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Some Additive and Multiplicative Results for Generalized Inverses

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Abstract. In this paper we give necessary and sufficient conditions for $A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\} \subseteq (A_1 + A_2 + ... + A_k)\{1,3\}$ and $A_1\{1,4\} + A_2\{1,4\} + ... + A_k\{1,4\} \subseteq (A_1 + A_2 + ... + A_k)\{1,4\}$ for regular operators on Hilbert space. We also consider similar inclusions for $\{1,2,3\}$- and $\{1,2,4\}$-i inverses. We give some new results concerning the reverse order law for reflexive generalized inverses.

1. Introduction

Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{I}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a given $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of operator A, respectively. The identity operator on $\mathcal{H}$ is denoted by $\mathcal{I}_{\mathcal{H}}$. For an operator A, by A_l^{-1} (A_r^{-1}) we denote the left (right) inverse of A and by $\mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ ($\mathcal{B}_r^{-1}(\mathcal{H},\mathcal{K})$) the set of all left (right) invertible operators from the space $\mathcal{B}(\mathcal{H},\mathcal{K})$. E_A and F_A stand for two orthogonal projectors $E_A = I_{\mathcal{K}} - AA^\dagger$ and $F_A = I_{\mathcal{H}} - A^\dagger A$. For given sets M, N, by MN or $M \cdot N$ we denote the set consisting of all products XY, where $X \in M$ and $Y \in N$.

Recall that $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ has a Moore-Penrose inverse if there exists an operator $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$(1)\ AXA = A \quad (2)\ XAX = X \quad (3)\ (AX)^* = AX \quad (4)\ (XA)^* = XA. \tag{1}$$

Moore-Penrose inverse of an operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ exists if and only if A has a closed range and in this case it is unique and is denoted by $A^\dagger$. If for $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ there exists a Moore-Penrose inverse, we say that A is regular operator.

For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, let $A\{i,j,\dots,k\}$ denote the set of all operators $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ which satisfy equations (i), (j), ..., (k) from among equations (1)–(4) of (1). In this case X is a $\{i,j,\dots,k\}$−inverse of A which we denote by $A^{(i,j,\dots,k)}$.

Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $i = \overline{1,k}$. We give necessary and sufficient conditions for the following inclusions

$$A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\} \subseteq (A_1 + A_2 + ... + A_k)\{1,3\},$$

$$A_1\{1,4\} + A_2\{1,4\} + ... + A_k\{1,4\} \subseteq (A_1 + A_2 + ... + A_k)\{1,4\},$$

$$(A_1 + A_2 + ... + A_k)\{1,3\} \subseteq A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\},$$

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$$(A_1 + A_2 + ... + A_k)\{1,4\} \subseteq A_1\{1,4\} + A_2\{1,4\} + ... + A_k\{1,4\},$$
$$A_1\{1,2,3\} + A_2\{1,2,3\} + ... + A_k\{1,2,3\} \subseteq (A_1 + A_2 + ... + A_k)\{1,2,3\}$$

and

$$A_1\{1,2,4\} + A_2\{1,2,4\} + ... + A_k\{1,2,4\} \subseteq (A_1 + A_2 + ... + A_k)\{1,2,4\}.$$

In [2] authors considered necessary and sufficient conditions for $B\{1,2\} \cdot A\{1,2\} \subseteq (AB)\{1,2\}$ in the case of bounded linear operators on Hilbert space. We derived necessary and sufficient conditions for the inclusion $C\{1,2\} \cdot B\{1,2\} \cdot A\{1,2\} \subseteq (ABC)\{1,2\}$.

2. Results

It is well-known that for $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ regular,

$$\begin{aligned} B \in A\{1,3\} \quad &\Leftrightarrow \quad A^*AB = A^*, \\ B \in A\{1,4\} \quad &\Leftrightarrow \quad BAA^* = A^*, \end{aligned} \tag{2}$$

and that sets of all $\{1,3\}$-and $\{1,4\}$-inverses of A are described by

$$\begin{aligned} A\{1,3\} &= \{A^\dagger + F_A V : V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}, \\ A\{1,4\} &= \{A^\dagger + V E_A : V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}. \end{aligned} \tag{3}$$

