



Polynomially Riesz Perturbations II

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Abstract. In this paper we investigate perturbations by holomorphically and polynomially Riesz operators concerning the sets of upper and lower semi-Fredholm operators, the sets of left and right Fredholm operators, as well as the sets of upper and lower semi-Browder operators and the sets of left and right Browder operators. We consider in particular perturbations of shifts by polynomially Riesz operators.

1. Introduction and Preliminaries

Let \mathbb{C} denote the set of all complex numbers and let X be an infinite dimensional Banach space. We denote by $B(X)$ the set of all linear bounded operators on X . For $A \in B(X)$, let $R(A)$ and $N(A)$ denote the range and the null-space of A , respectively, and let $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim X/R(A) = \text{codim } R(A)$. The sets of *upper and lower semi-Fredholm* operators, respectively, are defined as $\Phi_+(X) = \{A \in B(X) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed}\}$, and $\Phi_-(X) = \{A \in B(X) : \beta(A) < \infty\}$. Set $\Phi_{\pm}(X) = \Phi_+(X) \cup \Phi_-(X)$ for the set of semi-Fredholm operators and $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ for the set of Fredholm operators. For $A \in \Phi_{\pm}(X)$ the index is defined by $i(A) = \alpha(A) - \beta(A)$. The Calkin algebra over X is the quotient algebra $C(X) = B(X)/K(X)$ and let $\Pi : B(X) \rightarrow C(X)$ denote the natural homomorphism. It is well known that $A \in \Phi(X)$ if and only if $\Pi(A)$ is invertible in $C(X)$ [5, Theorem 3.2.8].

An operator $A \in B(X)$ is left Fredholm, if $\alpha(A) < \infty$ and $R(A)$ is a closed and complemented subspace of X , while A is right Fredholm if $\beta(A) < \infty$ and $N(A)$ is a complemented subspace of X . Set $\Phi_l(X)$ for the set of all left Fredholm operators on X , and $\Phi_r(X)$ for the set of all right Fredholm operators on X .

For $A \in B(X)$ denote by $\text{asc}(A)$ ($\text{dsc}(A)$) the ascent (the descent) of $A \in B(X)$, i.e. the smallest non-negative integer n such that $N(A^n) = N(A^{n+1})$ ($R(A^n) = R(A^{n+1})$). If such n does not exist, then $\text{asc}(A) = \infty$ ($\text{dsc}(A) = \infty$). An operator $A \in B(X)$ is upper semi-Browder if it is upper semi-Fredholm with finite ascent, while A is lower semi-Browder if it is lower semi-Fredholm with finite descent. Let $\mathcal{B}_+(X)$ ($\mathcal{B}_-(X)$) denote the set of all upper (lower) semi-Browder operators. We shall say that an operator $A \in B(X)$ is left Browder, if it is left Fredholm with finite ascent, while A is right Browder, if it is right Fredholm with finite descent. Let $\mathcal{B}_l(X)$ ($\mathcal{B}_r(X)$) denote the set of all left (right) Browder operators. An operator $A \in B(X)$ is Browder if it is Fredholm with finite ascent and finite descent. The set of all Browder operators on X is denoted by $\mathcal{B}(X)$. Clearly, $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X) = \mathcal{B}_l(X) \cap \mathcal{B}_r(X)$.

2010 *Mathematics Subject Classification.* 47A53; 47A55.

Keywords. upper and lower semi-Fredholm operator; left and right Fredholm and Browder operator; polynomially Riesz operator; perturbation.

Received: 05 February 2014; Accepted: 14 February 2014

Communicated by Dragan S. Djordjević

Research supported by the Ministry of Education and Science, Republic of Serbia, grant no. 174007.

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For $A, B \in B(X)$ recall that ([3, Theorem 1.46], [3, Theorem 1.54 (ii)])

$$BA \in \Phi_*(X) \implies A \in \Phi_*(X), \text{ for } * = +, l, \quad (1)$$

$$BA \in \Phi_*(X) \implies B \in \Phi_*(X), \text{ for } * = -, r.$$

Recall that for $A, B \in B(X)$ ([10, Theorem 7.10.2], [19, Lemma 2.5])

$$AB = BA, AB \in \mathcal{B}_*(X) \implies A, B \in \mathcal{B}_*(X), \quad (2)$$

for $* = +, -, l, r$.

An operator $A \in B(X)$ is Riesz, $A \in R(X)$, if $\{\lambda \in \mathbb{C} : A - \lambda \in \Phi(X)\} = \mathbb{C} \setminus \{0\}$. It is known for $A, B \in B(X)$ ([2, Theorem 3.112]):

$$\begin{aligned} A, B \in R(X), AB = BA &\implies A + B \in R(X), \\ B \in R(X), AB = BA &\implies AB \in R(X). \end{aligned} \quad (3)$$

For $K \subset \mathbb{C}$, ∂K denotes the topological boundary of K , $\text{iso } K$ denotes the set of the isolated points of K and $\text{acc } K$ denotes the set of all accumulation points of K .

