



## On the Convexity of the Solution Set of Symmetric Vector Equilibrium Problems

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**Abstract.** In this paper, without assumption of monotonicity and boundedness, we study existence results for a solution and the convexity of the solution set to the symmetric vector equilibrium problem for set-valued mappings in the setting of topological vector spaces. Our results improve the corresponding results in [9, 18, 19, 22, 28, 33, 36, 37].

### 1. Introduction

In 1980, vector variational inequality was first introduced and studied by Giannessi [20, 21]. Later on, vector variational inequality and its various extensions have been studied by Chen and Cheng [11], Chen, Huang and Yang [12], Yang[35] and other authors. Inspired by the study of vector variational inequalities, more general equilibrium problems (see [6, 7]) have been extended to the case of vector-valued bifunctions, known as vector equilibrium problems. The equilibrium problem contains as special cases, for instance, optimization problem, Nash equilibria problem, fixed point problem, variational inequalities, and complementarity problem (see, for examples,[1, 2, 8, 13, 25, 26] and references therein). Ansari et al. [3] introduced a system of vector equilibrium problems and established an existence result for a solution of the system. Moreover, they applied their results to study the vector optimization problem and the Nash equilibrium problem for vector-valued functions. The system of vector equilibrium problems contain system of scalar equilibrium problems, systems of vector variational inequalities, system of vector variational-like inequalities, system of optimization problems, fixed point problems and several related topics as special cases (see, for examples, [3–5, 14, 15, 24, 27, 28, 30–32] and references therein). On the other hand, the symmetric vector equilibrium problem is a generalization of the equilibrium problem which has been studied by many authors. A main topic of the current research is to establish existence theorems for a solution of the symmetric vector equilibrium problem(see, for examples, [17–19, 23]) and some important properties of the solution set to the symmetric vector equilibrium problem.

Recently, Zhong et al. [37] studied the connectedness and path-connectedness of the solution set of the symmetric vector equilibrium problem in the setting of locally convex topological vector spaces. It is worth noting that their results generalized the well known results of Chen et al. [10] and Lee et al. [29] ( the connectedness of weak efficient solutions set for single-valued vector variational inequalities in finite dimensional Euclidean space) Lee and Bu [28]( the connectedness of the solution sets for affine vector

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variational inequalities with noncompact polyhedral constraint sets and positive semidefinite (or monotone) matrices) Gong [22] (the connectedness and path-connectedness of various solution sets for single-valued vector equilibrium problems in a real locally convex Hausdorff topological vector space) Gong [22] (the connectedness of the set of efficient solutions for the generalized system with monotone bifunctions in real locally convex Hausdorff topological vector spaces) and Chen et al. [10] (the existence, connectedness, and the compactness of the weak efficient solutions set for set-valued vector equilibrium problems and the set-valued vector Hartman-Stampacchia variational inequalities in normed linear spaces). However, to the best of our knowledge, there is only one paper dealing with the connectedness and path-connectedness of the solution set for the symmetric vector equilibrium problem in the setting of locally convex spaces (see, [37]). So by considering this fact and the works mentioned in the above (especially [37]) the author of this paper is interested to extend the results obtained in [37] from locally convex topological vector spaces to topological vector spaces with mild assumptions as well as we investigate suitable conditions which guarantee the convexity of the solution set of symmetric vector equilibrium problem.

## 2. Preliminaries

Throughout the paper, let  $X, Y, E$  and  $Z$  be real Hausdorff topological vector spaces. Let  $A \subseteq X$  and  $B \subseteq E$  be nonempty closed convex sets,  $F : A \times B \times A \rightarrow 2^Y$  and  $G : A \times B \times A \rightarrow 2^Z$  be two set-valued mappings. Let  $C \subset Y$  and  $P \subset Z$  be two closed convex pointed cones with nonempty interiors, i.e.,  $\text{int}C \neq \emptyset$  and  $\text{int}P \neq \emptyset$ . Let  $Y^*$  and  $Z^*$  be the topological dual spaces of  $Y$  and  $Z$ , respectively. Let  $C^*$  and  $P^*$  be the dual cones of  $C$  and  $P$ , respectively, that is,

$$C^* = \{f \in Y^* : f(y) \geq 0, \forall y \in C\}, P^* = \{g \in Z^* : g(y) \geq 0, \forall y \in P\}$$