Also, for $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ regular,

$$\begin{aligned} B \in A\{1,2,3\} &\Leftrightarrow (A^*AB = A^* \wedge BAA^\dagger = B), \\ B \in A\{1,2,4\} &\Leftrightarrow (BAA^* = A^* \wedge A^\dagger AB = B), \end{aligned} \tag{4}$$

and sets of all $\{1,2,3\}$-and $\{1,2,4\}$-inverses of A are described by

$$\begin{aligned} A\{1,2,3\} &= \{A^\dagger + F_A V A A^\dagger : V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}, \\ A\{1,2,4\} &= \{A^\dagger + A^\dagger A V E_A : V \in \mathcal{B}(\mathcal{K},\mathcal{H})\}. \end{aligned} \tag{5}$$

Theorem 2.1. *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that A_i, $i = 1,2,\dots,k$ and $A = A_1 + A_2 + ... + A_k$ are regular operators. The following conditions are equivalent:*

(*i*) $A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\} \subseteq (A_1 + A_2 + ... + A_k)\{1,3\}$,

(*ii*) $A^*AF_{A_i} = 0$, $i = \overline{1,k}$ $A^*A \sum_{i=1}^k A_i^\dagger = A^*$.

Proof. (*ii*) $\Rightarrow$ (*i*) : Suppose that (*ii*) holds. We need to prove that for arbitrary $A_i^{(1,3)} \in A_i\{1,3\}$, $i = \overline{1,k}$, it follows that $A_1^{(1,3)} + A_2^{(1,3)} + ... + A_k^{(1,3)} \in A\{1,3\}$. Thus, given any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = \overline{1,k}$, we must show that

$$A^*A\Big(\sum_{i=1}^k A_i^\dagger + \sum_{i=1}^k F_{A_i} V_i\Big) = A^*, \tag{6}$$

which is satisfied by (*ii*).

(*i*) $\Rightarrow$ (*ii*): If (*i*) holds, then for any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = \overline{1,k}$, we have that (6) holds. Specially, for $V_i = 0$, $i = \overline{1,k}$, by (6) we get that $A^*A \sum_{i=1}^k A_i^\dagger = A^*$. Similarly, if for any $i \in \{1,2,...,k\}$, we take that $V_i = F_{A_i}$ and that $V_j = 0$, $j \neq i$, by (6) we will get that $A^*AF_{A_i} = 0$. Hence, (*ii*) holds. □

In the following theorem we present the necessary and sufficient condition for

$$(A_1 + A_2 + ... + A_k)\{1,3\} \subseteq A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\}.$$

Theorem 2.2. *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that A_i, $i = 1,2,\dots,k$, $A = A_1+A_2+...+A_k$ and $C = \begin{bmatrix} F_{A_1} & F_{A_2} & \dots & F_{A_k} \end{bmatrix}$ are regular operators. The following conditions are equivalent:*

(i) $(A_1 + A_2 + ... + A_k)\{1,3\} \subseteq A_1\{1,3\} + A_2\{1,3\} + ... + A_k\{1,3\}$,

(ii) $CC^\dagger F_A = F_A$, $CC^\dagger(A^\dagger - \sum_{i=1}^k A_i^\dagger) = A^\dagger - \sum_{i=1}^k A_i^\dagger$.

Proof. (ii) $\Rightarrow$ (i): Suppose that (ii) holds. We need to prove that for arbitrary $V \in \mathcal{B}(\mathcal{K},\mathcal{H})$ there exist $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = \overline{1,k}$ such that

$$A^\dagger + F_A V = \sum_{i=1}^k A_i^\dagger + \sum_{i=1}^k F_{A_i} V_i,$$

i.e.

$$\begin{bmatrix} F_{A_1} & F_{A_2} & \dots & F_{A_k} \end{bmatrix} \begin{bmatrix} V_1 \\ \dots \\ V_k \end{bmatrix} = F_A V + A^\dagger - \sum_{i=1}^k A_i^\dagger. \tag{7}$$

Hence, to show (i) we need to prove that the equation (7) is solvalable for any $V \in \mathcal{B}(\mathcal{K},\mathcal{H})$ which holds if and only if

$$CC^\dagger\Big(F_A V + A^\dagger - \sum_{i=1}^k A_i^\dagger\Big) = F_A V + A^\dagger - \sum_{i=1}^k A_i^\dagger. \tag{8}$$

Obviously, (8) is satisfied by (ii).