The spectrum of $A \in B(X)$ is

$$\sigma(A) = \sigma_l(A) \cup \sigma_r(A),$$

where

$$\begin{aligned} \sigma_l(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not left invertible}\}, \\ \sigma_r(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not right invertible}\}, \end{aligned}$$

are the left and the right spectrum of A , respectively. It is well-known that

$$\partial\sigma(A) \subset \sigma_l(A) \cap \sigma_r(A). \quad (4)$$

For $H = \Phi_+, \Phi_-, \Phi_l, \Phi_r, \Phi, \mathcal{B}_+, \mathcal{B}_-, \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}$, the corresponding spectrum of $A \in B(X)$ is defined by

$$\sigma_H(A) = \{\lambda : A - \lambda \notin H(X)\}.$$

From the punctured neighborhood theorem ([14, Theorem 18.7], [10, Theorem 7.8.5]) it follows for $* = l, r$

$$\partial\sigma_*(A) \cap (\mathbb{C} \setminus \sigma_*(A)) \subset \text{iso } \sigma_*(A),$$

and therefore,

$$\partial\sigma_*(A) \subset \sigma_*(A) \cup \text{iso } \sigma_*(A), \quad (5)$$

and analogously for the two-sided spectrum:

$$\partial\sigma(A) \subset \sigma_\Phi(A) \cup \text{iso } \sigma(A). \quad (6)$$

Recall that ([14, Corollary 20.20]),

$$\sigma_{\mathcal{B}}(A) = \sigma_\Phi(A) \cup \text{acc } \sigma(A), \quad (7)$$

and for $* = +, -, l, r$ (see [14, Corollary 20.20], [18, Theorems 5, 6 and the comment after Theorem 6])

$$\sigma_{\mathcal{B}_*}(A) = \sigma_\Phi(A) \cup \text{acc } \sigma_*(A). \quad (8)$$

It is known that ([13, Theorem 7])

$$\partial\sigma_{\Phi}(A) \subset \sigma_{\Phi_+}(A) \cap \sigma_{\Phi_-}(A). \quad (9)$$

Let S be a subset of a Banach space A . The perturbation class of S , denoted by $\text{Ptrb}(S)$, is the set

$$\text{Ptrb}(S) = \{a \in A : a + s \in S \text{ for every } s \in S\}.$$

Recall that ([11, Theorem 2.7], [6, Chapter 5.2, Corollary 3]),

$$\text{Ptrb}(\Phi_l(X)) = \text{Ptrb}(\Phi(X)) = \text{Ptrb}(\Phi_r(X)), \quad (10)$$

and $\text{Ptrb}(\Phi_+(X)) \cup \text{Ptrb}(\Phi_-(X)) \subset \text{Ptrb}(\Phi(X))$ ([5, Theorem 5.6.9]). Also, it is known that $\text{Ptrb}(\Phi(X))$, $\text{Ptrb}(\Phi_+(X))$ and $\text{Ptrb}(\Phi_-(X))$ are closed two-sided ideals [11, Theorem 2.4].

We recall the following result:

Theorem 1.1. [17, Corollary 2] *Let $A \in B(X)$ and let $D \in R(X)$. Then*

(i) *If $A \in \Phi_+(X)$ and $AD - DA \in \text{Ptrb}(\Phi_+(X))$, then $A - D \in \Phi_+(X)$ and $i(A) = i(A - D)$.*

(ii) *If $A \in \Phi_-(X)$ and $AD - DA \in \text{Ptrb}(\Phi_-(X))$, then $A - D \in \Phi_-(X)$ and $i(A) = i(A - D)$.*

From [18, Theorem 8] and the local constancy of the index it follows:

Theorem 1.2. *Let $A \in B(X)$ and let $D \in R(X)$.*

(i) *If $A \in \Phi_l(X)$ and $AD - DA \in \text{Ptrb}(\Phi(X))$, then $A - D \in \Phi_l(X)$ and $i(A) = i(A - D)$.*

(ii) *If $A \in \Phi_r(X)$ and $AD - DA \in \text{Ptrb}(\Phi(X))$, then $A - D \in \Phi_r(X)$ and $i(A) = i(A - D)$.*

V. Rakočević proved the stability of the upper and lower semi-Browder operators under commuting Riesz perturbations ([16, Corollary 2]):

Theorem 1.3. *Let $A \in B(X)$ and let $D \in R(X)$.*

(i) *If $A \in \mathcal{B}_+(X)$ and $AD = DA$, then $A - D \in \mathcal{B}_+(X)$.*

(ii) *If $A \in \mathcal{B}_-(X)$ and $AD = DA$, then $A - D \in \mathcal{B}_-(X)$.*

We need the following result which follows immediately from [18, Theorem 7]:

Theorem 1.4. *Let $A \in B(X)$ and let $D \in R(X)$.*

(i) *If $A \in \mathcal{B}_l(X)$ and $AD = DA$, then $A - D \in \mathcal{B}_l(X)$.*

(ii) *If $A \in \mathcal{B}_r(X)$ and $AD = DA$, then $A - D \in \mathcal{B}_r(X)$.*

Let A be a complex Banach algebra. We say that $S \subseteq A$ is a *commutative ideal* if

$$S +_{\text{comm}} S \subseteq S, \quad A \cdot_{\text{comm}} S \subseteq S.$$

where we write

$$H +_{\text{comm}} K = \{c + d : (c, d) \in H \times K, cd = dc\}$$

for the commuting sum and

$$H \cdot_{\text{comm}} K = \{c \cdot d : (c, d) \in H \times K, cd = dc\}$$

for the commuting product of subsets $H, K \subseteq A$.

From (3) it follows that the set $R(X)$ is a commutative ideal in $B(X)$.

If $S \subseteq A$ is an arbitrary set we shall write that $a \in \text{Poly}^{-1}(S)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in S$. If $S \subseteq A$ is a commutative ideal, the set

$$\mathcal{P}_a^S = \{p \in \text{Poly} : p(a) \in S\}$$

of polynomials p for which $p(a) \in S$ is an ideal of the algebra Poly . There exists a unique polynomial p of minimal degree with leading coefficient 1 contained in \mathcal{P}_a^S which we call the minimal polynomial of a ; we shall write $p = \pi_a \equiv \pi_a^S$. Then \mathcal{P}_a^S is generated by $p = \pi_a$, i.e. $\mathcal{P}_a^S = \pi_a \cdot \text{Poly}$.