The symmetric vector equilibrium problem (for short, SVEP) is the problem of finding  $(x, y) \in A \times B$  such that

$$(2.1) \quad \begin{cases} F(x, y, u) \cap (-\text{int}C) = \emptyset, \forall u \in A, \\ G(x, y, v) \cap (-\text{int}P) = \emptyset, \forall u \in B. \end{cases}$$

### Remark 2.1. Special cases:

(i) If we take  $C = P$ ,  $f : A \times B \rightarrow Y$  and  $g : A \times B \rightarrow Z$  are two single-valued mappings,

$$F(x, y, u) = \{f(u, y) - f(x, y)\}, \forall (x, y, u) \in A \times B \times A$$

and

$$G(x, y, v) = \{g(x, v) - g(x, y)\}, \forall (x, y, v) \in A \times B \times B,$$

then (2.1) reduces to a single-valued symmetric vector equilibrium problem considered in [17–19].

(ii) If  $G = 0$ ,  $F(x, y, u) = \{f(x, u)\}$ , for any  $(x, y, u) \in A \times B \times A$ , then (2.1) is the equilibrium problem which was considered and studied by many authors (see, for example, [1, 2, 6, 8, 13, 21, 25, 26] and the references therein).

(iii) If  $G = 0$  and  $T : A \rightarrow L(X, Y)$  is a mapping, where  $L(X, Y)$  denotes the space of all continuous linear operators from  $X$  to  $Y$ , and  $F(x, y, u) = T(x)(u - x) = \langle Tx, u - x \rangle$  for any  $(x, y, u) \in A \times B \times A$ , then (2.1) is the classic vector variational inequality problem which was introduced by Giannessi [21].

For any  $(f, g) \in (C^* \setminus \{0\}) \times (P^* \setminus \{0\})$ , we also consider the following problem scalar symmetric equilibrium problem (for short,  $(SSEP)_{f,g}$ ) which consists of finding  $(x, y) \in A \times B$ , such that

$$\begin{cases} f(F(x, y, u)) \geq 0, \forall u \in A, \\ g(G(x, y, v)) \geq 0, \forall u \in B. \end{cases}$$

We denote the solution sets of SVEP and  $(SSEP)_{f,g}$  by  $S(F, G)$  and  $S(f, g)$ , respectively.

**Definition 2.2.** Let  $X$  and  $Y$  be two topological spaces. A set-valued mapping  $T : X \rightarrow 2^Y$  is said to be

(i) **Closed** if its graph, that is

$$\text{Graph } T = \{(x, y) \in X \times Y : y \in T(x)\},$$

is closed in  $X \times Y$ .

(ii) **Compact** if the closure of the range  $T$ , i.e.,  $\overline{T(X)}$ , is compact, where

$$T(X) = \bigcup_{x \in X} T(x).$$

(iii) **Upper semicontinuous** (*u.s.c.*) if, for every  $x \in X$  and every open set  $V$  satisfying  $T(x) \subseteq V$ , there exists a neighborhood  $U$  of  $x$  such that

$$T(U) = \bigcup_{y \in U} T(y) \subseteq V.$$

(iv) **Lower semicontinuous** (*l.s.c.*) if, for any  $x \in X$ ,  $y \in T(x)$  and any neighborhood  $N(y)$  of  $y$ , there exists a neighborhood  $N(x)$  of  $x$  such that

$$T(z) \cap N(y) \neq \emptyset, \forall z \in N(x).$$

(v) **Continuous** if, for any  $x \in X$ , it is at the same time *u.s.c.* and *l.s.c.* on  $X$ .

**Remark 2.3.** ([34]) Let  $X$  and  $Y$  be two topological spaces. A set-valued mapping  $T : X \rightarrow 2^Y$ .  $T$  is *l.s.c.* at  $x \in X$  if and only if for any  $y \in T(x)$ , and any net  $\{x_\alpha\}$ ,  $x_\alpha \rightarrow x$ , there is a net  $\{y_\alpha\}$  such that  $y_\alpha \in T(x_\alpha)$  and  $y_\alpha \rightarrow y$ .

