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COHERENT HOMOTOPY IN INVERSE SYSTEMS

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ABSTRACT. *In this paper will be defined a new coherent category of coherent inverse systems which from the earlier known category COH, defined by the author, and S.Mardešić and the author in [6] and [8], differs in the definition of coherent homotopy.*

The phenomenon of coherence is presented best with the following definition

DEFINITION. *A coherent inverse system* $\mathcal{X} = (X_a, p_a, A)$ consists of the following data:

1) a directed set $(A, <)$ i.e. a set A with a transitive and non-reflexive relation $<$ such that for every $a, a' \in A$ there exists $a'' \in A$ such that $a'' > a$, $a'' > a'$.
2) for every $a \in A$ a topological space X_a, for $a_0 < a_1$ a map $p_{a_0a_1} : X_{a_1} \to X_{a_0}$ and for $n > 1$ and $\underline{a} = (a_0, \dots, a_n)$, $a_0 < \cdots < a_n$ a sequence in A, a map $p_{\underline{a}} : I^{n-1} \times X_{a_n} \to X_{a_0}$ such that

$$(1)\quad p_{\underline{a}}(t_1, \dots, t_n, x) = \begin{cases} p_{a_0\dots\hat{a}_i\dots a_n}(t_1, \dots, \hat{t}_i, \dots, t_{n-1}, x), & t_i = 0 \\ p_{a_0\dots a_i}(t_1, \dots, t_{i-1}, p_{a_i\dots a_n}(t_{i+1}, \dots, t_{n-1}, x)), & t_i = 1 \end{cases}$$

where $(t_1, \dots, t_{n-1}) \in I^{n-1}$, $x \in X_{a_n}$, and $1 \le i \le n-1$. As usually $\hat{a}_i$ means that a_i is omitted i.e.$(a_0, \dots, \hat{a}_i, \dots, a_n) = (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$.

For $n = 2$, $p_{a_0a_1a_2} : I \times X_{a_2} \to X_{a_0}$ is a homotopy connecting maps $p_{a_0a_2}$ and $p_{a_0a_1}p_{a_1a_2}$.

An example of a coherent inverse system is Cech system for a topological space.

The maps $f : \mathcal{X} \to \mathcal{Y}$ between coherent systems also consist of homotopies of all orders ,and their definition in this paper is actually the same as the earlier definition. Here instead of the space $I^{n-k} \times \Delta^k$,in the definition of coherent map it is used the same space with the permutated coordinates for combinatorial reasons. The advantage of the definition in this paper is the existence of complete analogy with the ordinary one-dimensional homotopy theory, not only in the composition formula, but also in the formulae which appear in the proofs of theorems.

The cruical difference is in the definition of the coherent homotopy between two coherent maps. The obtained category of coherent inverse systems with this new definition of coherent homotopy satisfies the requirements of the theory announced by Ju. Lisica in [1] and [2] without explicitely given formulas.

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The paper is devided in four sections:

In §1 we define a coherent category over a cofinite set $Coh(E)$. Although the construction of $Coh(E)$ is only a step in the constuction of Coh, it seems that category $Coh(E)$ itself is worth of attention.

In §2 it is defined a coherent map over a cofinite set, $p : \mathcal{X} \to \mathcal{X}$, which is cruical for the new definition of the coherent homotopy.

In §3 ,using the results of §1 and §2 it is constructed the category of coherent inverse systems – Coh. By associating to a topological space X, a coherent system $\mathcal{X}$ it is defined a coherent shape category of all topological spaces. The morphisms of this category $F : X \to Y$ are coherent homotopy classes of coherent maps $f : \mathcal{X} \to \mathcal{Y}$ between the associated coherent systems.

In §4 it is investigated the relation of Coh with the coherent category CPHTop of (commutative) inverse systems defined by Ju. Lisica and S. Mardešić in [4] and [3]. The objects of CPHTop are inverse system $\underline{X} = (X_a, p_{a_0a_1}, A)$ where maps $p_{a_0a_1} : X_{a_0a_11} \to X_{a_1}$ defined for $a_0 < a_1$, commute i.e. $p_{a_0a_2} = p_{a_0a_1}p_{a_1a_2}$ if $a_0 < a_1 < a_2$, but the coherence appear in the definition of maps $F : \underline{X} \to \underline{Y}$ i. e. they are defined using homotopies of all orders.

Also, in §4 it is given an explanation of the advantage of the new definition of coherent homotopy in relation with the old one.

§1.Coherent category over a cofinite set

A cofinite directed set is a directed set $(E, <)$ such that each element of E has only a finite number of predecessors.

We will define a category $Coh(E)$ - coherent category over E. Objects of $Coh(E)$ are ordered pairs $(\mathcal{X}, \alpha)$, where $\mathcal{X} = (X_a, p_{\underline{a}}, A)$ is a coherent inverse system and $\alpha : E \to A$ is a (strictly) increasing function.

Now, we will introduce the notion of a coherent map over E. For a given integer $n > 0$ and a sequence of integers $\underline{j} = (j_0, \ldots, j_k)$, $0 = j_0 < \cdots < j_k = n$, we define a subset $I^{\underline{j}}$ of the n-dimensional cube $I^n = [0, 1]^n$ by

$$I^{\underline{j}} = \{(t_1, \ldots, t_n) : 1 \geq t_{j_1} \geq t_{j_2} \geq \cdots \geq t_{j_k} \geq 0\}.$$

If $n = 0$ and $n = 1$ there is only one possible sequence, namely $\underline{j} = (0)$ and $\underline{j} = (0, 1)$. For this reason we will omit sequences (0) and (0,1).

If B is a directed set and $\phi : B \to A$ a (strictly) increasing function we put $\phi(b_0, \ldots, b_n) = (\phi(b_0), \ldots, \phi(b_n))$ for any sequence $(b_0, \ldots, b_n)$, $b_0 < \cdots < b_n \in B$.

Definition: *A coherent map over* E, $f : (\mathcal{X}, \alpha) \to (\mathcal{Y}, \beta)$ consists of: for every $\underline{e} = (e_0, \ldots, e_n)$ an increasing sequence in E, and $\underline{j} = (j_0, \ldots, j_k)$ of a map $f^{\underline{j}}_{\beta(\underline{e})} : I^{\underline{j}} \times X_{\alpha(e_n)} \to Y_{b(e_0)}$ satisfying the following boundary condition

$$f^{\underline{j}}_{\beta(\underline{e})}(t_1, \ldots, t_n, x) = \tag{2}$$

$$= \begin{cases} q_{\beta(e_0\dots e_{j_1})}(t_1,\dots,t_{j_1-1}, f^{j_1-j_1\dots j_k-j_1}_{\beta(e_{j_1}\dots e_n)}(t_{j_1+1},\dots,t_n,x)); & t_{j_1}=1\\ f^{j_0\dots\hat{j}_i\dots j_k}_{\beta(\underline{e})}(t_1,\dots,t_{j_i-1},1,t_{j_i+1},\dots,t_n,x); & t_{j_i}=t_{j_i+1},\ 0<i<k\\ f^{j_0\dots j_{k-1}}_{\beta(e_0\dots e_{j_{k-1}})}(t_1,\dots,t_{j_{k-1}}-1, p_{\alpha(e_{j_{k-1}}\dots e_n)}(t_{j_{k-1}+1},\dots,t_{n-1},x)); & t_n=0\\ f^{j_0\dots j_i j_{i+1}-1\dots j_k-1}_{\beta(e_0\dots\hat{e}_j\dots e_n)}(t_1,\dots,\hat{t}_j,\dots,t_n,x); & t_j=0,\ j_i<j<j_{i+1}. \end{cases}$$

Specially, for $n=1$ and $b_0<b_1$ the map $f_{\beta(e_0e_1)}: I\times X_{\alpha(e_1)}\to X_{\beta(e_0)}$ satisfies $f_{\beta(e_0e_1)}(1,x)=q_{\beta(e_0e_1)}f_{\beta(e_1)}(x), f_{\beta(e_0e_1)}(0,x)=f_{\beta(e_0)}p_{\alpha(e_0e_1)}(x)$.

In the general case, for n fixed integer, this condition enable us to stick all posible maps of order n i.e. $f^{\underline{j}}_{\beta(\underline{e})}$, $\underline{j}=(j_0,\dots,j_k)$, $0=j_0<\dots<j_k=n$ and to obtain one big homotopy of order n such that on the boundary appear all possible combinations of maps of type q,p,f of lower dimension.

To define the *composition of coherent maps* over E we define a partition of $I^{\underline{j}}$ into subpolyhedra defined by

$$K^{\underline{j}}_{j_m}=\left\{(t_1,\dots,t_n): t_{j_m}\ge 1/2\ge t_{j_{m+1}}\right\},\quad m=0,1,\dots,k.$$

Only for this type of definitions of partitions we formally put $t_{j_0}=1$ and $t_{j_{k+1}}=0$.

Composition of two coherent maps over E, $f:(\mathcal{X},\alpha)\to(\mathcal{Y},\beta)$ and $g:(\mathcal{Y},\beta)\to(\mathcal{Z},\gamma)$ given by maps $f^{\underline{j}}_{\beta(\underline{e})}$ and $g^{\underline{j}}_{\gamma(\underline{e})}$ respectively is a coherent map over E, $h=g\circ f:(\mathcal{X},\alpha)\to(\mathcal{Z},\gamma)$ given by maps $h^{\underline{j}}_{\gamma(\underline{e})}: I^{\underline{j}}\times X_{\alpha(e_n)}\to Z_{\gamma(e_0)}$ defined for $t\in K^{\underline{j}}_{j_m}$, $x\in X_{\alpha(\underline{e})}$ by

$$\begin{aligned} &h^{\underline{j}}_{\gamma(\underline{e})}(t_1,\dots,t_n,x)=\\ (3)\qquad &g^{j_0\dots j_m}_{\gamma(e_0\dots e_{j_m})}\Big(\sum_{i=0}^{m-1}(0,\dots,0_{j_i},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}}-1,0,\dots,0),\\ &f^{j_m-j_i\dots j_k-j_m}_{\beta(e_{j_m}\dots e_n)}\Big(\sum_{i=m}^{k-1}(0,\dots,0_{j_i-j_m},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}},0,\dots,0),x\Big). \end{aligned}$$

Coherent maps over E, $f,f':(\mathcal{X},\alpha)\to(\mathcal{Y},\beta)$ are *coherently homotopic* if there exists a coherent map over E, $F:(I\times\mathcal{X},\alpha)\to(\mathcal{Y},\beta)$, $(I\times\mathcal{X}=(I\times X_a,1\times p_{\underline{a}},A))$, given by the maps $F^{\underline{j}}_{\beta(\underline{e})}: I^{\underline{j}}\times I\times X_{\alpha(e_n)}\to Y_{\beta(e_0)}$ such that

$$(4)\qquad \begin{aligned} F^{\underline{j}}_{\beta(\underline{e})}(t_1,\dots,t_n,0,x)&=f^{\underline{j}}_{\beta(\underline{e})}(t_1,\dots,t_n,x)\\ F^{\underline{j}}_{\beta(\underline{e})}(t_1,\dots,t_n,1,x)&=f'^{\underline{j}}_{\beta(\underline{e})}(t_1,\dots,t_n,x). \end{aligned}$$

If f and f' are coherently homotopic maps over E we write $f\cong$f'.

