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ON THE EQUALITY $\|\cdot\|_e = \|\cdot\|_w$

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ABSTRACT. *In this note we prove the equality* $\|\cdot\|_e = \|\cdot\|_w$ [1, *Chapter IV, Exercise* 10] *on elementary way.*

Let X be an infinite-dimensional complex Hilbert space and denote the set of bounded (compact) linear operators on X by $B(X)$ $(K(X))$. The fact that $K(X)$ is a closed two-sided ideal in $B(X)$ enables us to define the Calkin algebra over X as the quotient algebra $C(X) = B(X)/K(X)$. $C(X)$ is itself a Banach algebra (in fact a C^*-algebra) in the quotient algebra norm

$$(1) \qquad \|T\|_e \equiv \|T + K(X)\| = \inf_{K \in K(X)} \|T + K\|, \quad T \in B(X).$$

For $T \in B(X)$ set

$$(2) \qquad \|T\|_w = \sup\{\limsup \|Tx_n\| : \|x_n\| = 1, \ x_n \to 0 \text{ weakly}\}.$$

We have seen [1, pp.47], that David Berg pointed out that

$$(3) \qquad \|T\|_e = \|T\|_w, \quad T \in B(X).$$

In [1, Chapter IV, Exercise 10] there are suggestions for the proof of (3). In this note we prove (3) on the different way, we think elementary.

Proof of (3). It is clear that $\|T\|_w \le \|T\|_e$, $T \in B(X)$, and that $\|T\|_w = 0 \iff T \in K(X)$. Further, it is also clear that $\|S+T\|_w \le \|S\|_w + \|T\|_w$, $S, T \in B(X)$. Now, by [4, Lemma] to prove (3) it is enough to prove that

$$(4) \qquad \|ST\|_w \le \|S\|_w \|T\|_w, \quad S, \ T \in B(X).$$

To prove (4), suppose that $\|ST\|_w > 0$ and $\epsilon > 0$ are such that $\|ST\|_w > \epsilon$. Now there exists a seguence (x_n) in X, such that $\|x_n\| = 1$, $n = 1, 2, \dots, x_n \to 0$ weakly and

$$(5) \qquad \|ST\|_w - \epsilon < \limsup_{n \to \infty} \|STx_n\|.$$

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From (5) it follows that there is a subsequence (x_{n_k}) of (x_n) such that

(6) $$\|ST\|_w - \epsilon < \|STx_{n_k}\|, \quad k = 1, 2, \dots .$$

Set $y_k = \frac{Tx_{n_k}}{\|Tx_{n_k}\|}$, $k = 1, 2, \dots$, and $a \equiv \frac{\|ST\|_w - \epsilon}{\|S\|}$. It is clear that $\|y_k\| = 1$, $k = 1, 2, \dots$,; that $y_k \to 0$ weakly follows from (of course $x \in X$)

(7) $$\begin{aligned} |(y_k, x)| &= |(\frac{Tx_{n_k}}{\|Tx_{n_k}\|}, x)| = \frac{1}{\|Tx_{n_k}\|}|(Tx_{n_k}, x)| = \frac{1}{\|Tx_{n_k}\|}|(x_{n_k}, T^* x)| \\ &< \frac{1}{a}|(x_{n_k}, T^* x)| \to 0, \quad k \to \infty. \end{aligned}$$

From (6) we have that

(8) $$\|ST\|_w - \epsilon < \|Sy_k\|\|Tx_{n_k}\|, \ k = 1, 2, \dots ,$$

and finally from (8) we get

(9) $$\begin{aligned} \|ST\|_w - \epsilon &\le \limsup_{k\to\infty} \|Sy_k\|\|Tx_{n_k}\| \\ &\le \limsup_{k\to\infty} \|Sy_k\| \limsup_{k\to\infty} \|Tx_{n_k}\| \le \|S\|_w\|T\|_w . \end{aligned}$$

This completes the proof.

Let us remark that [4, Lemma] is sometimes very useful (see for example [3] for anather application).

I would like to finish this note with the next problem (presented at the conference Filomat 92, problem section, University of Niš, Faculty of Philosophy). At the first glimpse our question is not connected with this note, but it is not true (see [2, pp. 29]).

Problem. Find a Banach space X and a subset $Q \subset X$ such that for every infinite dimensional subspace $Y \subset X$ there is an infinite dimensional subspace $Y_0 \subset Y$, such that $Q \cap Y_0$ is relatively compact.

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