**Definition 2.4.** ([37]) Let  $T : A \times B \times D \rightarrow 2^Y$  be a set valued mapping, where  $D$  is a convex set. For any fixed  $(x, y) \in A \times B$  the mapping  $z \rightarrow T(x, y, z)$  is said to be

(i)  **$C$ -convex** if, for every  $z_1, z_2 \in D$  and  $t \in [0, 1]$ ,

$$tT(x, y, z_1) + (1 - t)T(x, y, z_2) \subseteq T(x, y, tz_1 + (1 - t)z_2) + C.$$

(ii)  **$C$ -quasiconvex** if, for every  $z_1, z_2 \in D$  and  $t \in [0, 1]$ , either

$$T(x, y, z_1) \subseteq T(x, y, tz_1 + (1 - t)z_2) + C$$

or

$$T(x, y, z_2) \subseteq T(x, y, tz_1 + (1 - t)z_2) + C.$$

**Definition 2.5.** A topological space  $X$  is said to be

- (i) **connected** if, there exist no open subsets  $V_i \subseteq X$  with  $V_i \neq \emptyset$ , for  $i = 1, 2$  such that  $V_1 \cup V_2 = X$  and  $V_1 \cap V_2 = \emptyset$ .
- (ii) **path-connected** if, for each pair of points  $x$  and  $y$  in  $X$ , there exists a continuous mapping  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ .

**Definition 2.6.** Let  $S$  be a nonempty subset of a real linear space  $X$ .

(a) The algebraic interior of  $S$  is denoted by  $\text{cor}(S)$  and is defined by

$$\text{cor}(S) = \{\bar{x} \in S | \forall x \in X \quad \exists \bar{a} > 0; \quad \bar{x} + tx \in S, \quad \forall t \in [0, a]\},$$

(b) The set  $S$  with  $S = \text{cor}(S)$  is called algebraically open.

(c) The set of all elements of  $X$  which do not belong to  $\text{cor}(S)$  and  $\text{cor}(X \setminus S)$  is called the algebraic boundary of  $S$ .

(d) An element  $\bar{x} \in X$  is called linearly accessible from  $S$ , if there is an  $x \in S$ ,  $x \neq \bar{x}$ , with the property

$$tx + (1 - t)\bar{x} \in S, \quad \forall t \in (0, 1].$$

**Remark 2.7.** It is easy to verify that if  $S$  is convex then  $\text{cor}(S)$  is a convex set. Also if  $S$  is a subset of a topological vector space then  $\text{int}S \subseteq \text{cor}(S)$ . The following example shows that it is possible that  $\text{int}S = \emptyset$  and  $\text{cor}(S) \neq \emptyset$ .

**Example 2.8.** Let  $X = C_{00}$  be the space of all real sequences which have finite support, that is,

$$X = C_{00} = \{x = \{x_n\} : \text{the set } \{n \in N; x_n \neq 0\} \text{ is finite}\}$$

with  $\|x\| = \max_{n \in \mathbb{N}} |x_n|$ . Let

$$S = \{x = (x(n)) \in C_{00} : (\forall n), x(n) \leq \frac{1}{n}\}.$$

It is easy to check that  $\text{int}S = \emptyset$  and

$$\{x_n\}_{n \in N} = \{x_1 = 1, x_2 = 0, x_3 = 0, \dots\} \in \text{cor}(S).$$

Now, we can extend the SVEP (that is, the problem defined by 2.1) from topological vector spaces to vector spaces as follows:

The symmetric vector equilibrium problem, in the setting of vector spaces, is the problem of finding  $(x, y) \in A \times B$  such that

$$(2.1)' \quad \begin{cases} F(x, y, u) \cap (-\text{core}(C)) = \emptyset, \forall u \in A, \\ G(x, y, v) \cap (-\text{core}(P)) = \emptyset, \forall u \in B. \end{cases}$$

Similarly, the scalar SVEP with respect to  $(f, g) \in (C' \setminus \{0\}) \times (P' \setminus \{0\})$  consists of finding  $(x, y) \in A \times B$ , such that

$$\begin{cases} f(F(x, y, u)) \geq 0, \forall u \in A, \\ g(G(x, y, v)) \geq 0, \forall u \in B. \end{cases}$$

where

$$C' = \{x' \in X' : x'(c) \geq 0, \forall c \in C\}, \quad P' = \{z' \in Z' : z'(p) \geq 0, \forall p \in P\},$$

and  $X'$  and  $Z'$  denote all the linear mappings from  $X$  into the real line and all the linear mappings from  $Z$  into the real line, respectively.