An operator $A \in B(X)$ is called *polynomially Riesz* if $A \in \text{Poly}^{-1}(R(X))$.

We recall the following result [20, Theorem 11.1], [22, Theorem 2.3].

Theorem 1.5. *If $A \in B(X)$ is polynomially Riesz, then*

$$\sigma_\Phi(A) = \sigma_B(A) = \pi_A^{-1}(0).$$

If $K \subset \mathbb{C}$ is compact, then we write $f \in \text{Holo}(K)$ if f is a complex function which is holomorphic in a neighbourhood of K . We write $\text{Holo}_1(K) \subseteq \text{Holo}(K)$ for those holomorphic functions $g : U \rightarrow \mathbb{C}$ which are non constant on each connected component of open $U \supseteq K$.

Recall that for $A \in B(X)$, $f \in \text{Holo}(\sigma(A))$ and $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \Phi, \mathcal{B}_+, \mathcal{B}_l, \mathcal{B}_-, \mathcal{B}_r, \mathcal{B}$ ([8], [15, Theorem 3.4], [14, Corollary 22.8 (i) and the comment after Lemma 22.9], [19, Theorem 2.6])

$$f(\sigma_H(A)) = \sigma_H(f(A)). \quad (11)$$

We shall say that an operator $A \in B(X)$ is *holomorphically Riesz* if there exists an $f \in \text{Holo}(\sigma(A))$ such that $f(A)$ is Riesz. From [20, Theorem 12.1, the incusions (12.3)] it follows that A is polynomially Riesz if and only if $f(A) \in R(X)$ for some $f \in \text{Holo}_1(\sigma(A))$ (see also [12, Lemma 2.2]). So, the concept of holomorphically Riesz operator is a little more general than the concept of polynomially Riesz operator.

Our discussion about perturbation by polynomially Riesz elements of a Banach algebra concerning the set of (left, right) Fredholm elements was started in [20] (Theorems 11.2 and 12.3). This discussion is continued in [21] where we focused on the Banach algebra $B(X)$ and investigated perturbations of left(right) Fredholm, Weyl and Browder operators by polynomially Riesz operators using the concept of communicating operators, and, specially, perturbations of some shifts were considered. In this paper we investigate perturbations by holomorphically Riesz operators, and also by polynomially Riesz operators, concerning the sets of upper and lower semi-Fredholm and semi-Browder operators as well as the sets of left and right Fredholm and Browder operators. Our main result is Theorem 2.1 where we prove that if $A, D \in B(X)$ such that $AD - DA \in \text{Ptrb}(\Phi_+(X))$ ($AD - DA \in \text{Ptrb}(\Phi_-(X))$), and if there exists $f \in \text{Holo}(\sigma(A) \cup \sigma(D))$ such that $f(D)$ is Riesz and $f(A)$ is upper (lower) semi-Fredholm, then $A - D$ is upper (lower) semi-Fredholm. The similar result for left and right Fredholm operators is obtained. Under the hypothesis that $A, D \in B(X)$, $AD = DA$, $f(D)$ is Riesz and $f(A)$ is upper (lower) semi-Browder we get that $A - D$ is upper (lower) semi-Browder, and similarly the assertion holds if the set of upper (lower) semi-Browder operators is replaced by the set of left (right) Browder operators. We apply these results to investigate perturbations some shifts by polynomially Riesz operators. In that way we continue our discussion started in [21], where we considered perturbation of shifts by a polynomially Riesz operator such that zeros of its minimal polynomial are contained in the open unit ball $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. In the present paper we investigate perturbation of shifts by a polynomially Riesz operator such that zeros of its minimal polynomial are contained in the complement of the unit sphere $\mathcal{S} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

This paper is divided into three sections. Section 2 contains our main results about perturbations by holomorphically Riesz operators. As consequences the results concerning perturbations by polynomially Riesz operators are obtained. Section 3 is devoted to applying of the results obtained in Section 2 in order to get results about perturbations of some shifts by polynomially Riesz operators.

2. Perturbations

The following theorem refers to perturbations by holomorphically Riesz operators.

Theorem 2.1. *Let $A, D \in B(X)$ and $f \in \text{Holo}(\sigma(A) \cup \sigma(D))$.*

(i) *If $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \Phi$, then*

$$AD - DA \in \text{Ptrb}(H(X)) \quad \text{and} \quad f(D) \in R(X)$$

implies

$$f(A) \in H(X) \implies A - D \in H(X).$$

(ii) *If $H = \mathcal{B}_+, \mathcal{B}_l, \mathcal{B}_-, \mathcal{B}_r, \mathcal{B}$, then*

$$AD = DA \quad \text{and} \quad f(D) \in R(X)$$

implies

$$f(A) \in H(X) \implies A - D \in H(X).$$

Proof. Let f be a nonzero holomorphic function in a neighbourhood U of $\sigma(A) \cup \sigma(D)$. If Ω is an open set such that $\sigma(A) \cup \sigma(D) \subset \Omega \subset \overline{\Omega} \subset U$ and whose boundary $\partial\Omega$ consists of a finite numbers of simple closed rectifiable curves which do not intersect, then $f(A) = \frac{1}{2\pi i} \int_{\partial\Omega} (\lambda - A)^{-1} f(\lambda) d\lambda$ and $f(D) = \frac{1}{2\pi i} \int_{\partial\Omega} (\lambda - D)^{-1} f(\lambda) d\lambda$.