(i) $\Rightarrow$ (ii): If (i) is satisfied, then (8) holds for any $V \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Taking $V = 0$ and $V = F_A$ in (8), we get that the both equalities from (ii) hold. □

Theorem 2.3. *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that A_i, $i = 1,2,\dots,k$, $A = A_1+A_2+...+A_k$ and $D = \begin{bmatrix} E_{A_1} & E_{A_2} & \dots & E_{A_k} \end{bmatrix}$ are regular operators. The following conditions are equivalent:*

(i) $(A_1 + A_2 + ... + A_k)\{1,4\} \subseteq A_1\{1,4\} + A_2\{1,4\} + ... + A_k\{1,4\}$,

(ii) $DD^\dagger E_A = E_A$, $(A^\dagger - \sum_{i=1}^k A_i^\dagger)DD^\dagger = A^\dagger - \sum_{i=1}^k A_i^\dagger$.

Proof. For arbitrary matrix $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ we have from (2) that $B \in A\{1,4\} \Leftrightarrow B^* \in A^*\{1,3\}$, so it follows that $(A\{1,4\})^* = A^*\{1,3\}$. Now, condition ($i$) is equivalent to

$$\begin{aligned} &((A_1 + A_2 + ... + A_k)\{1,4\})^* \subseteq (A_1\{1,4\})^* + (A_2\{1,4\})^* + ... + (A_k\{1,4\})^* \\ \Leftrightarrow\ &(A_1 + A_2 + ... + A_k)^*\{1,3\} \subseteq A_1^*\{1,3\} + A_2^*\{1,3\} + ... + A_k^*\{1,3\} \\ \Leftrightarrow\ &(A_1^* + A_2^* + ... + A_k^*)\{1,3\} \subseteq A_1^*\{1,3\} + A_2^*\{1,3\} + ... + A_k^*\{1,3\} \end{aligned}$$

which is from the Theorem 2.2 equivalent to

$$CC^\dagger F_{A^*} = F_{A^*},\ CC^\dagger((A^*)^\dagger - \sum_{i=1}^k (A_i^*)^\dagger) = (A^*)^\dagger - \sum_{i=1}^k (A_i^*)^\dagger, \tag{9}$$

where $C = \begin{bmatrix} F_{A_1^*} & F_{A_2^*} & \dots & F_{A_k^*} \end{bmatrix}$. Since $F_{A^*} = E_A$ and $F_{A_i^*} = E_A$, it is easy to see that (9) is equivalent to (ii).

In an analogous way, necessary and sufficient condition for the opposite inclusion can be derived from the Theorem 2.1.

Theorem 2.4. *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that A_i, $i = 1,2,\dots,k$ and $A = A_1 + A_2 + ... + A_k$ are regular operators. The following conditions are equivalent:*

(i) $A_1\{1,4\} + A_2\{1,4\} + ... + A_k\{1,4\} \subseteq (A_1 + A_2 + ... + A_k)\{1,4\}$,

(ii) $AA^*E_{A_i} = 0$, $i = \overline{1,k}$ $(\sum_{i=1}^{k} A_i^\dagger)AA^* = A^*$.

Theorem 2.5. *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that A_i, $i = 1,2,\dots,k$ and $A = A_1 + A_2 + ... + A_k$ are regular operators. The following conditions are equivalent:*

(i) $A_1\{1,2,3\} + A_2\{1,2,3\} + ... + A_k\{1,2,3\} \subseteq (A_1 + A_2 + ... + A_k)\{1,2,3\}$,

(ii) $A^*AF_{A_i} = 0$, $i = \overline{1,k}$, $A^*A\sum_{i=1}^{k} A_i^\dagger = A^*$, $\sum_{i=1}^{k} A_i^\dagger F_A = 0$, *and* $F_{A_i} = 0$ *or* $A_iA_i^\dagger F_A = 0$ *for every* $i \in \{1,\dots k\}$.

Proof. (ii) $\Rightarrow$ (i) : Suppose that (ii) holds. We need to prove that for arbitrary $A_i^{(1,2,3)} \in A_i\{1,2,3\}$, $i = \overline{1,k}$ it follows that $A_1^{(1,2,3)} + A_2^{(1,2,3)} + ... + A_k^{(1,2,3)} \in A\{1,2,3\}$. Thus, given any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = \overline{1,k}$, we must show that

$$A^*A\Big(\sum_{i=1}^{k} A_i^\dagger + \sum_{i=1}^{k} F_{A_i}V_iA_iA_i^\dagger\Big) = A^*, \tag{10}$$

and

$$\Big(\sum_{i=1}^{k} A_i^\dagger + \sum_{i=1}^{k} F_{A_i}V_iA_iA_i^\dagger\Big)(I - AA^\dagger) = 0, \tag{11}$$

which is satisfied by (ii).

