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A CONVEXITY IN TOPOLOGICAL SPACES AND KKM MULTIFUNCTIONS

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ABSTRACT. *In this note using the method from [2], we prove a generalization of the well known KKM principle, for multifunctions in topological spaces. As application of this theorem, we give an extension of Ky Fan minimax inequalities.*

1. Introduction

H. Komiya introduced, in [2], the notion of convexity in an arbitrary topological space and obtained the generalizations of the well known Browder's fixed point theorem for multifunctions and Sion's minimax theorem. Other generalizations of this theorems to spaces without vector structures, have been given by O. Hadžic [1] and M. R. Tasković [3]. We shall give some definition's and results from [2].

Let X be a Hausdorff topological space, $P(X)$ the family all subsets of X and $F(X)$ the family of all finite subsets of X.

DEFINITION 1 [2]. *An H-operator on X is mapping $[]: P(X) \to P(X)$ satisfying the following conditions*:

a) $[\phi] = \phi$;
b) $[\{x\}] = \{x\}$, *for every* $x \in X$;
c) $[[A]] = [A]$, *for every* $A \in P(X)$;
d) $[A] = \{[F] : F \in F(X)\}$.

A set $A \subset P(X)$ is convex if $[A] = A$ and for $A \in P(X)$, the image $[A]$ of A is said to be the convex hull of A. In [2] the following proposition is proved.

PROPOSITION. 1) *An H-operator is monotone* $(A \subset B \Rightarrow [A] \subset [B]$).
2) *The convex hull* $[A]$ *of* A *is the smallest convex set containing* A.
3) $[X] = X$
4) *If* $[C_i] = C_i (i \in I)$ *then* $[\bigcap_{i \in I} C_i] = \bigcap_{i \in I} C_i$
5) *If* $[C_i] = C_i (i \in I)$ *and for every* $i_1, i_2 \in I$ *there exists* $i \in I$ *with* $(C_{i1} \bigcup C_{i2}) \subset C_i$ *then* $[\bigcup_{i \in I} C_i] = \bigcup_{i \in I} C_i$

Let $\mathcal{R}$ be the set of all functions$f : N \to R$ (N is a countably infinite set) such that card $\{x : x \in N, f(x) \neq 0\} < \infty$.

This implies that $\mathcal{R} = \sum R_i$, where $R_i = R$. The topology and linear structure on R are the usual ones.

Suppoose that an H-operator [ ] on a topological space X is given, and let $H(X) = \{[F] : F \in F(X)\}$.

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Definition 2 [2]. *For $H \in H(X)$, a mapping $\Phi : H \to \mathcal{R}$ is called a structure mapping on H, if it has the following properties:*
a) *The mapping Φ is an into-homeomorphism.*
b) *If $A \in H$ then $\Phi([A]) = [\Phi(A)]$, where $[\Phi(A)]$ is the usual convex hull of $\Phi(A)$ in $\mathcal{R}$.*

By SH the set of all structures mappings on H is denoted. If $SH = \phi$, for every $H \in H(X)$ then $\Phi \in SH$ is said to be a structure on X with respect to the H-operator $[\,]$.

Definition 3 [2]. *A convex space $(X, [\,], \Phi)$ is a triple consisting of a topological space X, an H-operator $[\,]$ on X and structure Φ on X with respect to $[\,]$.*

If $(X, [\,], \Phi)$ is a convex space,Y X and Y is convex then $(Y, [\,]_y, \Phi y)$ is subspace of $(X, [\,], \Phi)$ where the topology of Y is the induced topology, for A $P(Y) : [A]_y = [A]$ and $\Phi_y = \Phi \mid H(Y)$.

2. Main results

Let $(Y, [\,], \Phi)$ be an convex space and X be a non-empty subset of Y. A multifunction $G : X \to P(Y)$ is called KKM - map if

$$[F] \subseteq \bigcup_{x_k \in F} G_{x_k}, \quad \text{for all} \quad F \in F(X)$$

The following theorem is a generalisations of famous KKM principle.

Theorem 1. *Let $(X, [\,], \Phi)$ be an convex space and let $F : X \to P(X)$ be such that:*
a) *F is* KKM*-map;*
b) *for some $x_0 \in X$, $F(x_0)$ is compact and for each $x \in X$, $F(x)$ is closed in X. Then*

$$\bigcap_{x \in X} F(x) \neq 0$$

Proof. It is enough to show that the family $\{F(x) : x \in X\}$ has the finite intersection property. Let $\{x_1, ..., x_n\}$ be a finite set in X and $X_j* = F(x_j) \cap [x_1, ..., x_n]$ for $j = 1, n$. $[x_1, ... x_n] \subseteq H(X)$ wich implies that $\Phi([x_1, ..., x_n])$ is compact and convex set in $\mathcal{R}$. Let $Y_j = \Phi(X_j*)$ for $j = 1, n$ and let $G : \Phi([x_1, ..., x_n]) \to P(R)$ is a multifunction defined by $G = \Phi\ F\ \Phi^{-1}$. We can see that G satisfied conditions of clasical KKM principle (for Euclidian space) and so there exist

$$y_0 \in \bigcap_{j=1,n} Yj$$

Let $x_0 = \Phi^{-1}(y_0)$.Then

$$x_0 \in \bigcap_{k=1,n} F(x_k)$$

There family $\{F(x) : x \in X\}$ has the finite intersection property.

We recall that a real valued function f defined on a convex space X is said to be quasi-concave if for every real number t the set $\{x : f(x) > t\}$ is convex. The folowing theorem is generalisations of famous Ky Fan's minimax inequalities.

THEOREM 2. *Let X be a compact convex space. Let f be a real valued function defined on $X \times X$ such that:*

a) *For each fixed $x \in X$, $f(x,y)$ is a lower semi-continous function of y on X.*

b) *For each fixed $y \in X$, $f(x,y)$ is a quasi-concave functin of x on X.*

Then the minimax inequality

$$\min_{y\in X} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} f(x,x)$$

hold.

PROOF. Let $A = \sup_{x\in X} f(x,x)$ and $G(x) = \{y \in X : f(x,y) \leq A\}$ for all $x \in X$. We can see that G is a KKM multifunction. Hence by Theorem 1. , there is a point y_0 such that

$$y_0 \in \bigcap_{x\in X} G(x)$$

so $f(x, y_0) \leq A$ for all $x \in X$.

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