THEOREM 1. *The relation of homotopy $\cong$ of coherent maps over E is an equivalence relation.*

PROOF. Symmetry and reflexivety are obvious. If $f \cong f'$ with a homotopy F and $f' \cong f"$ with a homotopy F', then $f \cong f''$ with a homotopy F'' given by maps $F''^{\underline{j}}_{\beta(\underline{e})} : I^{\underline{j}} \times I \times X_{\alpha(e_n)} \to Y_{\beta(e_0)}$ defined by

$$F''^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, s, x) = \begin{cases} F''^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, 2s, x); & 0 \le s \le 1/2 \\ F'^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, 2s-1, x); & 1/2 \le s \le 1. \end{cases} \blacksquare$$

If $f : (\mathcal{X}, \alpha) \to (\mathcal{Y}, \beta)$, $g : (\mathcal{Y}, \beta) \to (\mathcal{Z}, \gamma)$ and $h : (\mathcal{Z}, \gamma) \to (\mathcal{W}, \delta)$ are coherent maps over E, in order to obtain an explicit formula for the map $h \circ (g \circ f) : (\mathcal{X}, \alpha) \to (\mathcal{W}, \delta)$ we define a partition of $I^{\underline{j}}$ to subpolyhedra $K^{\underline{j}}_{j_l j_m}$, $0 \le l \le m \le k$, defined by

$$K^{\underline{j}}_{j_l j_m} = \{(t_1, \dots, t_n) : t_{j_l} \ge \frac{1}{2} \ge t_{j_{l+1}}, t_{j_m} \ge \frac{1}{4} \ge t_{j_{m+1}}\}.$$

By applying the composition formula (3) twice we have for $(t_1, \dots, t_n) \in K^{\underline{j}}_{j_l j_m}$

$$h \circ (g \circ f)^{\underline{j}}_{\delta(\underline{e})}(t_1, \dots, t_n, x) =$$

$$h^{j_0 \dots j_l}_{\delta(e_0 \dots e_{j_l})}(\sum_{i=0}^{l-1}(0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 2t_{j_{i+1}} - 1, 0, \dots, 0),$$

$$(5) \qquad g^{j_l - j_l \dots j_m - j_l}_{\gamma(e_{j_l} \dots e_{j_m})}(\sum_{i=l}^{m-1}(0, \dots, 0_{j_i - j_l}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 4t_{j_{i+1}-1}, 0, \dots, 0)$$

$$f^{j_m - j_m \dots j_k - j_m}_{\beta(e_{j_m} \dots e_n)}(\sum_{i=m}^{k-1}(0, \dots, 0_{j_i - j_m}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 4t_{j_{i+1}}, 0, \dots, 0), x).$$

Similiarly, for the map $(h \circ g) \circ f : (\mathcal{X}, \alpha) \to (\mathcal{Y}, \beta)$ we define a partition of $I^{\underline{j}}$ to subpolyhedra $Q^{\underline{j}}_{j_l j_m}$, $0 \le l \le m \le k$ defined by

$$Q^{\underline{j}}_{j_l j_m} = \{(t_1, \dots, t_n) : t_{j_l} \ge 3/4 \ge t_{j_{l+1}}, t_{j_m} \ge 1/2 \ge t_{j_{m+1}}\}.$$

Then for $(t_1, \dots, t_n) \in Q^{\underline{j}}_{j_l j_m}$ we have

$$((h \circ g) \circ f)^{\underline{j}}_{\delta(\underline{e})}(t_1, \dots, t_n, x) =$$

$$h^{j_0 \dots j_l}_{\delta(e_0 \dots e_{j_l})}(\sum_{i=0}^{l-1}(0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 4t_{j_{i+1}} - 3, 0, \dots, 0),$$

$$(6) \qquad g^{j_l - j_l \dots j_m - j_l}_{\gamma(e_{j_l} \dots e_{j_m})}(\sum_{i=l}^{m-1}(0, \dots, 0_{j_i - j_l}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 4t_{j_{i+1}} - 2, 0, \dots, 0)$$

$$f^{j_m - j_m \dots j_k - j_m}_{\beta(e_{j_m} \dots e_n)}(\sum_{i=m}^{k-1}(0, \dots, 0_{j_i - j_m}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 2t_{j_{i+1}}, 0, \dots, 0), x).$$

THEOREM 2. *If* $f : (\mathcal{X},\alpha) \to (\mathcal{Y},\beta)$, $g : (\mathcal{Y},\beta) \to (\mathcal{Z},\gamma)$ *and* $h : (\mathcal{Z},\gamma) \to (\mathcal{W},\delta)$ *are coherent maps over* E *then* $h \circ (g \circ f) \cong (h \circ g) \circ f$.

PROOF. First we define a partition of $I^{\underline{j}} \times I$ to subpolyhedra $M^{\underline{j}}_{j_l j_m}$ for any pair of integers l, m such that $0 \le l \le m \le k$ by

$$M^{\underline{j}}_{j_l j_m} = \{(t_1,\ldots,t_n,s) : t_{j_l} \ge (2+s)/4 \ge t_{j_{l+1}}, t_{j_m} \ge (1+s)/4 \ge t_{j_{m+1}}\}.$$

Let f, g and h be given by maps $f^{\underline{j}}_{\beta(\underline{e})}, g^{\underline{j}}_{\gamma(\underline{e})}$ and $h^{\underline{j}}_{\delta(\underline{e})}$ respectively. We define a coherent map over E, $H : (\mathcal{X},\alpha) \to (\mathcal{W},\delta)$ given by maps $H^{\underline{j}}_{\delta(\underline{e})} : I^{\underline{j}} \times I \times X_{\alpha(e_n)} \to W_{\delta(e_0)}$ defined for $(t_1,\ldots,t_n,s) \in M^{\underline{j}}_{j_l j_m}$ by

$$(7)\qquad H^{\underline{j}}_{\delta(\underline{e})}(t_1,\ldots,t_n,s,x) =$$
$$h^{j_0\ldots j_l}_{\delta(e_0\ldots e_{j_l})}(\sum_{i=0}^{l-1}(0,\ldots,0_{j_i},t_{j_i+1},\ldots,t_{j_{i+1}-1},(4t_{j_{i+1}}-2-s)/(2-s),0,\ldots,0),$$
$$g^{j_l-j_l\ldots j_m-j_l}_{\gamma(e_{j_l}\ldots e_{j_m})}(\sum_{i=l}^{m-1}(0,\ldots,0_{j_i-j_l},t_{j_i+1},\ldots,t_{j_{i+1}-1},4t_{j_{i+1}}-1-s,0,\ldots,0)$$
$$f^{j_m-j_m\ldots j_k-j_m}_{\beta(e_{j_m}\ldots e_n)}(\sum_{i=m}^{k-1}(0,\ldots,0_{j_i-j_m},t_{j_i+1},\ldots,t_{j_{i+1}-1},(4t_{j_{i+1}})/(1+s),0,\ldots,0),x))).$$

To prove that H is a well defined and coherent map for $(t_1,\ldots,t_n,s) \in M^{\underline{j}}_{j_l j_m}$ we put $H^{\underline{j}}_{\delta(\underline{e})}(t_1,\ldots,t_n,s,x) = h^{j_0\ldots j_l}_{\delta(e_0\ldots e_{j_l})}(t'_1,\ldots,t'_l,z)$ where $z = g^{j_l-j_l\ldots j_m-j_l}_{\gamma(e_{j_l}\ldots e_{j_m})}(t'_{j_l+1},\ldots,t'_{j_m}, f^{j_m-j_m\ldots j_k-j_m}_{\beta(e_{j_m}\ldots e_n)}(t'_{j_m+1},\ldots,t'_n,x)$ and $(t'_1,\ldots,t'_n)$ are defined by the formula (7).

To check the well definition let $(t_1,\ldots,t_n,s) \in M^{\underline{j}}_{j_l j_m} \cap M^{\underline{j}}_{j_{l-1} j_m}$, in which case $H^{\underline{j}}_{\delta(\underline{e})}$ is defined in two ways. Then must $t_{j_l} = (2+s)/4$. If we compute the formula (7) for $(t_1,\ldots,t_n,s) \in M^{\underline{j}}_{j_l j_m}$ and $t_{j_l} = (2+s)/4$ we have $H^{\underline{j}}_{\delta(\underline{e})}(t_1,\ldots,t_n,s,x) = h^{j_0\ldots j_{l-1}}_{\delta(e_0\ldots e_{j_{l-1}})}(t'_1,\ldots,t'_{j_{l-1}}, r_{\gamma(e_{j_{l-1}}\ldots e_{j_l})}(t'_{j_{l-1}+1},\ldots,t'_{j_l-1},z))$.
The same expression is obtained if we compute the formula (7) for $(t_1,\ldots,t_n,s) \in M^{\underline{j}}_{j_{l-1} j_m}$ and $t_{j_l} = (2+s)/4$. Similiarly, we can check the well definition for $(t_1,\ldots,t_n,s) \in M^{\underline{j}}_{j_l j_m} \cap M^{\underline{j}}_{j_l j_{m-1}}$, and the other cases can be deduced to one of these two cases.

We have to check that H is a coherent map over E. If $(t_1,\ldots,t_n,s) \in M^{\underline{j}}_{j_l j_m}$ and $t_n = 0$ then

$$f^{j_m-j_m\ldots j_k-j_m}_{\beta(e_{j_m}\ldots e_n)}(t'_{j_m},\ldots,t'_n,x)$$
$$= f^{j_m-j_m\ldots j_{k-1}-j_m}_{\beta(e_{j_m}\ldots e_{j_{k-1}})}(t'_{j_m},\ldots,t'_{j_{k-1}}, p_{\alpha(e_{j_{k-1}}\ldots e_{j_k})}(t_{j_{k-1}+1},\ldots,t_{n-1},x)$$

and it follows that for $t_n = 0$

$$\begin{aligned} &H^{\underline{j}}_{\delta(\underline{e})}(t_1,\dots,t_n,x)\\ =&H^{j_m-j_m\dots j_{k-1}-j_m}_{\delta(e_{j_m}\dots e_{j_{k-1}})}(t_1,\dots,t_{j_{k-1}},p_{\alpha(e_{j_{k-1}}\dots e_{j_k})}(t_{j_{k-1}+1},\dots,t_{n-1},x).\end{aligned}$$

Similiarly is treated the case when $t_{j_1} = 1$.