(i) $\Rightarrow$ (ii): If (i) holds, then for any $V_i \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $i = \overline{1,k}$, we have that (10) and (11) hold. Specially, for $V_i = 0$, $i = \overline{1,k}$, by (10) we get that $A^*A\sum_{i=1}^{k} A_i^\dagger = A^*$ and by (11) we get that $\sum_{i=1}^{k} A_i^\dagger F_A = 0$. Similarly, if for any $i \in \{1,2,...,k\}$, we take that $V_i = F_{A_i}$ and that $V_j = 0$, $j \neq i$, by (10) we will get that $A^*AF_{A_i} = 0$ and by (11) we will get that $F_{A_i} = 0$ or $A_iA_i^\dagger F_A = 0$. Hence, (ii) holds. □

Theorem 2.6. *Let $A_i \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be such that A_i, $i = 1,2,\dots,k$ and $A = A_1 + A_2 + ... + A_k$ are regular operators. The following conditions are equivalent:*

(i) $A_1\{1,2,4\} + A_2\{1,2,4\} + ... + A_k\{1,2,4\} \subseteq (A_1 + A_2 + ... + A_k)\{1,2,4\}$,

(ii) $AA^*E_{A_i} = 0$, $i = \overline{1,k}$, $(\sum_{i=1}^{k} A_i^\dagger)AA^* = A^*$, $E_A\sum_{i=1}^{k} A_i^\dagger = 0$, *and* $E_{A_i} = 0$ *or* $E_AA_i^\dagger A_i = 0$ *for every* $i \in \{1,\dots k\}$.

Proof. For arbitrary matrix $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ we have from (4) that $B \in A\{1,2,4\} \Leftrightarrow B^* \in A^*\{1,2,3\}$, so it follows that $(A\{1,2,4\})^* = A^*\{1,2,3\}$. Now, condition ($i$) is equivalent to

$$\begin{aligned}
&(A_1\{1,2,4\})^* + ... + (A_k\{1,2,4\})^* \subseteq ((A_1 + A_2 + ... + A_k)\{1,2,4\})^* \\
&\Leftrightarrow A_1^*\{1,2,3\} + ... + A_k^*\{1,2,3\} \subseteq (A_1 + A_2 + ... + A_k)^*\{1,2,3\} \\
&\Leftrightarrow A_1^*\{1,2,3\} + ... + A_k^*\{1,2,3\} \subseteq (A_1^* + A_2^* + ... + A_k^*)\{1,2,3\}
\end{aligned}$$

which is from the Theorem 2.5 equivalent to

$$\begin{aligned}
&AA^*F_{A_i^*} = 0, \quad i = \overline{1,k}, \quad AA^*\sum_{i=1}^{k}(A_i^*)^\dagger = A, \\
&\sum_{i=1}^{k}(A_i^*)^\dagger F_{A^*} = 0, \\
&F_{A_i^*} = 0 \quad \text{or} \quad A_i^*(A_i^*)^\dagger F_{A^*} = 0, \quad \text{for every} \quad i \in \{1,\dots k\}
\end{aligned} \tag{12}$$

where $A = A_1 + A_2 + ... + A_k$. Since $F_{A^*} = E_A$ and $F_{A_i^*} = E_A$, it is easy to see that (12) is equivalent to (ii).

Theorem 2.7. *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{I},\mathcal{L})$ be such that A, B, C, AB, BC and ABC are regular operators. The following conditions are equivalent:*

(i) $C\{1,2\} \cdot B\{1,2\} \cdot A\{1,2\} \subseteq (ABC)\{1,2\}$,

(ii) $A = 0$ *or* $B = 0$ *or* $C = 0$,
or
$A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ *and* $B \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{H})$,
or
$B \in \mathcal{B}_r^{-1}(\mathcal{L},\mathcal{H})$ *and* $C \in \mathcal{B}_l^{-1}(\mathcal{I},\mathcal{L})$,
or
$A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ *and* $C \in \mathcal{B}_r^{-1}(\mathcal{I},\mathcal{L})$.