To prove (i) suppose that $f(D) \in R(X)$, $AD - DA \in \text{Ptrb}(H(X))$ and $f(A) \in H(X)$ where $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \Phi$. Since $\text{Ptrb}(H(X))$ is a two-sided ideal it follows that

$$(\lambda - A)^{-1}(\mu - D)^{-1} - (\mu - D)^{-1}(\lambda - A)^{-1} \in \text{Ptrb}(H(X)), \quad \lambda, \mu \in \partial\Omega. \quad (12)$$

As $\text{Ptrb}(H(X))$ is closed, from (12) we conclude that

$$f(A)f(D) - f(D)f(A) \in \text{Ptrb}(H(X)),$$

and by Theorem 1.1 and Theorem 1.2 it follows that $f(A) - f(D) \in H(X)$. Using again the fact that $\text{Ptrb}(H(X))$ is a two-sided ideal we get that for every $\lambda \in \partial\Omega$,

$$\begin{aligned} (\lambda - A)^{-1} - (\lambda - D)^{-1} &= (\lambda - A)^{-1}(A - D)(\lambda - D)^{-1} \\ &= (\lambda - A)^{-1}(\lambda - D)^{-1}(A - D) + P_1(\lambda) \\ &= (A - D)(\lambda - A)^{-1}(\lambda - D)^{-1} + P_2(\lambda), \end{aligned}$$

where $P_1(\lambda), P_2(\lambda) \in \text{Ptrb}(H(X))$, and therefore, from the fact that $\text{Ptrb}(H(X))$ is closed we get

$$\begin{aligned} f(A) - f(D) &= \frac{1}{2\pi i} \int_{\partial\Omega} ((\lambda - A)^{-1} - (\lambda - D)^{-1}) f(\lambda) d\lambda \\ &= A_1(A - D) + B_1 = (A - D)A_1 + B_2, \end{aligned}$$

where $A_1 = \frac{1}{2\pi i} \int_{\partial\Omega} ((\lambda - A)^{-1}(\lambda - D)^{-1}) f(\lambda) d\lambda \in A$ and

$$\begin{aligned} B_1 &= \frac{1}{2\pi i} \int_{\partial\Omega} P_1(\lambda) f(\lambda) d\lambda \in \text{Ptrb}(H(X)), \\ B_2 &= \frac{1}{2\pi i} \int_{\partial\Omega} P_2(\lambda) f(\lambda) d\lambda \in \text{Ptrb}(H(X)). \end{aligned}$$

Consequently, $A_1(A - D), (A - D)A_1 \in H(X)$ and from (1) it follows that $A - D \in H(X)$.

To prove (ii) suppose that $f(D) \in R(X)$, $AD = DA$ and $f(A) \in H(X)$ for $H = \mathcal{B}_+, \mathcal{B}_l, \mathcal{B}_-, \mathcal{B}_r, \mathcal{B}$. Then $(\lambda - A)^{-1}$ and $(\mu - D)^{-1}$ commute for every $\lambda, \mu \in \partial\Omega$, and hence, $f(A)f(D) = f(D)f(A)$. By Theorem 1.3 and Theorem 1.4, it follows that $f(A) - f(D) \in H(X)$. For every $\lambda \in \partial\Omega$, we have

$$\begin{aligned} (\lambda - A)^{-1} - (\lambda - D)^{-1} &= (\lambda - A)^{-1}(A - D)(\lambda - D)^{-1} \\ &= (\lambda - A)^{-1}(\lambda - D)^{-1}(A - D) \\ &= (A - D)(\lambda - A)^{-1}(\lambda - D)^{-1}, \end{aligned}$$

and therefore, we get

$$\begin{aligned} f(A) - f(D) &= \frac{1}{2\pi i} \int_{\partial\Omega} ((\lambda - A)^{-1} - (\lambda - D)^{-1})f(\lambda)d\lambda \\ &= A_1(A - D) = (A - D)A_1, \end{aligned}$$

where $A_1 = \frac{1}{2\pi i} \int_{\partial\Omega} ((\lambda - A)^{-1}(\lambda - D)^{-1})f(\lambda)d\lambda \in A$. From $A_1(A - D) = (A - D)A_1 \in H(X)$ according to (2) it follows that $A - D \in H(X)$. \square

Theorem 2.2. Let $A, D \in B(X)$ and $f \in \text{Holo}(\sigma(A) \cup \sigma(D))$.

(i) If $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \Phi$, then

$$AD - DA \in \text{Ptrb}(H(X)) \quad \text{and} \quad f(D) \in R(X)$$

implies

$$f^{-1}(0) \cap \sigma_H(A) = \emptyset \implies A - D \in H(X).$$

(ii) If $H = \mathcal{B}_+, \mathcal{B}_l, \mathcal{B}_-, \mathcal{B}_r, \mathcal{B}$, then

$$AD = DA \quad \text{and} \quad f(D) \in R(X)$$

implies

$$f^{-1}(0) \cap \sigma_H(A) = \emptyset \implies A - D \in H(X).$$

Proof. (i) Let $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \Phi$. Suppose that $AD - DA \in \text{Ptrb}(H(X))$, $f(D) \in R(X)$ and let $\sigma_H(A) \cap f^{-1}(0) = \emptyset$. Then $0 \notin f(\sigma_H(A))$ and from (11) it follows $0 \notin \sigma_H(f(A))$, that is $f(A) \in H(X)$. From Theorem 2.1 it follows that $A - D \in H(X)$.