If $t_{j_i} = t_{j_{i+1}}$, and $i < l$ then

$$\begin{aligned} &H^{\underline{j}}_{\delta(\underline{e}_0}(t_1,\dots,t_n,s,x) = h^{j_0\dots j_l}_{\delta(e_0\dots e_{j_l})}(t'_1,\dots,t'_l,z)\\ =&h^{j_0\dots\hat{j}_i\dots j_l}_{\delta(e_0\dots e_{j_l})}(t'_1,\dots,t'_{j_i-1},1,t'_{j_i+1},\dots,t'_{j_l},z)\\ =&H^{j_0\dots\hat{j}_i\dots j_k}_{\delta(e_0\dots e_{j_l})}(t_1,\dots,t_{j_i-1},1,t_{j_i+1},\dots,t_n,s,x).\end{aligned}$$

The cases $l < i < m$ and $m < i < k$ are treated similiarly.

If $t_j = 0$, $j_i < j < j_{i+1}$, and $m \leq i$ then

$$\begin{aligned} &f^{j_m-j_m\dots j_k-j_m}_{\beta(e_{j_m}\dots e_n)}(t'_{j_m+1},\dots,t'_n,x) =\\ &f^{j_m-j_m\dots j_i-j_m,j_{i+1}-1-j_m\dots j_k-1-j_m}_{\beta(e_{j_m}\dots\hat{e}_j\dots e_n)}(t'_{j_m+1},\dots,\hat{t}'_j,\dots,t'_n,x).\end{aligned}$$

It follows that for $t_j = 0$

$$H^{\underline{j}}_{\delta(\underline{e})}(t_1,\dots,t_n,s,x) = H^{j_0\dots j_i j_{i+1}-1\dots j_k-1}_{\delta(e_0\dots\hat{e}_j\dots e_n)}(t_1,\dots,\hat{t}_j,\dots,t_n,s,x).$$

The cases $i < l$ and $l \leq i \leq k$ are treated similiarly.

Mention that $K^{\underline{j}}_{j_l j_m} \times 0 = \{(t_1,\dots,t_n,0) : (t_1,\dots,t_n,0) \in M^{\underline{j}}_{j_l j_m}\}$ and $Q^{\underline{j}}_{j_l j_m} \times 1 = \{(t_1,\dots,t_n,1) : (t_1,\dots,t_n,1) \in M^{\underline{j}}_{j_l j_m}\}$. Then for $(t_1,\dots,t_n) \in K^{\underline{j}}_{j_l j_m}$ and for $(t_1,\dots,t_n) \in Q^{\underline{j}}_{j_l j_m}$ respectively, having in mind the formulae (5) and (6) we have

$$\begin{aligned} &H^{\underline{j}}_{\delta(\underline{e})}(t_1,\dots,t_n,0,x)= (h\circ(g\circ f))^{\underline{j}}_{\delta(\underline{e})}(t_1,\dots,t_n,x)\\ &H^{\underline{j}}_{\delta(\underline{e})}(t_1,\dots,t_n,1,x)= ((h\circ g)\circ f)^{\underline{j}}_{\delta(\underline{e})}(t_1,\dots,t_n,x). \quad ■\end{aligned}$$

THEOREM 3. *If $f, f' : (\mathcal{X},\alpha) \to (\mathcal{Y},\beta)$ and $g, g' : (\mathcal{Y},\beta) \to (\mathcal{Z},\gamma)$ are coherent maps over E such that $f \cong f'$ and $g \cong g'$ then $g\circ f \cong g\circ' f'$.*

PROOF. If $f \cong f'$ with a homotopy $F : (I\times\mathcal{X},\alpha) \to (\mathcal{Y},\beta)$ then $g\circ f \cong g\circ' f'$ with the homotopy $g\circ F : (I\times\mathcal{X},\alpha) \to (\mathcal{Z},\gamma)$. Also,if $g \cong g'$ with a homotopy $G : (I\times\mathcal{Y},\beta) \to (\mathcal{Z},\gamma)$ then $g\circ f' \cong g'\circ f'$ with the homotopy $G\circ(1\times f') : (I\times\mathcal{X},\alpha) \to (\mathcal{Z},\gamma)$. It follows $g\circ f \cong g\circ f' \cong g'\circ f'$. ■

The coherent identity map over E, $1_{(\mathcal{X},\alpha)} : (\mathcal{X},\alpha) \to (\mathcal{X},\alpha)$ is given by maps $1^{\underline{j}}_{\alpha(\underline{e})} : I^{\underline{j}} \times X_{\alpha(e_n)} \to Y_{\alpha(e_0)}$ defined for $n = 0$, $1_{\alpha(e_0)} = 1_{X_{\alpha(e_0)}}$, and for $n > 0$ by

$$\begin{aligned} &1^{\underline{j}}_{\alpha(\underline{e})}(t_1,\dots,t_n,x)\\ =&p_{\alpha(\underline{e})}(t_1,\dots,t_{j_1-1},1,t_{j_1+1},\dots,t_{j_{k-1}-1},1,t_{j_{k-1}+1},\dots,t_{n-1},x).\end{aligned} \tag{8}$$

THEOREM 4. *If* $f : (\mathcal{X},\alpha) \to (\mathcal{Y},\beta)$ *is a coherent map over* E, *then* $f \cong 1_{(\mathcal{Y},\beta)} \circ f$ *and* $f \cong f \circ 1_{(\mathcal{X},\alpha)}$.

PROOF. We will prove that $f \cong 1_{(\mathcal{Y},\beta)} \circ f$ and the second statement is treated in a similiar way.

First we define a partition of $I \times I^{\underline{j}}$ to subpolyhedra $L^{\underline{j}}_{j_l}, l = 0, 1, \dots, k$ defined by $L^{\underline{j}}_{j_l} = \{(t_1, \dots, t_n) : t_{j_l} \geq s/2 + 1/2 \geq t_{j_{l+1}}\}$. We define a coherent map $F : (I \times \mathcal{X}, \alpha) \to (\mathcal{Y}, \beta)$ given by maps $F^{\underline{j}}_{\beta(\underline{e})} : I^{\underline{j}} \times I \times X_{\alpha(e_n)} \to Y_{\beta(e_0)}$ defined for $(t_1, \dots, t_n) \in L^{\underline{j}}_{j_l}$ by

$$F^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, s, x) = q_{\beta(e_0 \dots e_{j_l})}(t_1, \dots, t_{j_1-1}, 1, t_{j_1+1}, \dots, t_{j_l-1},$$
$$f^{j_l-j_l \dots j_k-j_l}_{\beta(e_{j_l} \dots e_n)}(\sum_{i=l}^{k-1}(0, \dots, 0_{j_i-j_l}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 2t_{j_{i+1}}/(1+s), 0, \dots, 0), x).$$

Mention that $K^{\underline{j}}_{j_l} \times 0 = \{(t_1, \dots, t_n, 0) : (t_1, \dots, t_n, 0) \in L^{\underline{j}}_{jl}\}$ and $F^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, 0, x) = (1_{(\mathcal{Y},\beta)} \circ f)^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, x)$. Also, $\{((t_1, \dots, t_n, 1) : (t_1, \dots, t_n) \in I^{\underline{j}}\} = L^{\underline{j}}_{j_0}$, and $F^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, 1, x) = f^{\underline{j}}_{\beta(\underline{e})}(t_1, \dots, t_n, x)$.

The proof in all details of well definition and coherence is given in [4] and [8] in very similiar situation. ■

Now we can define the coherent category over a cofinite set E - $Coh(E)$ which objects are pairs $(\mathcal{X}, \alpha)$. The morphisms of $Coh(E)$ are homotopy classes of coherent maps over E, and the composition of morphisms is defined as composition of homotopy classes. The identity morphism is the homotopy class of the identity map.

Theorems 1,2,3 and 4 verify the category requierments.

§2. Map $p[\phi, \Phi] : (\mathcal{X}, \phi) \to (\mathcal{X}, \Phi)$

For two increasing functions $\phi, \Phi : B \to A$ such that $\phi < \Phi$, we define a special kind of coherent map over B, $p[\phi, \Phi] : \mathcal{X} \to \mathcal{X}$ given by maps $p^{\underline{j}}_{\phi(\underline{b})} : I^{\underline{j}} \times X_{\Phi(b_n)} \to Y_{\phi(b_0)}$ defined for $(t_1, \dots, t_n) \in K^{\underline{j}}_{j_m}$ by

$$\begin{aligned} p^{\underline{j}}_{\phi(\underline{b})}(t_1, \dots, t_n, x) = p_{\phi(b_0 \dots b_{j_m})\Phi(b_{j_m} \dots b_n)} \\ (\sum_{i=0}^{m-1}(0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 1, 0, \dots, 0)(2t_{j_{i+1}} - 1) \\ + \sum_{i=m}^{k-1}(0, \dots, 0_{j_i}, 1, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 0, \dots, 0)(1 - 2t_{j_{i+1}}), x). \end{aligned} \tag{9}$$

For $n = 0$, $p_{\phi(b_0)} : X_{\Phi(b_0)} \to X_{\phi(b_0)}$, $p_{\phi(b_0)}(x) = p_{\phi(b_0)\Phi(b_0)}(x)$. For $n = 1$, a map $p_{\phi(b_0 b_1)} : I \times X_{\Phi(b_1)} \to X_{\phi(b_0)}$ is given by

$$p_{\phi(b_0 b_1)}(t_1, x) = \begin{cases} p_{\phi(b_0 b_1)\Phi(b_1)}(2t_1 - 1, x); & t_1 \geq \frac{1}{2} \\ p_{\phi(b_0)\Phi(b_0 b_1)}(1 - 2t_1, x); & \frac{1}{2} \geq t_1. \end{cases}$$

To show that $p^{j}_{\underline{b}}$ is a well defined and coherent map, we put for $(t_1,\dots,t_n)\in K^{j}_{j_m}$, $p^{j}_{\phi(\underline{b})}(t_1,\dots,t_n,x) = p_{\phi(b_0\dots b_{j_m})}\Phi_{(b_{j_m}\dots b_n)}(t'_1,\dots,t'_n,x)$ where $t'_1,\dots,t'_n$ are defined by the formula (9).