We need the following lemma in the sequel.

**Lemma 2.9.** Let  $A$  and  $B$  be nonempty convex subsets of a real linear space  $X$ . If  $\text{cor}(A) \neq \emptyset$  then  $\text{cor}(A) \cap B = \emptyset$  if and only if there are a linear functional  $l \in X^* \setminus \{0_{X^*}\}$  and a real number  $a$  with

$$l(s) \leq a \leq l(t), \quad \forall s \in A, \quad \forall t \in B,$$

and

$$l(s) < a, \quad \forall s \in \text{cor}(A).$$

One can deduce the following result as a consequence of the Lemma 2.9.

**Corollary 2.10.** Let  $A$  and  $B$  be nonempty, disjoint, convex subsets of a topological vector space  $X$ . If  $A$  is open, then there exist continuous linear mapping  $x' \in X^* = L(X, \mathbb{R})$  and  $c \in \mathbb{R}$  such that

$$x'(x) = \langle x', x \rangle < cx'(y) = \langle x', y \rangle, \quad \forall (x, y) \in A \times B.$$

**Definition 2.11.** ([16]) Let  $K$  be a subset of a topological vector space  $E$ . A set-valued mapping  $F : K \rightarrow 2^E$  is said to be a KKM-mapping, if for any  $\{x_1, x_2, \dots, x_n\} \subset K$ ,  $\text{co}\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$ , where  $2^E \setminus \{\emptyset\}$  denotes the family of all nonempty subsets of  $E$ .

### 3. Main Results

In this section, we first present sufficient conditions which guarantee the equality of the solution sets of (SVEP) and the scalar (SVEP).

**Theorem 3.1.** Suppose that  $\text{cor}(C) \neq \emptyset$  and  $\text{cor}(P) \neq \emptyset$ . Let, for each  $(x, y) \in A \times B$ , the sets  $F(x, y, A) + C$  and  $G(x, y, B) + P$  be convex. Then

$$S(F, G) = \bigcup_{(f,g) \in ((C' \setminus \{0\}) \times (P' \setminus \{0\}))} S(f, g).$$

*Proof.* Let  $(x, y) \in \bigcup_{(f,g) \in ((C' \setminus \{0\}) \times (P' \setminus \{0\}))} S(f, g)$ . Then, for some  $(f, g) \in (C' \setminus \{0\}) \times (P' \setminus \{0\})$ , we have

$$\begin{cases} f(F(x, y, u)) \geq 0, \forall u \in A, \\ g(G(x, y, v)) \geq 0, \forall v \in B. \end{cases}$$

If  $(x, y) \notin S(F, G)$  then either there exists  $w_1 \in A$  such that  $F(x, y, w_1) \cap -\text{core}(C) \neq \emptyset$  or  $w_2 \in B$  such that  $F(x, y, w_2) \cap -\text{core}(P) \neq \emptyset$  and so either  $f(F(x, y, w_1)) \cap (-\infty, 0) \neq \emptyset$  or  $g(F(x, y, w_2)) \cap (-\infty, 0) \neq \emptyset$  which is a contraction. This proves one side the theorem. To see the opposite side of the theorem, let  $(x, y) \in S(F, G)$ . Then

$$\begin{cases} F(x, y, u) \cap -\text{cor}(C) = \emptyset, \forall u \in A, \\ G(x, y, v) \cap -\text{cor}(P) = \emptyset, \forall v \in B. \end{cases}$$

So

$$\begin{cases} F(x, y, A) \cap -\text{cor}(C) = \emptyset, \\ G(x, y, B) \cap -\text{cor}(P) = \emptyset. \end{cases}$$

Therefore, note  $C \cap (-\text{cor}(C)) = \emptyset$  and  $P \cap -\text{cor}(P) = \emptyset$ , we get

$$\begin{cases} (F(x, y, A) + C) \cap -\text{cor}(C) = \emptyset, \\ (G(x, y, B) + P) \cap -\text{cor}(P) = \emptyset. \end{cases}$$