Proof. $(i) \Rightarrow (ii)$: If (i) holds, then evidently $C^\dagger B^\dagger A^\dagger \in (ABC)\{1,2\}$, so

$$ABCC^\dagger B^\dagger A^\dagger ABC = ABC \tag{13}$$

and

$$C^\dagger B^\dagger A^\dagger ABCC^\dagger B^\dagger A^\dagger = C^\dagger B^\dagger A^\dagger. \tag{14}$$

Since, for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$, $A^\dagger + (I_\mathcal{H} - A^\dagger A)XAA^\dagger \in A\{1,2\}$ and $B^\dagger + B^\dagger BY(I_\mathcal{H} - BB^\dagger) \in B\{1,2\}$, we get that

$$ABCC^\dagger(B^\dagger + B^\dagger BY(I_\mathcal{H} - BB^\dagger))(A^\dagger + (I_\mathcal{H} - A^\dagger A)XAA^\dagger)ABC = ABC \tag{15}$$

holds for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$. For $X = 0$ in (15) we get

$$ABCC^\dagger(B^\dagger + B^\dagger BY(I_\mathcal{H} - BB^\dagger))A^\dagger ABC = ABC. \tag{16}$$

Substracting (16) from (15) we get that

$$ABCC^\dagger(B^\dagger + B^\dagger BY(I_\mathcal{H} - BB^\dagger))(I_\mathcal{H} - A^\dagger A)XAA^\dagger ABC = 0 \tag{17}$$

holds for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$. If we take $Y = 0$ in (17) we get

$$ABCC^\dagger B^\dagger(I_\mathcal{H} - A^\dagger A)XAA^\dagger ABC = 0. \tag{18}$$

Substracting (18) from (17) we get that

$$ABCC^\dagger B^\dagger BY(I_\mathcal{H} - BB^\dagger)(I_\mathcal{H} - A^\dagger A)XABC = 0 \tag{19}$$

holds for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$, so it follows that
$ABCC^\dagger B^\dagger B = 0$ or $(I_\mathcal{H} - BB^\dagger)(I_\mathcal{H} - A^\dagger A) = 0$ or $ABC = 0$ which is from (13) equivalent to

$$(I_\mathcal{H} - BB^\dagger)(I_\mathcal{H} - A^\dagger A) = 0 \quad \text{or} \quad ABC = 0. \tag{20}$$

Similarly, for any $Y \in \mathcal{B}(\mathcal{H},\mathcal{L})$, $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$, $B^\dagger + (I_\mathcal{L} - B^\dagger B)YBB^\dagger \in B\{1,2\}$ and $C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger) \in C\{1,2\}$, so

$$ABC(C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))(B^\dagger + (I_\mathcal{L} - B^\dagger B)YBB^\dagger)A^\dagger ABC = ABC.$$

In analogous way as above, by taking $Y = 0$ and $Z = 0$ we will obtain

$$(I_\mathcal{L} - CC^\dagger)(I_\mathcal{L} - B^\dagger B) = 0 \quad \text{or} \quad ABC = 0. \tag{21}$$

Since for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$, $A^\dagger + (I_\mathcal{H} - A^\dagger A)XAA^\dagger \in A\{1,2\}$ and $C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger) \in C\{1,2\}$, we have that

$$\begin{aligned} &ABC(C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger \cdot \\ &(A^\dagger + (I_\mathcal{H} - A^\dagger A)XAA^\dagger) \cdot ABC = ABC. \end{aligned} \tag{22}$$

Substituting $X = 0$ in (22) we get

$$ABC(C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger A^\dagger ABC = ABC. \tag{23}$$

Substracting (23) from (22) we get that

$$ABC(C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger(I_\mathcal{H} - A^\dagger A)XAA^\dagger ABC = 0 \tag{24}$$

holds for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$. If we take $Z = 0$ in (24) we get

$$ABCC^\dagger B^\dagger(I_\mathcal{H} - A^\dagger A)XAA^\dagger ABC = 0 \tag{25}$$

Substracting (25) from (24) we get that

$$ABCZ(I_\mathcal{L} - CC^\dagger)B^\dagger(I_\mathcal{H} - A^\dagger A)XABC = 0 \tag{26}$$

holds for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$, so it follows that

$$(I_\mathcal{L} - CC^\dagger)B^\dagger(I_\mathcal{H} - A^\dagger A) = 0 \quad \text{or} \quad ABC = 0. \tag{27}$$