To prove well definition it is enough to check the case when $(t_1,\dots,t_n)\in K^{j}_{j_m}\cap K^{j}_{j_{m-1}}$, and this is possible if and only if $t_{j_m}=1/2$. Then for $(t_1,\dots,t_n)\in K^{j}_{j_{m-1}}$ and $t_{j_m}=1/2$ we have

$$\begin{aligned}
&p^{j}_{\phi(\underline{b})}(t_1,\dots,t_n,x)\\
=&p_{\phi(b_0\dots b_{j_{m-1}})\Phi(b_{j_{m-1}}\dots b_{j_m}\dots b_n)}(t'_1,\dots,t'_{j_{m-1}},0,\dots,0,t'_{j_m},\dots,t'_n,x)\\
=&p_{\phi(b_0\dots b_{j_{m-1}})\Phi(b_{j_m}\dots b_n)}(t'_1,\dots,t'_{j_{m-1}},t'_{j_m},\dots,t'_n,x)
\end{aligned}$$

and the same expression is obtained if we compute the formula (9) for $(t_1,\dots,t_n)\in K^{j}_{j_m}$ and $t_{j_m}=1/2$.

Now we check that $p^{j}_{\beta(\underline{e})}$ is a coherent map. For $t_n=0$ and $(t_1,\dots,t_n)\in K^{j}_{j_m}$ we have

$$\begin{aligned}
&p^{j}_{\phi(\underline{b})}(t_1,\dots,t_n,x)\\
=&p_{\phi(b_0\dots b_{j_m}\Phi(b_{j_m}\dots b_{j_{k-1}})}(t'_1,\dots,t'_{j_{k-1}})p_{\Phi(b_{j_{k-1}}\dots b_n)}(t_{j_{k-1}+1},\dots,t_{n-1},x)\\
=&p^{j}_{\phi(b_0\dots b_{j_{k-1}})}(t_1,\dots,t_{j_{k-1}},p_{\Phi(b_{j_{k-1}}\dots b_n)}(t_{j_{k-1}+1},\dots,t_{n-1},x)).
\end{aligned}$$

Similarly is treated the case $t_{j_1}=1$.

For $l\le m-1$, and $(t_1,\dots,t_n)\in K^{j}_{j_m}$ and $t_{j_{l+1}}=t_{j_l}$ we have
$p^{j_0\dots j_k}_{\phi(\underline{b})}(t_1,\dots,t_n,x) = p_{\phi(b_0\dots b_{j_m})\Phi(b_{j_m}\dots b_n)}((t'_1,\dots,t'_{j_{l-1}},0,\dots,0,t_{j_{l+1}+1},\dots,t'_n)$
$+(0,\dots,0_{j_{l-1}},t_{j_{l-1}+1},\dots,t_{j_l-1},1,t_{j_l+1},\dots,t_{j_{l+1}-1},1,0,\dots,0)(t_{j_{l+1}}-1),x)$
$=p^{j_0\dots\hat{j}_l\dots j_k}_{\phi(\underline{b})}(t_1,\dots,t_{j_l-1},1,t_{j_l+1},\dots,t_n,x)$.

Similiarly is treated the case when $m-1<l$.

Finally, for $j_l<j<j_{l+1}$, $l\le m-1$, and $(t_1,\dots,t_n)\in K^{j}_{jm}$ and $t_j=0$ we have
$p^{j}_{\beta(\underline{e})}(t_1,\dots,t_n,x) = p_{\phi(b_0\dots\hat{b}_j\dots b_{j_m})\Phi(b_{j_m}\dots b_n)}((t'_1,\dots,t'_{j_l},0,\dots,0,t'_{j_l+1},\dots,t'_n)$
$+(0,\dots,0_{j_l},t_{j_l+1},\dots,\hat{t}_j,\dots,t_{j_{l+1}-1},1,0,\dots,0)(2t_{j_{l+1}}-1),x)$
$=p^{j_0\dots j_l j_{l+1}-1\dots j_k}_{\phi(b_0\dots\hat{b}_j\dots b_n}(t_1,\dots,\hat{t}_j,\dots,t_n,x)$.

The case $l\ge m$ is treated in the same way. ■

THEOREM 5. *If $\phi,\psi,\chi: B\to A$ are increasing functions such that $\phi<\psi<\chi$, then coherent maps over B, $p[\phi,\psi]\circ p[\psi,\chi]$ and $p[\phi,\chi]$ are homotopic.*

PROOF. First we will define a partition of $I^{\underline{j}}\times I$ to subpolyhedra $S^{j}_{j_l j_r}\subseteq I^{\underline{j}}\times I$ for any pair of integers l,r such that $0\le l\le r\le k$ with

$S^{j}_{j_l j_m j_r} = S^{j}_{j_l j_r} \cap (K^{j}_{j_m} \times I)$
$= \{(t_1, \ldots, t_n, s) : t_{j_l}(2+s)/4 \geq t_{j_{l+1}},\ t_{j_m} \geq 1/2 t_{j_{m+1}},\ t_{j_r} \geq (2-s)/4 \geq t_{j_{r+1}}\}.$

Now we define a coherent map $P[\phi, \chi]$ given by maps $P^{j}_{\phi(\underline{b})} : I^{j} \times I \times X_{\chi(b_n)} \to X_{\phi(b_0)}$ defined for $(t_1, \ldots, t_n, s) \in S^{j}_{j_l j_m j_r}$ by

$$P_{\phi(\underline{b})}(t_1, \ldots, t_n, s, x) = p_{\phi(b_0 \ldots b_{j_l})\psi(b_{j_l} \cdots_{j_r})\chi(b_{j_r} \ldots b_n)}$$
$$\Big(\frac{1}{s-2}\sum_{i=0}^{l-1}(0, \ldots, 0_{j_i}, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 1, 0, \ldots, 0)[s - 2(2t_{j_{i+1}} - 1)]$$
$$+ \sum_{i=l}^{m-1}(0, \ldots, 0_{j_i}, 1, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 0, \ldots, 0)[s - 2(2t_{j_{i+1}} - 1)]$$
$$+(0, \ldots, 0_{j_m}, s, 0, \ldots, 0)$$
$$+ \sum_{i=m}^{r-1}(0, \ldots, 0_{j_i+1}, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 1, 0, \ldots, 0)[s - 2(1 - 2t_{j_{i+1}})]$$
$$+ \frac{1}{s-2}\sum_{i=r}^{k-1}(0, \ldots, 0_{j_i+1}, 1, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 0, \ldots, 0)[s - 2(1 - 2t_{j_{i+1}})], x\Big).$$

For $n = 0$, the map $P_{\phi(b_0)} : I \times X_{\chi(b_0)} \to X_{\phi(\beta_0)}$ is defined by $P_{\phi(b_0)}(s, x) = p_{\phi(b_0)\psi(b_0)\chi(b_0)}(s, x)$.

For $n = 1$, the map $P_{\phi(b_0 b_1)} : I \times I \times X_{\chi(b_1)} \to X_{\phi(\beta_0)}$ is defined by

$$p_{\phi(b_0 b_1)}(t_1, s, x) = \begin{cases} p_{\phi(b_0)\psi(b_0)\chi(b_0 b_1)}(s, \frac{s-2(1-2t_1)}{s-2}, x), & (t_1, s) \in S_{000} = S_{00} \\ p_{\phi(b_0)\psi(b_0 b_1)\chi(b_1)}(s, s - 2(1 - 2t_1), x), & (t_1, s) \in S_{001} \subseteq S_{01} \\ p_{\phi(b_0)\psi(b_0 b_1)\chi(b_1)}(s - 2(2t_1 - 1), s, x), & (t_1, s) \in S_{011} \subseteq S_{01} \\ p_{\phi(b_0 b_1)\psi(b_1)\chi(b_1)}\frac{s-2(2t_1-1)}{s-2}, s, x), & (t_1, s) \in S_{111} = S_{11}. \end{cases}$$

We have to show that

$$P^{j}_{\phi(\underline{b})}(t_1, \ldots, t_n, 0, x) = (p[\phi, \chi])^{j}_{\phi(\underline{b})}(t_1, \ldots, t_n, x)$$
$$P^{j}_{\phi(\underline{b})}(t_1, \ldots, t_n, 1, x) = (p[\phi, \psi] \circ p[\psi, \chi])^{j}_{\phi(\underline{b})}(t_1, \ldots, t_n, x).$$

If $s = 0$, then there exists m such that $(t_1, \ldots, t_n, 0) \in S^{j}_{j_m j_m j_m}$. Also $(t_1, \ldots, t_n, 0) \in S^{j}_{j_m j_m j_m}$ if and only if $(t_1, \ldots, t_n) \in K^{j}_{j_m}$ and

$$P^{j}_{\phi(\underline{b})}(t_1, \ldots, t_n, 0, x) = p_{\phi(b_0 \ldots b_{j_m})\psi(b_{j_m})\chi(b_{j_m} \ldots b_n)}$$
$$\Big(\sum_{i=0}^{m-1}(0, \ldots, 0_{j_i}, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 1, 0, \ldots, 0)(2t_{j_{i+1}} - 1) + (0, \ldots, 0_{j_m}, s, 0, \ldots, 0)$$
$$+ \sum_{i=m}^{k-1}(0, \ldots, 0_{j_i+1}, 1, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 0, \ldots, 0)(1 - 2t_{j_{i+1}}), x\Big)$$
$$= p_{\phi(b_0 \ldots b_{j_m})\chi(b_{j_m} \ldots b_n)}\Big(\sum_{i=0}^{m-1}(0, \ldots, 0_{j_i}, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 1, 0, \ldots, 0)(2t_{j_{i+1}} - 1)$$

$$+\sum_{i=m}^{k-1}(0,\dots,0_{j_i},1,t_{j_i+1},\dots,t_{j_{i+1}-1},0,\dots,0)(1-2t_{j_{i+1}}),x)$$
$$=(p[\phi,\chi])^{\underline{j}}_{\phi(\underline{b})}(t_1,\dots,t_n,x)$$

If $s=1$, and $(t_1,\dots,t_n,x)\in S^{\underline{j}}_{j_l j_r}\cap(K^{\underline{j}}_{j_m}\times 1)$ we have

$$P^{\underline{j}}_{\phi(\underline{b})}(t_1,\dots,t_n,1,x)=p_{\phi(b_0\dots b_{j_l})\psi(b_{j_l}\dots b_{j_m})}$$
$$(\sum_{i=0}^{l-1}(0,\dots,0_{j_i},t_{j_i+1},\dots,t_{j_{i+1}-1},1,0,\dots,0)[2(2t_{j_{i+1}}-1)-1]$$
$$+\sum_{i=l}^{m-1}(0,\dots,0_{j_i},1,t_{j_i+1},\dots,t_{j_{i+1}-1},0,\dots,0)[1-2(2t_{j_{i+1}}-1)],$$
$$p_{\psi(b_{j_m}\dots b_{j_r})\chi(b_{j_r}\dots b_n)}(\sum_{i=m}^{r-1}(0,\dots,0_{j_i-j_m},t_{j_i+1},\dots,t_{j_{i+1}-1},1,0,\dots,0)$$
$$(4t_{j_{i+1}}-1)+\sum_{i=r}^{k-1}(0,\dots,0_{j_i-j_m},1,t_{j_i+1},\dots,t_{j_{i+1}-1},0,\dots,0)(1-4t_{j_{i+1}}),x)$$
$$=p[\phi,\psi]^{j_0\dots j_m}_{\phi(b_0\dots b_{j_m})}(\sum_{i=0}^{m-1}(0,\dots,0_{j_i},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}}-1,0,\dots,0),$$
$$p[\psi,\chi]^{j_m-j_m\dots j_k-j_m}_{\psi(b_{j_m}\dots b_n)}(\sum_{i=m}^{k-1}(0,\dots,0_{j_i-j_m},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}},0,\dots,0),x)$$
$$=(p[\phi,\psi]\circ p[\psi,\chi])^{\underline{j}}_{\phi(\underline{b})}(t_1,\dots,t_n,x).$$

§3. Coherent category *Coh* and coherent shape of topological spaces

DEFINITION: A *coherent map* $f:\mathcal{X}\to\mathcal{Y}=(Y_b,q_{\underline{b}},B)$ consists of the following data:

1) A (strictly) increasing function $\phi:B\to A$.