Hence it follows from Lemma 2.8 that there exist  $(x', y') \in Y' \times Z'$ , and real numbers  $a, b$  such that

$$x'(s) \leq a \leq x'(t), \quad \forall s \in C, \quad \forall t \in -F(x, y, A) - C, \quad (3.1)$$

$$x'(s) < a, \quad \forall s \in \text{cor}(C),$$

and

$$y'(s) \leq b \leq y'(t), \quad \forall s \in P, \quad \forall t \in -F(x, y, B) - P, \quad (3.2)$$

$$y'(s) < b, \quad \forall s \in \text{cor}(P).$$

Hence, by taking  $s = 0$  in (3.1) and (3.2) we get that  $a$  and  $b$  are nonnegative. Also by taking  $f = -x'$ ,  $g = -y'$ , it follows from (3.1), (3.2),  $F(x, y, A) \subset F(x, y, A) + C$  and  $G(x, y, P) \subset G(x, y, B) + P$  that  $f(F(x, y, A)) \geq 0$  and  $g(G(x, y, B)) \geq 0$ . This completes the proof.  $\square$

As an application of Theorem 3.1, we give the next result which extends the result given in Lemma 2.1 of [37] from locally convex spaces to topological vector spaces.

**Corollary 3.2.** Suppose that  $\text{int}C \neq \emptyset$  and  $\text{int}P \neq \emptyset$ . If  $F(x, y, A) + C$  and  $G(x, y, B) + P$  are convex sets for each  $(x, y) \in A \times B$ , then

$$S(F, G) = \bigcup_{(f,g) \in ((C^* \setminus \{0\}) \times (P^* \setminus \{0\}))} S(f, g).$$

*Proof.* The proof follows directly from Theorem 3.1 and Corollary 2.9  $\square$

The following lemma plays a key role in the next result.

**Lemma 3.3.** ([16]) Let  $K$  be a nonempty subset of a topological vector space  $X$  and  $F : K \rightarrow 2^X$  be a KKM mapping with closed values in  $K$ . Assume that there exists a nonempty compact convex subset  $B$  of  $K$  such that  $\bigcap_{x \in B} F(x)$  is compact. Then

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

The following theorem is the main goal of this paper which guarantees the convexity of the solution set of SVEP (note that each convex set is connected and path-connected) as well as an existence theorem for a solution of the scalar symmetric equilibrium problem and symmetric vector equilibrium problem under suitable assumptions. Moreover, the proof presented for the next result is different from those proofs given in [37] for Lemma 2.3 and Theorem 3.1 by relaxing the boundedness assumption considered in [37].

**Theorem 3.4.** Let  $A \subseteq X$  and  $B \subseteq E$  be nonempty convex subsets, let  $C \subset Y$  and  $P \subset Z$  be closed convex pointed cone with  $\text{int } C \neq \emptyset$  and  $\text{int } P \neq \emptyset$ . Suppose  $F : A \times B \times A \rightarrow 2^Y$  and  $G : A \times B \times B \rightarrow 2^Z$  are two set-valued mappings which satisfy the following conditions:

- (i) for each  $(x, y) \in A \times B$ ,  $F(x, y, x) \subseteq C$ ,  $G(x, y, y) \subseteq P$ ;
- (ii) for each  $(z, w) \in A \times B$ , the set-valued mappings  $(x, y) \rightarrow F(x, y, z)$  and  $w \rightarrow G(x, y, w)$  are lower semicontinuous on  $A \times B$ ;
- (iii) for each  $(x, y) \in A \times B$ , the set-valued mappings  $z \rightarrow F(x, y, z)$  and  $w \rightarrow G(x, y, w)$  are  $C$ -quasiconvex on  $A$  and  $P$ -quasiconvex on  $B$ , respectively;
- (iv) there exist nonempty compact convex set  $D_1 \times D_2 \subset A \times B$  and compact set  $M_1 \times M_2 \subset A \times B$  such that for each  $(x, y) \in (A \times B) \setminus (M_1 \times M_2)$ , there exists  $(x_1, y_1) \in D_1 \times D_2$  such that  $F(x, y, x_1) \cap -\text{int } C \neq \emptyset$  or  $G(x, y, y_1) \cap -\text{int } P \neq \emptyset$ .