Suppose that $ABC \neq 0$. Then from (20),(21) and (27) we have

$$\begin{aligned} &(I_\mathcal{H} - BB^\dagger)(I_\mathcal{H} - A^\dagger A) = 0, \\ &(I_\mathcal{L} - CC^\dagger)(I_\mathcal{L} - B^\dagger B) = 0, \\ &(I_\mathcal{L} - CC^\dagger)B^\dagger(I_\mathcal{H} - A^\dagger A) = 0. \end{aligned} \tag{28}$$

Equalities (28) are equivalent to

$$I_\mathcal{H} - BB^\dagger - A^\dagger A - BB^\dagger A^\dagger A = 0, \tag{29}$$

$$I_\mathcal{L} - CC^\dagger - B^\dagger B - CC^\dagger B^\dagger B = 0, \tag{30}$$

$$B^\dagger - CC^\dagger B^\dagger - B^\dagger A^\dagger A + CC^\dagger B^\dagger A^\dagger A = 0. \tag{31}$$

Since for any $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$, $C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger) \in C\{1,2\}$, we have that

$$\begin{aligned} &(C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger A^\dagger ABC(C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger A^\dagger \\ &= (C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger A^\dagger. \end{aligned}$$

which is from (28) equivalent to

$$\begin{aligned} &C^\dagger B^\dagger A^\dagger ABC(C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger A^\dagger \\ &= (C^\dagger + C^\dagger CZ(I_\mathcal{L} - CC^\dagger))B^\dagger A^\dagger. \end{aligned} \tag{32}$$

Since (32) holds for any $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$, we get

$$(I_\mathcal{L} - CC^\dagger)B^\dagger A^\dagger = 0 \quad \text{or} \quad C^\dagger B^\dagger A^\dagger ABC = C^\dagger C. \tag{33}$$

Since for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $A^\dagger + (I_\mathcal{H} - A^\dagger A)XAA^\dagger \in A\{1,2\}$, we have that

$$C^\dagger B^\dagger(A^\dagger + (I_{\mathcal{H}} - A^\dagger A)XAA^\dagger)ABCC^\dagger B^\dagger(A^\dagger + (I_{\mathcal{H}} - A^\dagger A)XAA^\dagger)$$
$$= C^\dagger B^\dagger(A^\dagger + (I_{\mathcal{H}} - A^\dagger A)XAA^\dagger)$$

which us from (28) equivalent to

$$\begin{aligned} &C^\dagger B^\dagger(A^\dagger + (I_{\mathcal{H}} - A^\dagger A)XAA^\dagger)ABCC^\dagger B^\dagger A^\dagger \\ &= C^\dagger B^\dagger(A^\dagger + (I_{\mathcal{H}} - A^\dagger A)XAA^\dagger). \end{aligned} \tag{34}$$

Since (34) holds for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, using (14) we get

$$ABCC^\dagger B^\dagger A^\dagger = AA^\dagger \quad \text{or} \quad C^\dagger B^\dagger(I_{\mathcal{H}} - A^\dagger A) = 0. \tag{35}$$

Suppose now that the first equality in (33) is satisfied. From (31) we get that $B^\dagger - CC^\dagger B^\dagger = 0$, which is from (30) equivalent to $CC^\dagger = I_{\mathcal{L}}$, i.e. $C \in \mathcal{B}_r^{-1}(\mathcal{I}, \mathcal{L})$. If first equality in (35) is satisfied, then from $CC^\dagger = I_{\mathcal{L}}$ we get $ABB^\dagger A^\dagger = AA^\dagger$ i.e. $BB^\dagger A^\dagger A = A^\dagger A$ which from (29) implies $BB^\dagger = I_{\mathcal{H}}$, i.e. $B \in \mathcal{B}_r^{-1}(\mathcal{L}, \mathcal{H})$. Since $ABC \neq 0$ by assumption and $B \in \mathcal{B}_r^{-1}(\mathcal{L}, \mathcal{H})$ and $C \in \mathcal{B}_r^{-1}(\mathcal{I}, \mathcal{L})$, (*ii*) obviously holds. If second equality in (35) is satisfied, then multiplying it by C from the left we get $B^\dagger(I_{\mathcal{H}} - A^\dagger A) = 0$, i.e. $BB^\dagger(I_{\mathcal{H}} - A^\dagger A) = 0$ which from (29) implies $A^\dagger A = I_{\mathcal{H}}$, i.e. $A \in \mathcal{B}_l^{-1}(\mathcal{H}, \mathcal{K})$. Since $A \in \mathcal{B}_l^{-1}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}_r^{-1}(\mathcal{I}, \mathcal{L})$, (*ii*) holds.