2) For $n=0$, and $b_0\in B$ of a map $f_{b_0}:X_{\phi(b_0)}\to Y_{b_0}$.

For $n>1$, and $\underline{b}=(b_0,\dots,b_n),b_0<\dots<b_n$ a sequence in B, and $\underline{j}=(j_0,\dots,j_k),0=j_o<\dots<j_k=n$, a sequence of integers, of a map $f^{\underline{j}}_{\underline{b}}:I^{\underline{j}}\times X_{\phi(b_n)}\to Y_{b_0}$ satisfying the following boundary conditions

$$f^{\underline{j}}_{\underline{b}}(t_1,\dots,t_n,x)=$$
$$(10)\qquad =\begin{cases} q_{b_0\dots b_{j_1}}(t_1,\dots,t_{j_1-1},f^{j_1-j_1\dots j_k-j_1}_{b_{j_1}\dots b_n}(t_{j_1+1},\dots,t_n,x));\ t_{j_1}=1\\ f^{j_0\dots\hat{j}_i\dots j_k}_{\underline{b}}(t_1,\dots,t_{j_i-1},1,t_{j_i+1},\dots,t_n,x);\ t_{j_i}=t_{j_{i+1}};\ 0<i<k\\ f^{j_0\dots j_{k-1}}_{b_0\dots b_{j_{k-1}}}(t_1,\dots,t_{j_{k-1}-1},p_{\phi(b_{j_{k-1}}\dots b_n)}(t_{j_{k-1}+1},\dots,t_{n-1},x));\ t_n=0\\ f^{j_0\dots j_i j_{i+1}-1\dots j_k-1}_{b_0\dots\hat{b}_j\dots b_n}(t_1,\dots,\hat{t}_j,\dots,t_n,x);\ t_j=0,\ j_i<j<j_{i+1}.\end{cases}$$

REMARK 1. a) Mention that for example for $n=0$, and $\alpha(e)=\alpha(e')=\alpha(e'')=\dots$ for a coherent map over E, $f:(\mathcal{X},\alpha)\to(\mathcal{Y},\beta)$ we may have many maps from a fixed set $X_{\alpha(e)}$ of $\mathcal{X}$, i.e. $f_{\beta(e)} : X_{\alpha(e)} \to Y_{\beta(e)}$, $f_{\beta(e')} : X_{\alpha(e)} \to Y_{\beta(e')}$, $f_{\beta(e'')} : X_{\alpha(e)} \to Y_{\beta(e'')}, \ldots$. while in the case of coherent maps there is only one map from a set $X_{\phi(b)}$ of $\mathcal{X}$.

b) If $f : \mathcal{X} \to \mathcal{Y}$ is a coherent map given by function ϕ and maps $f_{\underline{b}}^{\underline{j}}$, and $\psi : E \to B$ is an increasing function,then maps $f_{\psi(\underline{e})}^{\underline{j}} : X_{\phi\psi(e_n)} \to Y_{\psi(e_0)}$ define a coherent map over E, $f : (\mathcal{X}, \phi\psi) \to (\mathcal{Y}, \psi)$. In the special case when $E = B$, $\psi = id$,then $\phi : B \to A$, and the coherent map over B, $f : (\mathcal{X}, \phi) \to (\mathcal{Y}, id)$ given by maps $f_{\underline{b}}^{\underline{j}}$ we will identify with the coherent map $f : \mathcal{X} \to \mathcal{Y}$.

c) This definition of a coherent map in fact is the same as the cubical-simplicial definition of a coherent map in [6] and [8] where instead of the space $I^{\underline{j}}$ appears $I^{n-k} \times \Delta^k$. Here instead of the simplex $\Delta^k = \{(s_0, \ldots, s_k) : s_0 \geq 0, \ldots, s_k \geq 0,\ s_0 + \cdots + s_k = 1\}$ is used the subspace of the cube I^k, $\nabla^k = \{(t_1'', \ldots, t_k'') : 1 \geq t_1'' \geq \cdots \geq t_k'' \geq 0\}$.

From the definition of a coherent map in this paper to the earlier definition we can pass by a permutation of coordinates $I^{\underline{j}} \ni (t_0, \ldots, t_n) \to (t_1', \ldots, t_{n-k}', s_0, \ldots, s_k) \in I^{n-k} \times \Delta^k$ where $t_{j-i}' = t_j$, $j_i < j < j_{i+1}$ and $s_i = t_{j_i}$, $1 \leq i < k$ and ∇^k and Δ^k are naturally homeomorphic by the mapping given by $t_1'' = s_1 + s_2 + \cdots + s_k, \ldots, t_{k-1}'' = s_{k-1} + s_k$, $t_k'' = s_k$.

Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z} = (Z_c, r_{\underline{c}}, C)$ be coherent maps. Let g be given by the function ψ and maps $g_{\underline{c}}^{\underline{j}} : I^{\underline{j}} \times Y_{\psi(c_n)} \to Z_{c_0}$.

Then the composition $h = gf : \mathcal{X} \to \mathcal{Z}$ is given by the function $\chi = \psi\phi$ and maps $h_{\underline{c}}^{\underline{j}} : I^{\underline{j}} \times X_{\phi\psi(c_n)} \to Z_{c_0}$ defined for $n = 0$ with $h_{c_0} = g_{c_0} f_{\psi(c_0)}$ and for $n > 0$ and $(t_1, \ldots, t_n) \in K_{\underline{j}_m}^{\underline{j}}$ with

$$
\begin{aligned}
&h_{\underline{c}}^{\underline{j}}(t_1, \ldots, t_n, x) = \\
(11)\qquad &g_{c_0 \ldots c_{j_m}}^{j_0 \ldots j_m}\Big(\sum_{i=0}^{m-1}(0, \ldots, 0_{j_i}, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 2t_{j_{i+1}} - 1, 0, \ldots, 0), \\
&f_{\psi(c_{j_m} \ldots c_n)}^{j_m - j_m \ldots j_k - j_m}\Big(\sum_{i=m}^{k-1}(0, \ldots, 0_{j_i - j_m}, t_{j_i+1}, \ldots, t_{j_{i+1}-1}, 2t_{j_{i+1}}, 0, \ldots, 0), x\Big).
\end{aligned}
$$

REMARK 2. If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are coherent maps given by the functions ϕ and ψ and by maps $f_{\underline{b}}^{\underline{j}}$ and $g_{\underline{c}}^{\underline{j}}$, then their composition $gf : \mathcal{X} \to \mathcal{Z}$ is the same map (in the sense of Remark 1b)) as a composition $g \circ f : (\mathcal{X}, \phi\psi) \to (\mathcal{Z}, id)$ of coherent maps over C, $f : (\mathcal{X}, \phi\psi) \to (\mathcal{Y}, \psi)$ and $g : (\mathcal{Y}, \psi) \to (\mathcal{Z}, id)$.

If a coherent map $g : \mathcal{Y} \to \mathcal{Z}$ is considered as a coherent map over C i.e $g : (\mathcal{Y}, \beta) \to (\mathcal{Z}, \text{id})$, and $f : (\mathcal{X}, \alpha) \to (\mathcal{Y}, \beta)$ is an arbitrary coherent map over C, then the composition of these two coherent maps over C, $g \circ f : (\mathcal{X}, \alpha) \to (\mathcal{Z}, id)$ is also a coherent map in the sense of Remark 1. This allows us to give the following definition of a homotopy of two coherent maps.

Coherent maps $f, f' : \mathcal{X} \to \mathcal{Y}$ given by functions ϕ, ϕ' and maps $f_{\underline{b}}^{\underline{j}}, f'^{\underline{j}}_{\underline{b}}$ respectively are *coherently homotopic* if

1) there exists an increasing function $\Phi : B \to A$ such that $\Phi > \phi$, $\Phi > \phi'$

2) There exists a coherent map $F : I \times \mathcal{X} \to \mathcal{Y}$, $(I \times \mathcal{X} = (I \times X_a, 1 \times p_{\underline{a}}, A))$ given by the function Φ and maps $F^j_{\underline{b}} : I \times I^{\underline{j}} \times X_{\Phi(b_n)} \to Y_{b_0}$ such that

$$
\begin{aligned}
F^j_{\underline{b}}(t_1,\dots,t_n,1,x) &= (f \circ p[\phi,\Phi])^j_{\underline{b}}(t_1,\dots,t_n,x)\\
F^j_{\underline{b}}(t_1,\dots,t_n,0,x) &= (f' \circ p[\phi',\Phi])^j_{\underline{b}}(t_1,\dots,t_n,x).
\end{aligned}
\tag{12}
$$

If f and f' are coherently homotopic we write $f \cong f'$.

REMARK 3. Let coherent maps $f, f' : \mathcal{X} \to \mathcal{Y}$ be defined by the same function ϕ and by maps $f^j_{\underline{b}}$, $f'^j_{\underline{b}}$ respectively. If there exists a coherent map $H : I \times \mathcal{X} \to \mathcal{Y}$ given by ϕ and maps $H^j_{\underline{b}} : I \times I^{\underline{j}} \times X_{\phi(b_n)} \to Y_{b_0}$ such that

$$
\begin{aligned}
H^j_{\underline{b}}(t_1,\dots,t_n,0,x) &= f^j_{\underline{b}}(t_1,\dots,t_n,x)\\
H^j_{\underline{b}}(t_1,\dots,t_n,1,x) &= f'^j_{\underline{b}}(t_1,\dots,t_n,x)
\end{aligned}
$$

then f and f' are coherently homotopic, because for an arbitrary function $\Phi > \phi$, the coherent map $F : I \times \mathcal{X} \to \mathcal{Y}$ defined by $F = H \circ p[\phi,\Phi]$ satisfies the formulas (12).

b) In the paper [10] it is avoided the notion of a coherent map over a cofinite set,and $p[\phi,\Phi]$ is a coherent map by the following extra requierment:the function pair $\phi < \Phi$ satisfy, if $\phi(b) = \phi(b')$ then $\Phi(b) = \Phi(b')$. In this way it is avoided situation from Remark 1a), but the theory of inverse systems allways deal with arbitrary functions and in order to consider the general case ,the notion of a coherent map over a cofinite set ocurs naturally.

