsetneq M$, $I \subsetneq N$. But then, according to the choice of I, $J \subsetneq \bigoplus_{a \in A^*} M \cap A(a)$ and $J \subsetneq \bigoplus_{a \in A^*} N \cap A(a)$, and so, $\left(\bigoplus_{a \in A^*} M \cap A(a)\right)\left(\bigoplus_{a \in A^*} N \cap A(a)\right) \nsubseteq J$. Since the left-hand side of the previous expression is obviously contained in MN, we obtain $MN \nsubseteq I$, and so, I is a prime ideal.
iii) This is a corollary to *i*) and *ii*). □

In next theorems, we depart from the homogeneous aspect, since, in these cases, the assertions are more clear when stated from the nonhomogeneous aspect. Also, in the next theorem, P denotes the prime radical, while P^g denotes the graded prime radical.

Theorem 8.17. *Let $R = \bigoplus_{\delta \in \Delta} R_\delta$ be a graded ring. Then $P(R) \cap R_e = P^g(R) \cap R_e \subseteq P(R_e)$, for every idempotent $e \in \Delta^*$.*

Proof. Lemma 8.16*iii*) implies that $P(R) \cap R_e = P^g(R) \cap R_e$. Let $a \in P^g(R) \cap R_e$. It is enough to prove that a is a strongly nilpotent element in R_e (see e.g. [28]). Let $a_0 = a$, $a_{i+1} \in a_i R_e a_i$ $(i = 0, 1, \dots)$ be an arbitrary sequence in R_e. However, $a_0 \in R$, $a_{i+1} \in a_i R a_i$ $(i = 0, 1, \dots)$ and $a_0 = a \in P^g(R)$, and so, there exists i such that $a_i = a_{i+k} = 0$ $(k \geq 1)$. Hence, $a \in P(R_e)$. □

Let us now examine the homogeneity of the prime radical of a graded ring with the totally ordered grading set (see Section 7). We obtain analogous result to one from [9] which asserts that the prime radical of a u.p.-semigroup-graded ring is homogeneous.

Theorem 8.18. *Let $R = \bigoplus_{\delta \in \Delta} R_\delta$ be a regular proper graded ring with the totally ordered grading set Δ such that the order is compatible with the induced operation on Δ.*

i) *If R is not prime and if I is a nonzero prime ideal of R, then I contains a nonzero homogeneous ideal.*

ii) *All minimal prime ideals of R are homogeneous.*

Hence, the prime radical of R is homogeneous.

Proof. *i*) Since R is not prime, there exist nonzero elements $a = a_{\xi_1} + \cdots + a_{\xi_n}$ and $b = b_{\eta_1} + \cdots + b_{\eta_m}$ with $a_{\xi_i} \in R_{\xi_i}$ and $b_{\eta_j} \in R_{\eta_j}$, $i \in \{1, \ldots, n\}$, $j = \{1, \ldots, m\}$, respectively, such that $aRb = 0$. Let $\delta \in \Delta$ be arbitrary. If $r \in R_\delta$, then $arb = a_{\xi_1} rb_{\eta_1} + \cdots + a_{\xi_1} rb_{\eta_m} + \cdots + a_{\xi_n} rb_{\eta_1} + \cdots + a_{\xi_n} rb_{\eta_m}$. For those $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ for which $R_{\xi_i} R_\delta R_{\eta_j} \neq 0$, we have $(\xi_i \delta)\eta_j = \xi_i(\delta\eta_j)$ and $\xi_i\delta\eta_j$ is uniquely presented element of Δ under our assumptions on R and its grading set. Hence, for all such i, j, $a_{\xi_i} R_\delta b_{\eta_j} = 0$, and therefore, $a_{\xi_i} R_\delta b_{\eta_j} = 0$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Element δ was arbitrary, so $a_{\xi_i} Rb_{\eta_j} = 0$ for all i and j, and particularly, $a_{\xi_n} Rb_{\eta_m} = 0 \subseteq I$, and since I is prime, it follows that $a_{\xi_n}, b_{\eta_m} \in I$. Therefore, I contains a nonzero homogeneous ideal.
ii) If I is a minimal prime ideal of R, and J the homogeneous ideal $\bigoplus_{\delta\in\Delta} I \cap R_\delta$, then if $J \neq I$, it suffices to apply *i*) to R/J and I/J in order to conclude that J must be equal to I (cf. [9]). □