Then

$$S(f, g) \neq \emptyset, \forall (f, g) \in (C^* \setminus \{0\}) \times (P^* \setminus \{0\}).$$

Furthermore, the  $S(f, g)$  is a compact subset of  $A \times B$  and the solution set of the symmetric vector equilibrium problem (that is  $S(F, G)$ ) is nonempty and compact when the sets  $F(x, y, A) + C$  and  $G(x, y, B) + P$  are convex, for each  $(x, y) \in A \times B$ . Also if the mappings  $(x, y) \in F(x, y, z)$  and  $(x, y) \rightarrow G(x, y, w)$  are  $C$ -convex and  $P$ -convex, respectively, for all  $(z, w) \in A \times B$  then the solution set of SVEP, i.e.,  $S(F, G)$  is convex and so connected and path-connected.

*Proof.* Let  $(f, g) \in (C^* \setminus \{0\}) \times (P^* \setminus \{0\})$ . Define a set-valued mapping  $\Gamma : A \times B \rightarrow A \times B$  by

$$\Gamma(z, w) = \{(x, y) \in A \times B : f(F(x, y, z)) \leq 0, g(G(x, y, w)) \leq 0\}.$$

It is obvious that the solution set of the scalar symmetric equilibrium problem equals to the intersection  $\bigcap_{(x,y) \in A \times B} \Gamma(x, y)$ .

We assert that the set-valued mapping  $\Gamma$  fulfills all the assumptions of Lemma 2.12. Indeed:

- (a)  $\Gamma$  is a KKM mapping.

Otherwise, there exist a subset  $\{(x_1, y_1), \dots, (x_n, y_n)\} \subset A \times B$  and  $(z, w) \in A \times B$  such that

$$(z, w) \in \text{co}\{(x_1, y_1), \dots, (x_n, y_n)\} \setminus \bigcup_{i=1}^n \Gamma(x_i, y_i).$$

Hence there exist nonnegative real numbers  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that

$$\sum_{i=1}^n \alpha_i = 1 \quad \text{and} \quad (z, w) = \sum_{i=1}^n \alpha_i (x_i, y_i),$$

and so  $z = \sum_{i=1}^n \alpha_i x_i$ ,  $w = \sum_{i=1}^n \alpha_i y_i$ . Consequently, for each  $i = 1, 2, \dots, n$ . Then

$$f(F(z, w, x_i)) < 0 \quad \text{or} \quad g(G(z, w, y_i)) < 0$$

which is contradicted by (iii) because it follows from  $(z, w) \in \text{co}\{(x_1, y_1), \dots, (x_n, y_n)\}$  and (iii) that, for each  $i = 1, 2, 3, \dots, n$ ,

$$F(z, w, x_i) \subseteq F(z, w, z) + C$$

or

$$G(z, w, y_i) \subseteq G(z, w, w) + P$$

and so

$$f(F(z, w, x_i)) \subseteq f(F(z, w, z)) + f(C) \subseteq [0, \infty) + [0, \infty) = [0, \infty)$$

or

$$g(G(z, w, y_i)) \subseteq g(G(z, w, w)) + g(P) \subseteq [0, \infty) + [0, \infty) = [0, \infty).$$

(b) For each  $(z, w) \in A \times B$ , the set  $\Gamma(z, w)$  is closed.

To verify this, let  $(z, w) \in A \times B$  and  $(z_i, w_i) \in \Gamma(z, w)$  be a convergent net to  $(z_1, w_2)$ . Then

$$f(F(z, w, z_i)) \geq 0, \quad g(G(z, w, w_i)) \leq 0.$$

Now let  $(h_1, h_2) \in f(F(z, w, z_1)) \times g(G(z, w, w_2))$ . Hence there exists  $(z_2, w_3) \in F(z, w, z_1) \times G(z, w, w_2)$  such that  $(h_1, h_2) = (f(z_2), g(w_3))$  and so it follows from (iii) and Remark 2.3 that there is net  $(t_i, s_i) \in F(z, w, z_i) \times G(z, w, w_i)$  with  $(t_i, s_i) \rightarrow (z_2, w_3)$ . Hence the continuity of the mappings  $f, g$  and  $f(t_i) \geq 0, g(s_i) \geq 0$  we deduce that  $f(h_1) \geq 0, g(h_2) \geq 0$ . Consequently, the set-valued mapping  $\Gamma$  fulfills all the assumptions of Lemma 2.12 and so by using Lemma 2.12 the intersection

$$\bigcap_{(x,y) \in A \times B} \Gamma(x, y)$$

is nonempty and so the solution set of the scalar symmetric equilibrium problem is nonempty.