THEOREM 6. *If B is a cofinite set, then the relation of coherent homotopy $\cong$ of coherent maps $f : \mathcal{X} \to \mathcal{Y} = (Y_b, q_{\underline{b}}, B)$ is an equivalence relation.*

PROOF. Reflexivity follows from Remark 3a) and symmetry is obvious.

To prove the transitivity let $f, f', f'' : \mathcal{X} \to \mathcal{Y}$ be defined by functions ϕ, ϕ', ϕ'' and maps $f^j_{\underline{b}}, f'^j_{\underline{b}}, f''^j_{\underline{b}}$ respectively. Let F be the coherent homotopy connecting f and f',and F be given by Φ and $F^j_{\underline{b}}$, and F' be the homotopy connecting f' and f'',and F' be given by Φ' and $F'^j_{\underline{b}}$.

We define by induction on

$$\eta(b) = \max\{n : b_0 < \cdots < b_n = b,\, b_0 \in B, \dots, b_n \in B\}$$

an increasing function $\Phi_* : B \to A$ such that $\Phi_* > \Phi$ and $\Phi_* > \Phi'$.

If $b \in B$ is such that $\eta(b) = 0$, then there exists an index $\Phi_*(b)$ in A such that $\Phi_*(b) > \Phi(b), \Phi'(b)$.

Let $\Phi_*(b)$ be defined for all $b \in B$ with $\eta(b) = 0, 1, \dots n-1$. Now let $b \in B$ be with $\eta(b) = n$, and and let $\{b_1, b_2, \dots, b_r\}$ be all the predecessors of b.

Then there exists an index $\Phi_*(b)$ in A such that $\Phi_*(b) > \Phi(b), \Phi'(b)$ and $\Phi_*(b) > \Phi_*(b_1), \Phi_*(b_2), \ldots, \Phi_*(b_r)$.

The coherent map $f : \mathcal{X} \to \mathcal{Y}$ we can consider as a coherent map over B, $f : (\mathcal{X}, \phi) \to (\mathcal{Y},\text{id})$, and also f', f'', F and F' may be cosidered as coherent maps over B. Applying theorems 1,2,3 from §1, and theorem 5 we have

$$\begin{aligned}
& f \circ p[\phi, \Phi_*] \cong f \circ (p[\phi, \Phi] \circ p[\Phi, \Phi_*]) \\
\cong & (f \circ p[\phi, \Phi]) \circ p[\Phi, \Phi_*] \cong (f' \circ p[\phi', \Phi]) \circ p[\Phi, \Phi_*]) \cong f' \circ (p[\phi', \Phi] \circ p[\Phi, \Phi_*]) \\
\cong & f' \circ p[\phi', \Phi_*]) \cong f' \circ (p[\phi', \Phi'] \circ p[\Phi', \Phi_*]) \cong (f' \circ p[\phi', \Phi']) \circ p[\Phi', \Phi_*] \\
\cong & (f'' \circ p[\phi'', \Phi']) \circ p[\Phi', \Phi_*] \cong f'' \circ (p[\phi'', \Phi']) \circ p[\Phi', \Phi_*]) \cong f'' \circ p[\phi'', \Phi_*]).
\end{aligned}$$

It follows that $f \circ p[\phi, \Phi_*])$ and $f'' \circ p[\phi'', \Phi_*])$ are homotopic, with a homotopy over B, $F_* : (\mathcal{X}, \Phi_*) \to Y, id)$ i.e F_* is a coherent map, and consequently $f \circ p[\phi, \Phi_*]) \cong f'' \circ p[\phi'', \Phi_*])$. ■

THEOREM 7. *If $f : \mathcal{X} - \mathcal{Y}$, $g : \mathcal{Y} \to CalZ$ and $h : \mathcal{Z} - \mathcal{W} = (W_d, w_{\underline{d}}, D)$ are coherent maps and D is cofinite ,then $h(gf) \cong (hg)f$*

PROOF. Let $f : \mathcal{X} - \mathcal{Y}, g : \mathcal{Y} - \mathcal{Z}$ and $h : \mathcal{Z} - \mathcal{W} = (W_d, w_{\underline{d}}, D)$ be given by functions ϕ, ψ, χ and by maps $f^{j}_{\underline{b}}, g^{j}_{\underline{b}}, h^{j}_{\underline{b}}$, respectively. Then, for coherent maps over D, $f : (\mathcal{X}, \phi\psi\chi) - (\mathcal{Y}, \psi\lambda)$, $g : (\mathcal{Y}, \psi\chi) - (\mathcal{Z}, \chi)$. and $h : (\mathcal{Z}, \chi) - (\mathcal{W}, id)$ given by maps $f^{j}_{\phi\psi\chi(\underline{b})}, g^{j}_{\psi\chi(\underline{b})}, h^{j}_{\chi(\underline{b})}$ it is satisfied $h \circ (g \circ f) \cong (h \circ g) \circ f$ with the homotopy $H : (I \times \mathcal{X}, \phi\psi\chi) \to (\mathcal{W}, id)$ given by maps $H^{j}_{\phi\psi\chi(\underline{d})}$. It follows that for coherent maps f, g and h it is satisfied $h(gf) \cong (hg)f$ with a homotopy $H : I \times \mathcal{X} \to \mathcal{W}$ given by the function $\phi\psi\chi$ ant the maps $H^{j}_{\phi\psi\chi(\underline{d})}$. ■

The following tehnical result is needed in the proof of the next theorem

THEOREM 8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a coherent map given by a function $\phi : B \to A$ and maps $f^{j}_{\underline{b}}$, and $\psi, \Psi : C \to B$ be increasing functions such that $\psi < \Psi$, for some cofinite set C. Then for coherent maps over $C, f[\psi, \phi\psi] : (\mathcal{X}, \phi\psi) \to (\mathcal{Y}, \psi)$ given by maps $f^{j}_{\psi(\underline{c})}$ and $f[\Psi, \phi\Psi] : \mathcal{X}, \phi\Psi) \to (\mathcal{Y}, \Psi)$ given by maps $f^{j}_{\Psi(\underline{c})}$ it is satisfied $f[\psi, \phi\psi] \circ p[\phi\psi, \phi\Psi] \cong q[\psi, \Psi] \circ f[\Psi, \phi\Psi]$ i.e the following diagram commutes*

$$\begin{array}{ccc}
(\mathcal{X}, \phi\Psi) & \xrightarrow{p[\phi\psi, \phi\Psi]} & (\mathcal{X}, \phi\psi) \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow \\
(\mathcal{Y}, \Psi) & \xrightarrow{q[\psi, \Psi]} & (\mathcal{Y}, \psi).
\end{array}$$

PROOF. For any $\underline{j} = (j_0, \ldots, j_k)$, and j_m a member of $\underline{j}$, $0 \leq m \leq k$ we define polyhedra $T_{jm}(\underline{j}) \subseteq I^{\underline{j}} \times I$ by

$$T_{j_m}(\underline{j}) = \{(t_1, \ldots, t_n, s) : t_{j_m} \geq s/2 + 1/4 \geq t_{j_{m+1}}\}.$$

Then $\cup_{m=0}^{k} T_{j_m}(\underline{j}) = I^{\underline{j}} \times I$. Further on, for any pair of integers l, r such that $0 \leq l \leq m \leq r \leq k$ we define subpolyhedra of $T_{jm}(\underline{j})$ by

$$\begin{aligned}
& T_{jm}^{j_0 \ldots j_l (j_r+1) \ldots (j_k+1)} \\
= & \{(t_1, \ldots, t_n, s) \in I_{j_m}(\underline{j}) : t_{j_l} \geq s/2 + 1/2 \geq t_{j_{l+1}}, t_{j_r} \geq s/2 \geq t_{j_{r+1}}\}.
\end{aligned}$$

Then $\cup_{l,r} T_{j-m}^{j_0 \dots j_l (j_r+1) \dots (j_k+1)} = T_{j_m}(\underline{j})$.

Now we define a coherent map over C, $H : (\mathcal{X} \times I, \phi\Psi) \to (\mathcal{Y}, \psi)$ defining maps $H_{\psi(\underline{c})}^{\underline{j}} : I^{\underline{j}} \times I \times X_{\phi\Psi(c_n)} \to Y_{\psi(c_0)}$ for $(t_1, \dots, t_n, s) \in T_{jm}^{j_0 \dots j_l (j_r+1) \dots (j_k+1)}$, $0 \le l \le m \le r \le k$, and $x \in X_{\phi\Psi(c_n)}$ by

$$
\begin{aligned}
&H_{\psi(\underline{c})}^{\underline{j}}(t_1, \dots, t_n, s, x) = f_{\psi(c_0 \dots c_{j_m})\Psi(c_{j_m} \dots c_n)}^{j_0 \dots j_l (j_r+1) \dots (j_k+1)} \\
&(\sum_{i=0}^{l-1} (0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 1, 0, \dots, 0)(2t_{j_{i+1}} - 1) \\
&+ \sum_{i=l}^{m-1} (0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 1, 0, \dots, 0)(4t_{j_{i+1}} - 2s - 1) \\
&+ \sum_{i=m}^{r-1} (0, \dots, 0_{j_i+1}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 1, 0, \dots, 0)(2s + 1 - 4t_{j_{i+1}}) \\
&+ (0, \dots, 0_{j_r}, s, 0, \dots, 0) \\
&+ \sum_{i=r}^{k-1} (0, \dots, 0_{j_i+1}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 1, 0, \dots, 0) 2t_{j_{i+1}}, x).
\end{aligned}
\tag{13}
$$

For $n = 0$, $H_{\psi(c_0)} : I \times X_{\phi\Psi(c_0)} \to Y_{\psi(c_0)}$ is a map $H_{\psi(c_0)}(s, x) = f_{\psi(c_0)\Psi(c_0)}(s, x)$. For $n = 1$, the map $H_{\psi(c_0 c_1)} : I \times I \times X_{\phi\Psi(c_1)} \to Y_{\psi(c_0)}$ is defined by

$$
H_{\psi(c_0 c_1)}(t_1, s, x) = \begin{cases}
f_{\psi(c_0)\Psi(c_0 c_1)}^{012}(s, 2t_1, x); & (t_1, s) \in T_0^{012} \\
f_{\psi(c_0)\Psi(c_0 c_1)}^{02}(2s + 1 - 4t_1, s, x); & (t_1, s) \in T_0^{02} \\
f_{\psi(c_0 c_1)\Psi(c_1)}^{02}(4t_1 - 2s - 1, s, x); & (t_1, s) \in T_1^{02} \\
f_{\psi(c_0 c_1)\Psi(c_1)}^{012}(2t_1 - 1, s, x); & (t_1, s) \in T_1^{012}.
\end{cases}
$$