Suppose now that the second equality in (33) is satisfied. If the second equality in (35), i.e. $C^\dagger B^\dagger = C^\dagger B^\dagger A^\dagger A$ is satisfied, then we get $C^\dagger B^\dagger BC = C^\dagger C$, i.e. $CC^\dagger B^\dagger B = CC^\dagger$ which using (30) implies $B^\dagger B = I$, i.e. $B \in \mathcal{B}_l^{-1}(\mathcal{L}, \mathcal{H})$. Substituting $C^\dagger B^\dagger = C^\dagger B^\dagger A^\dagger A$ in (31) we get

$$B^\dagger - CC^\dagger B^\dagger - B^\dagger A^\dagger A + CC^\dagger B^\dagger = 0.$$

Multiplying last equality by B from the right and using $B^\dagger B = I$ we get $B^\dagger A^\dagger AB = I$, i.e. $A^\dagger ABB^\dagger = BB^\dagger$. Now from (29) we obtain $A^\dagger A = I$, i.e. $A \in \mathcal{B}_l^{-1}(\mathcal{H}, \mathcal{K})$. Since $A \in \mathcal{B}_l^{-1}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}_l^{-1}(\mathcal{L}, \mathcal{H})$ and $ABC \neq 0$ by assumption, (*ii*) holds. Now, suppose that first equality in (35) is satisfied. Multiplying (31) by BC from the right we get

$$B^\dagger BC - CC^\dagger B^\dagger BC - B^\dagger A^\dagger ABC + CC^\dagger B^\dagger A^\dagger ABC = 0. \tag{36}$$

Substituting $C^\dagger B^\dagger A^\dagger ABC = C^\dagger C$ in (36) and using $CC^\dagger B^\dagger B = B^\dagger BCC^\dagger$ we get

$$-B^\dagger A^\dagger ABC + C = 0. \tag{37}$$

Multiplying (37) by B from the left we get

$$A^\dagger ABC = BC. \tag{38}$$

Substituting (38) in $C^\dagger B^\dagger A^\dagger ABC = C^\dagger C$ we get $C^\dagger B^\dagger BC = C^\dagger C$, i.e. $CC^\dagger B^\dagger B = CC^\dagger$. Now from (30) we get $B^\dagger B = I$, i.e. $B \in \mathcal{B}_l^{-1}(\mathcal{L}, \mathcal{H})$. Multiplying (31) by AB from the left and using $CC^\dagger B^\dagger B = B^\dagger BCC^\dagger$ we get

$$ABCC^\dagger B^\dagger = ABCC^\dagger B^\dagger A^\dagger A. \tag{39}$$

Substituting $ABCC^\dagger B^\dagger A^\dagger = AA^\dagger$ in (39) we get

$$ABCC^\dagger B^\dagger = A. \tag{40}$$

Multiplying (40) by B from the left and using $B^\dagger B = I$ we get

$$ABCC^\dagger = AB. \tag{41}$$

Multiplying (41) by $B^\dagger A^\dagger$ from the right and using $ABCC^\dagger B^\dagger A^\dagger = AA^\dagger$ we get

$$ABB^\dagger A^\dagger = AA^\dagger,$$

i.e.

$$A^\dagger ABB^\dagger = A^\dagger A.$$

Now from (29) we have $BB^\dagger = I$, i.e. $B \in \mathcal{B}_r^{-1}(\mathcal{L},\mathcal{H})$. Since $B \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{H})$, it follows that B is invertible, i.e. $B \in \mathcal{B}^{-1}(\mathcal{L},\mathcal{H})$. Now, $B\{1,2\} = \{B^{-1}\}$ and it is easy to see that $(AB)\{1,2\} = B^{-1}A\{1,2\}$. Now, $C\{1,2\} \cdot B\{1,2\} \cdot A\{1,2\} \subseteq (ABC)\{1,2\}$ is equiavlent to $C\{1,2\} \cdot (AB)\{1,2\} \subseteq (ABC)\{1,2\}$. From [2] we have that this is satisfied if and only if $AB \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{K})$ or $C \in \mathcal{B}_r^{-1}(\mathcal{I},\mathcal{L})$. If $AB \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{K})$ then