Now, we want to prove that the solution set of the scalar symmetric equilibrium problem is convex and hence connected and path-connected.

It follows from (iv) that

$$(c) \bigcap_{(z,w) \in D_1 \times D_2} \Gamma(z, w) \subseteq M_1 \times M_2,$$

and so it completes the proof of the first part of the theorem. The second part follows directly from Corollary 2.11. Finally, the  $C$ -convexity and  $P$ -convexity of the mappings  $(x, y) \in F(x, y, z)$  and  $(x, y) \rightarrow G(x, y, w)$ , respectively, imply that for all  $(z, w) \in A \times B$  the set  $\Gamma(z, w)$  is convex and so and the set  $S(F, G) = \bigcap_{(z,w) \in A \times B} \Gamma(z, w) \neq \emptyset$  is convex (the intersection of the convex sets is convex) and so is connected and path-connected. This completes the proof.  $\square$

**Remark 3.5.** It is easy to verify that the result of the previous theorem is still valid if we replace condition (i) by

$$(i') \quad f(F(x, y, x)) \geq 0, \quad g(G(x, y, y)) \geq 0, \quad \forall (f, g) \in (C^* \setminus \{0\}) \times (P^* \setminus \{0\}).$$

One can check that the condition (i) implies condition (i') while the example  $Y = E = L^p(0, 1)$  (all the measurable functions) on  $[0, 1]$  with quasi-norm  $\int_{[0,1]} \frac{|f|^p}{1+|f|^p}$ , where  $0 < p < 1$ , shows that the converse may fail (note  $Y^* = E^* = 0$ ). It is worth noting that  $L^p(0, 1)$  is a topological vector space which is not a locally convex space. Also, we can replace condition (ii) by the weaker assumption:

(ii') for each  $(z, w) \in A \times B, (f, g) \in (C^* \setminus \{0\}) \times (P^* \setminus \{0\})$ , the set-valued mappings  $(x, y) \rightarrow f(F(x, y, z))$  and  $w \rightarrow g(G(x, y, w))$  are lower semicontinuous on  $A \times B$ .

Further, it is straightforward to check that if  $T$  is  $C$ -quasiconvex and  $f \in C^*$  then  $f \circ T$  (the composition of  $f$  and  $T$ ) is  $[0, \infty)$ -quasiconvex and so we can replace condition (iii) with the following weaker condition:

(iii') for each  $(x, y) \in A \times B$ , the set-valued mappings  $z \rightarrow F(x, y, z)$  and  $w \rightarrow G(x, y, w)$  are, respectively,  $C$ -quasiconvex on  $A$  and  $P$ -quasiconvex on  $B$ , respectively.

Notice that the condition (iv) collapses to the condition (iv) of Lemma 2.3 and Theorem 3.1 of [37] when we take  $D_1 \times D_2 = M_1 \times M_2$ .

Finally, if the mappings  $(x, y) \in F(x, y, z)$  and  $(x, y) \rightarrow G(x, y, w)$  are  $C$ -convex and  $P$ -convex, for all  $(z, w) \in A \times B$ , respectively, then the set

$$\Gamma(z, w) = \{(x, y) \in A \times B : f(F(x, y, z)) \leq 0, g(G(x, y, w)) \leq 0\}$$

is convex, for each  $(f, g) \in (C^* \setminus \{0\}) \times (P^* \setminus \{0\})$ . While the simple example  $Y = Z = \mathbb{R}$  with the topology induced by the semi-norm  $p(x) = 0$ , for all  $x \in \mathbb{R}$  shows that the converse is not true in general. Hence, we can replace in Theorem 3.4 the following condition: The mappings  $(x, y) \rightarrow F(x, y, z)$  and  $(x, y) \rightarrow G(x, y, w)$  are  $C$ -convex and  $P$ -convex, respectively, for all  $(z, w) \in A \times B$  by the convexity of the set  $\Gamma(w, z)$  for all  $(w, z) \in Y \times Z$ .

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