We have $\{(t_1, \dots, t_n, 0) \in T_{j_m}^{j_0 \dots j_l (j_r+1) \dots (j_k+1)}\} \subseteq \{(t_1, \dots, t_n) \in K_{j_l j_m}^{\underline{j}}\}$ Specially, $(t_1, \dots, t_n, 0) \in T_{j_m}^{j_0 \dots j_l (j_k+1)}$ if and only if $(t_1, \dots, t_n) \in K_{j_l j_m}^{\underline{j}}$ and in this case

$$
\begin{aligned}
&H_{\psi(\underline{c})}^{\underline{j}}(t_1, \dots, t_n, 0, x) = f_{\psi(c_0 \dots c_{j_m})\Psi(c_{j_m} \dots c_n)}^{j_0 \dots j_l (j_k+1)}(t_1', \dots, t_n', 0, x) = \\
&= f_{\psi(c_0 \dots c_{j_l})}^{j_0 \dots j_l}(\sum_{i=0}^{l-1} (0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 2t_{j_{i+1}} - 1, 0, \dots, 0), \\
&p_{\phi\psi(c_0 \dots c_{j_m})\phi\Psi(c_{j_m} \dots c_n)}(\sum_{i=l}^{m-1} (0, \dots, 0_{j_i - j_l}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 1, 0, \dots, 0) \\
&(4t_{j_{i+1}} - 1) + \sum_{i=m}^{k-1} (0, \dots, 0_{j_i - j_l}, 1, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 0, \dots, 0)(1 - 4t_{j_{i+1}}), x) \\
&= f_{\psi(c_0 \dots c_{j_l})}^{j_0 \dots j_l}(\sum_{i=0}^{l-1} (0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 2t_{j_{i+1}} - 1, 0, \dots, 0),
\end{aligned}
$$

$$p^{j_l-j_l\dots j_k-j_l}_{\phi\psi(c_{j_l}\dots c_n)}(\sum_{i=l}^{k-1}(0,\dots,0_{j_i-j_l},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}},0,\dots,0),x))$$

$$=(f[\psi,\phi\psi]\circ p[\phi\psi,\phi\Psi])^{\underline{j}}_{\psi(\underline{c})}(t_1,\dots,t_n,x)$$

where $t'_1,\dots,t'_n$ are defined by the formula (13).

Also, $\{(t_1,\dots,t_n,1)\in T^{j_0\dots j_l(j_r+1)\dots(j_k+1)}_{j_m}\}\subseteq\{(t_1,\dots,t_n)\in Q^{\underline{j}}_{j_mj_r}\}$.

Specially, $(t_1,\dots,t_n,1)\in T^{j_0(j_r+1)\dots(j_k+1)}_{j_m}$ if and only if $(t_1,\dots,t_n)\in Q^{\underline{j}}_{j_mj_r}$ and in this case

$$H^{\underline{j}}_{\psi(\underline{c})}(t_1,\dots,t_n,1,x)=f^{j_0(j_r+1)\dots(j_k+1)}_{\psi(c_0\dots c_{j_m})\Psi(c_{j_m}\dots c_n)}(t'_1,\dots,t'_{j_r},1,\dots,t'_n,x)=$$

$$=q_{\psi(c_0\dots c_{j_m})\Psi(c_{j_m}\dots c_{j_r})}(\sum_{i=0}^{m-1}(0,\dots,0_{j_i},t_{j_i+1},\dots,t_{j_{i+1}-1},1,0,\dots,0)$$

$$(4t_{j_{i+1}}-3)+\sum_{i=m}^{r-1}(0,\dots,0_{j_i},1,t_{j_i+1},\dots,t_{j_{i+1}-1},0,\dots,0)(3-4t_{j_{i+1}}),$$

$$f^{j_r-j_r\dots j_k-j_r}_{\Psi(c_{j_r}\dots c_n)}(\sum_{i=r}^{k-1}(0,\dots,0_{j_i-j_r},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}},0,\dots,0),x))$$

$$=q_{\psi(c_0\dots c_{j_r})}(\sum_{i=0}^{r-1}(0,\dots,0_{j_i},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}}-1,0,\dots,0),$$

$$f^{j_r-j_r\dots j_k-j_r}_{\Psi(c_{j_r}\dots c_n)}(\sum_{i=r}^{k-1}(0,\dots,0_{j_i-j_r},t_{j_i+1},\dots,t_{j_{i+1}-1},2t_{j_{i+1}},0,\dots,0),x))$$

$$=(q[\psi,\Psi]\circ f[\Psi,\phi\Psi])^{\underline{j}}_{\psi(\underline{c})}(t_1,\dots,t_n,x).\ \blacksquare$$

THEOREM 9. *If $f,f':\mathcal{X}\to\mathcal{Y}$ and $g,g':\mathcal{Y}\to\mathcal{Z}=(Z_c,r_{\underline{c}},C)$ are coherent maps such that $f\cong f'$ and $g\cong g'$ and C is cofinite, then $gf\cong g'f'$.*

PROOF. It is enough to show that $gf\cong$ gf' and $gf\cong$ g'f . If $f\cong f'$ with a coherent homotopy F, then $gf\cong gf'$ with the coherent homotopy gF.

To show the second statement let g,g' be given by functions ψ,ψ' and maps $g^{\underline{j}}_{\underline{c}},g'^{\underline{j}}_{\underline{c}}$ respectively, and let g,g' be homotopic with a coherent homotopy $G:(I\times Y_b,1\times q_{\underline{b}},B)\to\mathcal{Z}$. Further on, let G be given by a function Ψ and maps $G^{\underline{j}}_{\underline{c}}$. Then we can define a coherent homotopy $H:(I\times X_a,1\times p_{\underline{a}},A)\to\mathcal{Z}$ given by the function $\phi\Psi$ and maps $H^{\underline{j}}_{\underline{c}}:(I\times X_a,1\times p_{\underline{a}},A)\to\mathcal{Z}$ defined by

$$H^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,s,x)=(G(1\times f))^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,s,x).$$

Then

$$H^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,0,x)=((g\circ q[\psi,\Psi])f)^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,s,x)$$
$$H^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,1,x)=((g'\circ q[\psi',\Psi])f)^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,s,x).$$

If we consider $g:\mathcal{Y}\to\mathcal{Z}$ and $H:I\times\mathcal{X}\to\mathcal{Z}$ as coherent maps over C, $g:(\mathcal{Y},\psi)\to(\mathcal{Z},\text{id})$ and $H:(I\times\mathcal{X},\phi\Psi)\to(\mathcal{Z},id)$, then by the maps $h^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,x)=H^{\underline{j}}_{\underline{c}}(t_1,\dots,t_n,0,x)$ it is defined a coherent map over C, $h:(\mathcal{X},\phi\Psi)\to(\mathcal{Z},id)$. Also, we consider the coherent maps over $C, f[\psi, \phi\psi] : (\mathcal{X}, \phi\psi) \to (\mathcal{Y}, \psi)$ and $f[\psi, \phi\psi] : (\mathcal{X}, \phi\psi) \to (\mathcal{Y}, \psi)$. Then the coherent map $(g \circ q[\psi, \Psi])f$ is the same as the coherent map over C $(g \circ q[\psi, \Psi]) \circ f[\Psi, \phi\Psi]$ in the sense of Remark 1. By theorems 1,2 and 3 from §1 and theorem 8 we have

$$h \cong g \circ (q[\psi, \Psi] \circ f[\Psi, \phi\Psi]) \cong g \circ (f[\psi, \phi\psi] \circ p[\phi\psi, \phi\Psi]) \cong (g \circ f[\psi, \phi\psi]) \circ p[\phi\psi, \phi\Psi].$$

In the same way, if we define a coherent map over C, $h' : (\mathcal{X}, \phi\Psi) \to (\mathcal{Z}, id)$ with the maps $h'^{j}_{\underline{c}}(t_1, \ldots, t_n, x) = H^{j}_{\underline{c}}(t_1, \ldots, t_n, 1, x)$ then $h' \cong (g' \circ f[\psi', \phi\psi']) \circ p[\phi\psi', \phi\Psi]$.

It follows $(g' \circ f[\psi', \phi\psi']) \circ p[\phi\psi', \phi\Psi] \cong (g \circ f[\psi, \phi\psi]) \circ p[\phi\psi, \phi\Psi])$ with a homotopy over C, $(I \times \mathcal{X}, \phi\Psi) \to (\mathcal{Z}, id)$ i.e. a coherent map in the sense of Remark 1. Also the composition of coherent maps over C, $g \circ f[\psi, \phi\psi] : (\mathcal{X}, \phi\psi) \to (\mathcal{Z}, id)$ is the same with the coherent map $gf : \mathcal{X} \to \mathcal{Z}$ and from Remark 3 we have $gf \cong g'f$. ■

DEFINITION: The *category Coh* has as objects coherent inverse systems $\mathcal{X} = (X_a, p_{\underline{a}}, A)$ where A is a cofinite directed set. The morphisms are coherent homotopy classes of coherent maps. The composition of morphisms is defined as the composition of homotopy classes. The identity morphism is a coherent homotopy class of the identity map defined as follows:

The *coherent identity map* $1_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ consists of the identity function 1_A and of the maps $1^{j}_{\underline{a}} : I^{\underline{j}} \times Y_{a_n} \to Y_{a_0}$. For $n = 0$ we have $1_{a_0} = 1_{X_{a_0}}$ and for $n > 0$

$$1^{j}_{\underline{a}}(t_1, \ldots, t_n, x) = p_{\underline{a}}(t_1, \ldots, t_{j_1 - 1}, 1, t_{j_1 + 1}, \ldots, t_{j_{k-1} - 1}, 1, t_{j_{k-1} + 1}, \ldots, t_{n-1}, x).$$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a coherent map given by a map ϕ and maps $f^{j}_{\underline{c}}$. We consider f as a coherent map over B, $f : (\mathcal{X}, \Phi) \to (\mathcal{Y}, id)$. Then by Theorem 4, $f \cong 1_{(\mathcal{Y}, id)} \circ f$ and $f \cong f \circ 1_{(\mathcal{X}, \phi)}$ with a coherent homotopies over B, $(I \times \mathcal{X}, \phi) \to (\mathcal{Y}, id)$ i.e. coherent maps. It follows that $f \cong 1_{\mathcal{Y}} f$ and $f \cong f 1_{\mathcal{X}}$.