(ii) Follows from Theorem 2.1 (ii) and (11), analogously to the proof of (i). \square

The following theorem refers to perturbations by polynomially Riesz operators.

Theorem 2.3. Let $A, D \in B(X)$.

(i) If $H = \Phi_+, \Phi_l, \Phi_-, \Phi_r, \Phi$, then

$$AD - DA \in \text{Ptrb}(H(X)) \quad \text{and} \quad D \in \text{Poly}^{-1}R(X)$$

implies

$$\sigma_H(A) \cap \pi_D^{-1}(0) = \emptyset \implies A - D \in H(X).$$

(ii) If $H = \mathcal{B}_+, \mathcal{B}_l, \mathcal{B}_-, \mathcal{B}_r, \mathcal{B}$, then

$$AD = DA \quad \text{and} \quad D \in \text{Poly}^{-1}R(X)$$

implies

$$\sigma_H(A) \cap \pi_D^{-1}(0) = \emptyset \implies A - D \in H(X).$$

Proof. Follows from Theorem 2.2. \square

Theorem 2.4. Let $A, D \in B(X)$.

(i) Then

$$AD - DA \in \text{Ptrb}(\Phi(X)) \quad \text{and} \quad A, D \in \text{Poly}^{-1}R(X)$$

implies

$$\pi_A^{-1}(0) \cap \pi_D^{-1}(0) = \emptyset \implies A - D \in \Phi(X).$$

(ii) Then

$$AD = DA \quad \text{and} \quad A, D \in \text{Poly}^{-1}R(X)$$

implies

$$\pi_A^{-1}(0) \cap \pi_D^{-1}(0) = \emptyset \implies A - D \in \mathcal{B}(X).$$

Proof. Follows from Theorem 2.3, since from $A \in \text{Poly}^{-1}R(X)$ we have $\sigma_\Phi(A) = \sigma_{\mathcal{B}}(A) = \pi_A^{-1}(0)$ by Theorem 1.5. \square

3. Application to Shifts

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathbb{C}^{\mathbb{N}_0}$ be the linear space of all complex sequences $x = (x_k)_{k=0}^\infty$. Let ℓ_∞ , c and c_0 denote the set of bounded, convergent and convergent sequences with null limit. We write $\ell_p = \{x \in \mathbb{C}^{\mathbb{N}_0} : \sum_{k=0}^\infty |x_k|^p < \infty\}$ for $1 \leq p < \infty$. For $n = 0, 1, 2, \dots$, let $e^{(n)}$ denote the sequences such that $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. The forward and the backward unilateral shifts U and V are linear operators on $\mathbb{C}^{\mathbb{N}_0}$ defined by

$$Ue^{(n)} = e^{(n+1)} \quad \text{and} \quad Ve^{(n+1)} = e^{(n)}, \quad n = 0, 1, 2, \dots$$

Invariant subspaces for U and V include c_0 , c , ℓ_∞ and ℓ_p , $p \geq 1$. Recall that for each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $1 \leq p < \infty$, $U, V \in B(X)$ and

$$\|U\| = \|V\| = 1. \tag{13}$$

On the Hilbert space ℓ_2 we also have that $V = U^*$.

We shall write $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\mathbb{S} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Lemma 3.1. If $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, then for the forward unilateral shift $U \in B(X)$ there are the equalities

$$\sigma_l(U) = \mathbb{S}, \tag{14}$$

$$\sigma_r(U) = \mathbb{D} \tag{15}$$

and

$$\sigma(U) = \mathbb{D} \tag{16}$$

Proof. From (13) it follows that

$$\sigma_r(U) \subset \sigma(U) \subset \mathbb{D}. \tag{17}$$

Since U is not surjective, it follows that $0 \in \sigma_r(U)$. Suppose that $\lambda \in \mathbb{C}$ and $0 < |\lambda| < 1$. We show that $e_0 \notin R(\lambda - U)$. If there exists $x = (x_k)_{k=0}^\infty$ such that $(\lambda - U)x = e_0$, then

$$(\lambda x_0, \lambda x_1 - x_0, \lambda x_2 - x_1, \dots) = (1, 0, 0, \dots)$$

and hence

$$x = \left(\frac{1}{\lambda}, \frac{1}{\lambda^2}, \frac{1}{\lambda^3}, \dots\right),$$

which is not a bounded sequence and so, it is not in X . Therefore, $\lambda \in \sigma_r(U)$ and hence, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_r(U)$. Since $\sigma_r(U)$ is closed, it follows that

$$\mathbb{D} \subset \sigma_r(U). \quad (18)$$

From (17) and (18) we get $\sigma_r(U) = \sigma(U) = \mathbb{D}$.

For $\lambda \neq 0$, $|\lambda| < 1$, since $|\lambda|^{-1} > 1 = \|V\|$ it follows that $V - \lambda^{-1}I \in B(X)^{-1}$. From

$$V(\lambda - U) = \lambda V - I = \lambda(V - \lambda^{-1}I)$$

we conclude that $\lambda - U \in B(X)_I^{-1}$, and so $\lambda \notin \sigma_l(U)$. Therefore, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \cap \sigma_l(U) = \emptyset$, and from $\sigma_l(U) \subset \sigma(U) = \mathbb{D}$ it follows that $\sigma_l(U) \subset \mathbb{S}$. From (4) we get the opposite inclusion $\mathbb{S} = \partial\sigma(U) \subset \sigma_l(U)$, and therefore $\sigma_l(U) = \mathbb{S}$. \square