The lower and upper weakly solvable radical of an anneid A generally do not coincide, since from the ordinary ring theory, we know that there exists a ring whose prime and Levitzki radical do not coincide (see e.g. [5]), and every ring is an anneid.
However, there are some classes of anneids for which we have coincidence of the upper and lower weakly solvable radicals, just like in the case of ordinary rings. The following result is just the corollary to Theorem 8.14, since, it may be proved by standard means that if an anneid A satisfies the ascending chain condition for right annihilators $ann(a) = \{x \in A \mid ax = 0\}$ $(a \in A)$, then any left or right nil ideal is contained in the prime radical of A (see e.g. [23]).

Theorem 8.19. *If A is a Noetherian anneid, i.e., an anneid with maximum condition on ideals [2, 20], then the prime radical of A coincides with the Levitzki radical of A.*

Finally, we observe one particular example of an anneid which comes from the notion of an *incidence ring of group automata* defined by A. V. Kelarev in [14]. We use notation presented in [15].
A *group automation* is an algebraic system $\mathcal{A} = (X, G, \delta)$, where

1. X is a nonempty set of *states*;
2. G is a group of *input symbols*;
3. $\delta : X \times G \to X$, given by $\delta(x, g) = xg$ is a *transition function* satisfying the equality $x(gh) = (xg)h$ for all $x \in X$, $g, h \in G$.

If F is a ring with 1, then the *incidence ring* of the automation $\mathcal{A} = (X, G, \delta)$ over F is the ring $I_{\mathcal{A}} = I_{\mathcal{A}}(F)$ that is spanned as a free left F-module by the set $T_{\mathcal{A}} = \{(x, g, xg) \mid x \in X, g \in G\}$ with multiplication defined by the distributive law and the rules

$$\begin{aligned} (x_1, g_1, x_1g_1) \cdot (x_2, g_2, x_2g_2) &= \begin{cases} (x_1, g_1g_2, x_1g_1g_2), & \text{if } x_1g_1 = x_2, \\ 0, & \text{otherwise}, \end{cases} \\ (x, g, xg) \cdot r &= r \cdot (x, g, xg), \end{aligned}$$

for all $x, x_1, x_2 \in X$, $g, g_1, g_2 \in G$, $r \in F$.
Clearly, $T^0_{\mathcal{A}} = T_{\mathcal{A}} \cup 0$ is a semigroup, and $I_{\mathcal{A}}$ is a $T^0_{\mathcal{A}}$-graded ring, since $I_{\mathcal{A}}$ is isomorphic to the contracted semigroup ring $F_0[T^0_{\mathcal{A}}]$ (see [12, 13]). Moreover, this graded ring is regular (see Section 7). Denote by $A_{\mathcal{A}}$ the corresponding regular anneid.

Theorem 8.20. *If F is a field, and if both X and G are finite, then the prime radical and the Levitzki radical of $A_{\mathcal{A}}$ coincide.*

Proof. According to Corollary 3.1(ii) of [15], if F is a field, then $I_{\mathcal{A}}$ is right or left Artinian if and only if both X and G are finite. Hence, $I_{\mathcal{A}}$ is right or left Artinian, which implies that $A_{\mathcal{A}}$ is also right or left Artinian, i.e., it satisfies the descending chain condition on right or left ideals. In that case, since $A_{\mathcal{A}}$ is regular, the Jacobson radical $J(A_{\mathcal{A}})$ is nilpotent, according to [6, 7]. It follows that $J(A_{\mathcal{A}})$ is contained in the prime radical $P(A_{\mathcal{A}})$ of $A_{\mathcal{A}}$. On the other hand, $P(A_{\mathcal{A}})$ is contained in the Levitzki radical of $A_{\mathcal{A}}$, which is contained in the nil radical of $A_{\mathcal{A}}$, which is contained in $J(A_{\mathcal{A}})$, according to [7], and so, the prime radical and the Levitzki radical of $A_{\mathcal{A}}$ coincide. □

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