$$\mathcal{N}(AB) = \{0\}. \tag{42}$$

Since

$$\begin{aligned}
\mathcal{N}(A) &= \mathcal{N}(A) \cap \mathcal{H} \\
&= \mathcal{N}(A) \cap \mathcal{R}(B) \\
&= \mathcal{N}(A) \cap \mathcal{R}(B) \cup \mathcal{N}(B) \\
&= \mathcal{N}(AB) \\
&= \{0\},
\end{aligned}$$

it follows that $A^\dagger A = I$, i.e. $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$. Since $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{H})$, (ii) is satisfied.

Now, let us consider the case $ABC = 0$. It is obvious that $(ABC)\{1,2\} = \{0\}$. From [2] we have that $(AB)\{1,2\} \subseteq B\{1,2\}A\{1,2\}$ always holds so $C\{1,2\}(AB)\{1,2\} \subseteq C\{1,2\}B\{1,2\}A\{1,2\} \subseteq (ABC)\{1,2\} = \{0\}$. From [2] this is satisfied if and only if

$$AB = 0, \tag{43}$$
$$\text{or} \quad C = 0, \tag{44}$$
$$\text{or} \quad AB \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{K}), \tag{45}$$
$$\text{or} \quad C \in \mathcal{B}_r^{-1}(\mathcal{I},\mathcal{L}). \tag{46}$$

Case 1. Suppose that (43) is satisfied. Let $A^{(1,2)} \in A\{1,2\}$ and $B^{(1,2)} \in B\{1,2\}$ be arbitrary. Then $C^\dagger B^{(1,2)}A^{(1,2)} \subseteq (ABC)\{1,2\} = \{0\}$, i.e.
$C^\dagger B^{(1,2)}A^{(1,2)} = 0$. Since for arbitrary $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$, $C^\dagger + C^\dagger CZ(I_{\mathcal{L}} - CC^\dagger) \in C\{1,2\}$ we have that

$$(C^\dagger + C^\dagger CZ(I_{\mathcal{L}} - CC^\dagger))B^{(1,2)}A^{(1,2)} = 0, \tag{47}$$

is satisfied for any $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$. Since $C^\dagger B^{(1,2)}A^{(1,2)} = 0$ from (47) we get that

$$C^\dagger CZB^{(1,2)}A^{(1,2)} = 0 \tag{48}$$

holds for any $Z \in \mathcal{B}(\mathcal{L},\mathcal{I})$. From (48) we have that either $C = 0$ or $B^{(1,2)}A^{(1,2)} = 0$. If $C \neq 0$ it follows that $B\{1,2\}A\{1,2\} = \{0\}$. Now we have $B\{1,2\}A\{1,2\} = \{0\} = (AB)\{1,2\}$ so from [2] we have that $A = 0$ or $B = 0$ or $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ or $B \in \mathcal{B}_r^{-1}(\mathcal{H},\mathcal{K})$. If $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ then from $AB = 0$ it follows that $B = 0$. If $B \in \mathcal{B}_r^{-1}(\mathcal{H},\mathcal{K})$ then from $AB = 0$ we get that $A = 0$.

Case 2. In this case we have $C = 0$.

Case 3. Suppose that (45) is satisfied. Then from $ABC = 0$ it follows that $C = 0$.

Case 4. Suppose that (46) holds. Then from $ABC = 0$ it follows that $AB = 0$ which is the same as Case 1.

So $ABC = 0$ implies $A = 0$ or $B = 0$ or $C = 0$.

$(ii) \Rightarrow (i)$: If A or B or C is zero, it is evident that (i) holds.

Suppose that $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{H})$. Then since $B \in \mathcal{B}_l^{-1}(\mathcal{L},\mathcal{H})$ it follows from [2] that $C\{1,2\}B\{1,2\} \subseteq (BC)\{1,2\}$. Since $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ from [2] it follows that $(BC)\{1,2\}A\{1,2\} \subseteq (ABC)\{1,2\}$. Now we have

$$C\{1,2\}B\{1,2\}A\{1,2\} \subseteq (BC)\{1,2\}A\{1,2\} \subseteq (ABC)\{1,2\}.$$

The rest can be proved in the same manner. □

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