Lemma 3.2. *If $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, then for the backward unilateral shift $V \in B(X)$ there are the equalities*

$$\sigma_l(V) = \mathbb{D}, \quad (19)$$

$$\sigma_r(V) = \mathbb{S} \quad (20)$$

and

$$\sigma(V) = \mathbb{D} \quad (21)$$

Proof. From (13) it follows that

$$\sigma_l(V) \subset \sigma(V) \subset \mathbb{D}. \quad (22)$$

Since $V(I - UV) = 0 \neq I - UV$, then

$$N(V) = (I - UV)X \neq \{0\}. \quad (23)$$

From (13) it is clear that

$$|\lambda| < 1 \implies I - \lambda U \in B(X)^{-1}. \quad (24)$$

Since $V - \lambda = V - \lambda VU = V(I - \lambda U)$, from (23) and (24) it follows

$$N(V - \lambda) = (V - \lambda)^{-1}(0) = (I - \lambda U)^{-1}V^{-1}(0) \neq \{0\}, \quad (25)$$

and hence, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_l(V)$. Now, since $\sigma_l(V)$ is closed we obtain

$$\mathbb{D} \subset \sigma_l(V). \quad (26)$$

From (22) and (26) it follows (19) and (21).

From $VU = I$ it follows that $0 \notin \sigma_r(V)$. For $\lambda \neq 0$, $|\lambda| < 1$, since $|\lambda|^{-1} > 1 = \|U\|$ it follows that $\lambda^{-1} - U \in B(X)^{-1}$. From

$$V - \lambda = V - \lambda VU = \lambda V(\lambda^{-1} - U)$$

we conclude that $V - \lambda \in B(X)_r^{-1}$, and so $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \cap \sigma_r(V) = \emptyset$ which together with $\sigma_r(V) \subset \sigma(V) = \mathbb{D}$ implies $\sigma_r(V) \subset \mathbb{S}$. From (4) we get $\mathbb{S} = \partial\sigma(V) \subset \sigma_r(V)$, and hence, $\sigma_r(V) = \mathbb{S}$. \square

For $X = \ell_2$ it is known that $\sigma_\Phi(U) = \mathbb{S}$ [4, Example 1.2], [7, Proposition 27.7(b)]. In [1, Remark 2.9] it was shown that $\sigma_\Phi(V) = \mathbb{S}$ for $X = \ell_p$, $p \geq 1$. This equality holds also if $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$:

Lemma 3.3. ([21, Theorem 3.2]) *For each $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, and the forward and backward unilateral shifts $U, V \in B(X)$ there are equalities*

$$\sigma_\Phi(U) = \sigma_\Phi(V) = \mathbb{S}. \quad (27)$$

Now we find the left and right Fredholm spectrum, the left, right and two-sided Browder spectrum of the forward unilateral shift, and also of the backward unilateral shift.

Theorem 3.4. *If $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, then for the forward unilateral shift $U \in B(X)$ there are the equalities*

$$\sigma_{\Phi_l}(U) = \sigma_{\mathcal{B}_l}(U) = \mathfrak{S}, \quad (28)$$

and

$$\mathfrak{S} = \sigma_{\Phi_r}(U) \subset \sigma_{\mathcal{B}_r}(U) = \mathbb{D}. \quad (29)$$

Proof. From (14) and (5) it follows that $\mathfrak{S} = \partial\sigma_l(U) \subset \sigma_{\Phi_l}(U) \subset \sigma_{\mathcal{B}_l}(U) \subset \sigma_l(U) = \mathfrak{S}$ and we get (28).

From (8) and (15) it follows that $\mathbb{D} = \text{acc } \sigma_r(U) \subset \sigma_{\mathcal{B}_r}(U) \subset \sigma_r(U) = \mathbb{D}$, and hence, $\sigma_{\mathcal{B}_r}(U) = \mathbb{D}$. According to (5) and (27) we have $\mathfrak{S} = \partial\sigma_r(U) \subset \sigma_{\Phi_r}(U) \subset \sigma_{\Phi}(U) = \mathfrak{S}$, and so $\sigma_{\Phi_r}(U) = \mathfrak{S}$. \square

Theorem 3.5. *If $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, then for the backward unilateral shift $V \in B(X)$ there are the equalities*

$$\mathfrak{S} = \sigma_{\Phi_l}(V) \subset \sigma_{\mathcal{B}_l}(V) = \mathbb{D}, \quad (30)$$

and

$$\sigma_{\Phi_r}(V) = \sigma_{\mathcal{B}_r}(V) = \mathfrak{S}. \quad (31)$$

Proof. Since $\mathbb{D} = \text{acc } \sigma_l(V) \subset \sigma_{\mathcal{B}_l}(V) \subset \sigma_l(V) = \mathbb{D}$ by (8) and (19), we obtain $\sigma_{\mathcal{B}_l}(V) = \mathbb{D}$. From (5) and (27) it follows $\mathfrak{S} = \partial\sigma_l(V) \subset \sigma_{\Phi_l}(V) \subset \sigma_{\Phi}(V) = \mathfrak{S}$, and hence $\sigma_{\Phi_l}(V) = \mathfrak{S}$.

Since $\mathfrak{S} = \partial\sigma_r(V) \subset \sigma_{\Phi_r}(V) \subset \sigma_{\mathcal{B}_r}(V) \subset \sigma_r(V) = \mathfrak{S}$ by (5) and (20), we get (31). \square

From (28) and (29) it follows

$$\sigma_{\mathcal{B}}(U) = \sigma_{\mathcal{B}_l}(U) \cup \sigma_{\mathcal{B}_r}(U) = \mathbb{D}, \quad (32)$$

and from (30) and (31) we obtain

$$\sigma_{\mathcal{B}}(V) = \sigma_{\mathcal{B}_l}(V) \cup \sigma_{\mathcal{B}_r}(V) = \mathbb{D}. \quad (33)$$

According to (9), from (27) we have $\mathfrak{S} = \partial\sigma_{\Phi}(U) \subset \sigma_{\Phi_+}(U) \subset \sigma_{\Phi}(U) = \mathfrak{S}$, and so $\sigma_{\Phi_+}(U) = \mathfrak{S}$. Similarly, $\sigma_{\Phi_-}(U) = \mathfrak{S}$, as well as $\sigma_{\Phi_+}(V) = \sigma_{\Phi_-}(V) = \mathfrak{S}$.

From $\mathfrak{S} = \sigma_{\Phi_+}(U) \subset \sigma_{\mathcal{B}_+}(U) \subset \sigma_{\mathcal{B}_l}(U) = \mathfrak{S}$ it follows $\sigma_{\mathcal{B}_+}(U) = \mathfrak{S}$. Hence from $\mathbb{D} = \sigma_{\mathcal{B}}(U) = \sigma_{\mathcal{B}_+}(U) \cup \sigma_{\mathcal{B}_-}(U)$ and $\mathfrak{S} = \sigma_{\Phi_-}(U) \subset \sigma_{\mathcal{B}_-}(U)$ we get $\sigma_{\mathcal{B}_-}(U) = \mathbb{D}$. Similarly, $\sigma_{\mathcal{B}_+}(V) = \mathbb{D}$ and $\sigma_{\mathcal{B}_-}(V) = \mathfrak{S}$.

In [21, Theorem 3.3 and Theorem 3.4] it is proved that if $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$, $T \in \text{Poly}^{-1}R(X)$, $\pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, then from $UT - TU \in \text{Ptrb}(\Phi(X))$ it follows that $U - T \in \Phi(X)$ and $i(U - T) = -1$, while from $VT - TV \in \text{Ptrb}(\Phi(X))$ it follows that $V - T \in \Phi(X)$ and $i(V - T) = 1$. Now we prove:

Theorem 3.6. *Let $X \in \{c_0, c, \ell_\infty, \ell_p\}$, $p \geq 1$. Then for the forward and backward unilateral shifts $U, V \in B(X)$ and $T \in B(X)$,*

$$T \in \text{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \cap \mathfrak{S} = \emptyset$$

implies

$$UT - TU \in \text{Ptrb}(\Phi(X)) \implies U - T \text{ is Fredholm,}$$

and

$$VT - TV \in \text{Ptrb}(\Phi(X)) \implies V - T \text{ is Fredholm.}$$

Proof. Follows from Theorem 2.3 (i) and (27). \square

In [21, Theorem 3.3 and Theorem 3.4] it is proved that if $X \in \{c_0, c, \ell_\infty, \ell_p\}, p \geq 1, T \in \text{Poly}^{-1}R(X), \pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, then from $UT = TU$ it follows that $U - T$ is Fredholm and left Browder, i.e. $U - T$ is Fredholm with finite ascent, and $i(U - T) = -1$, while from $VT = TV$ it follows that $V - T$ is Fredholm and right Browder, i.e. $V - T$ is Fredholm with finite descent, and $i(V - T) = 1$. Now we prove:

Theorem 3.7. *Let $X \in \{c_0, c, \ell_\infty, \ell_p\}, p \geq 1$. Then for the forward and backward unilateral shifts $U, V \in B(X)$ and $T \in B(X)$,*

$$T \in \text{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \cap \mathfrak{S} = \emptyset$$

implies

$$UT = TU \implies U - T \text{ is left Browder and Fredholm,}$$

and

$$VT = TV \implies V - T \text{ is right Browder and Fredholm.}$$

Proof. Follows from Theorem 2.3, since $\mathfrak{S} = \sigma_\Phi(U) = \sigma_{\mathfrak{B}}(U) = \sigma_\Phi(V) = \sigma_{\mathfrak{B}}(V)$ ((27), (28), (31)). \square

Theorem 3.8. *Let $X \in \{c_0, c, \ell_\infty, \ell_p\}, p \geq 1$. Then for the forward and backward unilateral shifts $U, V \in B(X)$ and $T \in B(X)$,*

$$T \in \text{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \cap \mathbb{D} = \emptyset$$

implies

$$UT = TU \implies U - T \text{ is Browder,}$$

and

$$VT = TV \implies V - T \text{ is Browder.}$$

Proof. Follows from Theorem 2.3 (ii), (32) and (33). \square

Let $\mathbb{C}^{\mathbb{Z}}$ be the linear space of all complex sequences $x = (x_k)_{k=-\infty}^{\infty}$. Let $c_0(\mathbb{Z})$ be the set of all sequences $x = (x_k)_{k=-\infty}^{\infty}$ such that $\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} x_{-k} = 0$, i.e. $x_k \rightarrow 0$ when $|k| \rightarrow \infty$. For $x = (x_k)_{k=-\infty}^{\infty} \in c_0(\mathbb{Z})$ set $\|x\| = \sup_k |x_k|$. We write $\ell_p(\mathbb{Z}) = \{x \in \mathbb{C}^{\mathbb{Z}} : \sum_{k=-\infty}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$, and for $x = (x_k)_{k=-\infty}^{\infty} \in \ell_p(\mathbb{Z})$, $\|x\| = (\sum_{k=-\infty}^{\infty} |x_k|^p)^{1/p}$. Remark that $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$ are Banach spaces.

For $k = \dots, -2, -1, 0, 1, 2, \dots$, let $\delta^{(k)}$ denote the sequences such that $\delta_k^{(k)} = 1$ and $\delta_i^{(k)} = 0$ for $i \neq k$. The forward and the backward bilateral shifts W_1 and W_2 are linear operators on $\mathbb{C}^{\mathbb{Z}}$ defined by

$$W_1 \delta^{(k)} = \delta^{(k+1)} \quad \text{and} \quad W_2 \delta^{(k+1)} = \delta^{(k)}, \quad k = \dots, -2, -1, 0, 1, 2, \dots$$

Obviously, $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z}), p \geq 1$ are invariant subspaces for W_1 and W_2 , and $W_1^{-1} = W_2$. For each $X \in \{c_0(\mathbb{Z}), \ell_p(\mathbb{Z})\}$, W_1 and W_2 are isometries. On the Hilbert space $\ell_2(\mathbb{Z})$ we have that $W_2 = W_1^*$, that is W_1 and W_2 are unitary.

For $X = \ell_2(\mathbb{Z})$ it is known that ([9, Solution 68], [7, Proposition 27.7 (c)])

$$\sigma(W_1) = \sigma(W_2) = \mathfrak{S}. \tag{34}$$

The last equalities hold also if X is one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z}), p \geq 1$ [21, Theorem 3.5].

Thus, for the forward and backward bilateral shifts $W_1, W_2 \in B(X), X \in \{c_0(\mathbb{Z}), \ell_p\}, p \geq 1$, according to (34) and (6) it follows that

$$\sigma_\Phi(W_1) = \sigma_\Phi(W_2) = \mathfrak{S}. \tag{35}$$

The inclusions $\sigma_\Phi(W_i) \subset \sigma_{\mathfrak{B}}(W_i) \subset \sigma(W_i), i = 1, 2$, (34) and (35) imply equalities

$$\sigma_{\mathfrak{B}}(W_1) = \sigma_{\mathfrak{B}}(W_2) = \mathfrak{S}. \tag{36}$$

According to (9) and (35) it follows $\mathfrak{S} = \partial\sigma_{\Phi}(W_i) \subset \sigma_{\Phi_+}(W_i) \subset \sigma_{\Phi}(W_i) = \mathfrak{S}$, $i = 1, 2$, and so,

$$\sigma_{\Phi_+}(W_1) = \sigma_{\Phi_+}(W_2) = \mathfrak{S}, \quad (37)$$

and analogously,

$$\sigma_{\Phi_-}(W_1) = \sigma_{\Phi_-}(W_2) = \mathfrak{S}. \quad (38)$$

From $\mathfrak{S} = \sigma_{\Phi_+}(W_i) \subset \sigma_{\Phi_1}(W_i) \subset \sigma_{\mathcal{B}_1}(W_i) \subset \sigma(W_i) = \mathfrak{S}$ and $\mathfrak{S} = \sigma_{\Phi_+}(W_i) \subset \sigma_{\mathcal{B}_+}(W_i) \subset \sigma(W_i) = \mathfrak{S}$, $i = 1, 2$, it follows that $\sigma_{\Phi_1}(W_i) = \sigma_{\mathcal{B}_1}(W_i) = \mathfrak{S}$ and $\sigma_{\mathcal{B}_+}(W_i) = \mathfrak{S}$. Analogously, $\sigma_{\Phi_-}(W_i) = \sigma_{\mathcal{B}_-}(W_i) = \sigma_{\mathcal{B}_-}(W_i) = \mathfrak{S}$, $i = 1, 2$.

In [21, Theorem 3.3 and Theorem 3.4] it is proved that if $X \in \{c_0(\mathbb{Z}), \ell_p\}$, $p \geq 1$, $T \in \text{Poly}^{-1}R(X)$, $\pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, then from $W_1T - TW_1 \in \text{Ptrb}(\Phi(X))$ it follows that $W_1 - T$ is Fredholm and $i(W_1 - T) = 0$, while from $W_2T - TW_2 \in \text{Ptrb}(\Phi(X))$ it follows that $W_2 - T$ is Fredholm and $i(W_2 - T) = 0$. Now we prove:

Theorem 3.9. *Let X be one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, then for the forward and backward bilateral shifts $W_1, W_2 \in B(X)$ and $T \in B(X)$,*

$$T \in \text{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \cap \mathfrak{S} = \emptyset$$

implies

$$W_1T - TW_1 \in \text{Ptrb}(\Phi(X)) \implies W_1 - T \text{ is Fredholm,}$$

and

$$W_2T - TW_2 \in \text{Ptrb}(\Phi(X)) \implies W_2 - T \text{ is Fredholm.}$$

Proof. Follows from Theorem 2.3 (i) and (35). \square

In [21, Theorem 3.7 and Theorem 3.8] it is proved that if X is one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, $T \in \text{Poly}^{-1}R(X)$, $\pi_T^{-1}(0) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, then from $W_1T = TW_1$ it follows that $W_1 - T$ is Browder, while from $W_2T = TW_2$ it follows that $W_2 - T$ is Browder. We can improve on this:

Theorem 3.10. *Let X be one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geq 1$, then for the forward and backward bilateral shifts $W_1, W_2 \in B(X)$ and $T \in B(X)$,*

$$T \in \text{Poly}^{-1}R(X) \quad \text{and} \quad \pi_T^{-1}(0) \cap \mathfrak{S} = \emptyset$$

implies

$$W_1T = TW_1 \implies W_1 - T \text{ is Browder,}$$

and

$$W_2T = TW_2 \implies W_2 - T \text{ is Browder.}$$

Proof. Follows from (36) and Theorem 2.3 (ii). \square

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