The theorems 6,7 and 9 verify other requierments for category.

Now we can define a coherent shape category which objects are all topological spaces.The definition is analogous to the definition of the shape category in [5] and [7] and strong shape category in [4].

The role of H-Top expansions in [5] and of coherent expansions in a commutative inverse system in [4] here is played by coherent expansions in a coherent inverse system.

A single topological space X is presented as coherent inverse system $(X_m, id_{\underline{m}}, \mathbb{N})$ where $\mathbb{N} = \{1, 2, \ldots\}$, $X_m = X$ for all $m \in \mathbb{N}$, and $id_{\underline{m}}(t_1, \ldots, t_{n-1}, x) = x$. We will identify the topological space X and the coresponding coherent inverse system $(X_m, id_{\underline{m}}, \mathbb{N})$. Then we mention that for an arbitrary coherent inverse system $\mathcal{X} = (X_a, p_{\underline{a}}, A)$ a coherent map $\pi : X \to \mathcal{X}$ is given by maps $\pi^{j}_{\underline{a}} : I^{\underline{j}} \times I \times X \to X_{a_0}$ and does not depend on the choice of an increasing function.

DEFINITION: A coherent map $\pi : X \to \mathcal{X}$ is a *coherent expansion (in a coherent inverse system)* if the following conditions hold:

1) If $\mathcal{Y} = (Y_b, q_{\underline{b}}, B)$ is a coherent inverse system where B is cofinite and all Y_b are ANR for metric spaces, then for a coherent map $f : \mathcal{X} \to \mathcal{Y}$ there exists a coherent map $h : \mathcal{X} \to \mathcal{Y}$ such that $f \cong h\pi$.

2) If $h, h' : \mathcal{X} \to \mathcal{Y}$ are coherent maps such that $h\pi \cong h'\pi$, then $h \cong h'$.

Now, let X, Y be topological spaces and $\pi : X \to \mathcal{X}$, $\rho : Y \to \mathcal{Y}$ be coherent expansions and $f : \mathcal{X} \to \mathcal{Y}$ a coherent map. If we choose another expansions $\pi' : X \to \mathcal{X}$, $\rho' : Y \to \mathcal{Y}$ and map $f' : \mathcal{X} \to \mathcal{Y}$, then by the definition of the coherent expansion, there exist (iso)morphisms in $Coh[i] : \mathcal{X} \to \mathcal{X}'$ and $[j] : \mathcal{Y} \to \mathcal{Y}'$ such that $[i][\pi] = [\pi']$, $[j][\rho] = [\rho']$ (here [ ] denotes the coherent homotopy class).

We define an equivalence relation $\backsim$ with $(\pi, \rho, [f]) \backsim (\pi', \rho', [f'])$ if $[f'][i] = [j][f]$.

The *coherent shape category* has as objects all topological spaces. Morphisms $F : X \to Y$ are equivalence classes of $(\pi, \rho, [f])$. We can allways suppose that two morphisms $F : X \to Y$ and $G : Y \to Z$ are given by $(\pi, \rho, [f])$ and $(\rho, \sigma, [g])$ and we define a composition morphism as equivalence class of $(\pi, \sigma, [g][f])$ and the identity map is a class of $(\pi, \pi, [1_{\mathcal{X}}])$.

§4. Commutative inverse systems

The coherent category CPHTop of commutative inverse systems is constructed in details by Yu.Lisica and S.Mardešić in [4]. There coherent maps are defined by use of the standard simplex Δ^n. Here we will give the transfered definition of the category CPHTop using simplex $\nabla^n = \{(t_1, \dots, t_n) \in I^n : 1 \geq t_1 \geq \dots t_n \geq 0\}$.

It seems that in this way not only the composition formula, but more of the proofs of theorems become simpler. Also, here we will use directed sets $(A, <)$, instead of $(A, \leq)$. For the equivalence of these two coherent theories see N. Šekutkovski [9].

In inverse system $\underline{X} = (X_a, p_{a_0a_1}, A)$ for a pair of indices $a_0 < a_1$ there is a map $p_{a_0a_1} : X_{a_0} \to X_{a_1}$ such that if $a_0 < a_1 < a_2$ then $p_{a_0a_2} = p_{a_0a_1}p_{a_1a_2}$. For this last equation often we use the terminology commutative inverse system. Any commutative inverse system $\underline{X} = (X_a, p_{a_0a_1}, A)$, may be considered as coherent inverse system $\mathcal{X} = (X_a, p_{\underline{a}}, A)$ defining for $\underline{a} = (a_0, \dots, a_n)$, a map $p_{\underline{a}} : I^{n-1} \times X_{a_n} \to X_{a_0}$ by

$$p_{a_0 \dots a_n}(t_1, \dots, t_{n-1}, x) = p_{a_0a_n}(x). \tag{14}$$

A special coherent map $\underline{f} : \underline{X} \to \underline{Y} = (Y_b, q_{b_0b_1}, B)$ cosists of

1) an increasing function $\phi : B \to A$

2) for any increasing sequence $\underline{b} = (b_0, \dots, b_n)$ in B of a map $f_{\underline{b}} : \nabla^n \times X_{\phi(b_n)} \to Y_{b_0}$ satisfying the following boundary conditions

$$f_{\underline{b}}(t_1, \dots, t_n, x) = \begin{cases} f_{b_0b_1}f_{b_1 \dots b_n}(t_2, \dots, t_n, x); & t_1 = 1 \\ f_{b_0 \dots \hat{b}_i \dots b_n}(t_1, \dots, \hat{t}_i, \dots, t_n, x); & t_i = t_{i+1} \\ f_{b_0 \cdots _{n-1}}(t_1, \dots, t_{n-1}, p_{\phi(b_{n-1}b_n)}(x)); & {}_n = 0. \end{cases} \tag{15}$$

If inverse systems $\underline{X}$ and $\underline{Y}$ are interpreted as coherent inverse systems $\mathcal{X} = (X_a, p_{\underline{a}}, A)$ and $\mathcal{Y} = (Y_b, q_{\underline{b}}, B)$ then special coherent map $\underline{f} : \underline{X} \to \underline{Y}$ may be interpreted as coherent map $f : \mathcal{X} \to \mathcal{Y}$ given by function ϕ and maps $f^{j}_{\underline{b}} : I^{\underline{j}} \times X_{\phi(b_n)} \to Y_{b_0}$ defined by

$$f^{j}_{\underline{b}}(t_1, \dots, t_n, x) = f_{\underline{b}}(0, \dots, 0, t_{j1}, 0, \dots, 0, t_{j_i}, 0, \dots, 0, t_{j_k}, x). \tag{16}$$

Special coherent maps $\underline{f}, \underline{f}' : \underline{X} \to \underline{Y}$ given by functions ϕ, ϕ' and maps $f_{\underline{b}}, f'_{\underline{b}}$ respectively are coherently homotopic if:

1) there exists an increasing function $\Phi : B \to A$, such that $\Phi > \phi$ and $\Phi > \phi'$.

2) there exists a special coherent map $\underline{F} : I \times \underline{X} \to \underline{Y}$ given by Φ and by maps $F_{\underline{b}} : \nabla^n \times I \times X_{\phi(b_n)} \to Y_{b_0}$ such that

$$\begin{aligned} F_{\underline{b}}(t_1, \dots, t_n, 0, x) &= f_{\underline{b}}(t_1, \dots, t_n, p_{\phi(bn)\Phi(b_n)}(x)) \\ F_{\underline{b}}(t_1, \dots, t_n, 1, x) &= f'_{zab}(t_1, \dots, t_n, p_{\phi'(b_n)\Phi(b_n)}(x)). \end{aligned} \tag{17}$$

If $\underline{f}, \underline{f}'$ are homotopic we put $\underline{f} \cong f'$.

REMARK 4. We mention that by function Φ and by maps $h_{\underline{b}}(t_1, \dots, t_n, x) = f_{\underline{b}}(t_1, \dots, t_n, p_{\phi(b_n)\Phi(b_n)}(x))$ which appear in the definition of homotopy it is defined a coherent map $\underline{h} : \underline{X} \to \underline{Y}$.

The coherent homotopy between two arbitrary coherent maps $f, f' : \mathcal{X} \to \mathcal{Y}$ in [6] and [8] is defined by formula (17) (only one has to replace $f_{\underline{b}}$ and $f'_{\underline{b}}$ with $f^{j}_{\underline{b}}$ and $f'^{j}_{\underline{b}}$), instead of formula (12). But, in this case the map $g : \mathcal{X} \to \mathcal{Y}$ defined by function Φ and by maps $g^{j}_{\underline{b}}(t_1, \dots, t_n, x) = f^{j}_{\underline{b}}(t_1, \dots, t_n, p_{\phi(bn)\Phi(bn)}(x))$ it is not coherent in the general situation.

The composition of special coherent maps $\underline{f} : \underline{X} \to \underline{Y}$ and $\underline{g} : \underline{Y} \to \underline{Z} = (Z_c, r_{\underline{c}}, C)$ given by function ψ and maps $g^{j}_{\underline{c}}$, is a special coherent map $\underline{h} = \underline{g}f : \underline{X} \to \underline{Z}$ given by function $\chi = \phi\psi$ and maps $h_{\underline{c}} : \nabla^n \times X_{\chi(c_n)} \to Z_{c_0}$ defined by

$$h_{\underline{c}}(t_1, \dots, t_n, x) = g_{c_0 \dots c_i}(2t_1 - 1, \dots, 2t_i - 1, f_{\psi(ci \dots c_n)}(2t_{i+1}, \dots, 2t_n, x) \tag{18}$$

for $(t_1, \dots, t_n) \in K^n_i = \{(t_1, \dots, t_n) : t_i \geq \frac{1}{2} \geq t_{i+1}\}$, $i = 0, 1, \dots, k$.

The coherent identity map $\underline{1} : \underline{X} \to \underline{X}$ is given by the identity function $1_A : A \to A$ and by maps $1_{\underline{a}} : \nabla^n \times X_{b_n} \to X_{b_0}$ defined by $1_{\underline{b}}(t_1, \dots, t_n, x) = p_{b_0 b_n}(x)$.

The category CPHTop has as objects commutative inverse systems $\underline{X} = (X_a, p_{a_0 a_1}, A)$ where A is a cofinite set and morphisms are homotopy classes of special coherent maps.

We define a functor $\mathcal{F}$:CPHTop $\to$ $\mathcal{C}$ *Coh* with $\mathcal{F}(\underline{X}) = \mathcal{X}$ where $\mathcal{X}$ is the interpretation of $\underline{X}$ as coherent inverse system, and if $\underline{f} : \underline{X} \to \underline{Y}$ is a special coherent map, then $\mathcal{F}([\underline{f}]) = [f]$ where $f : \mathcal{X} \to \mathcal{Y}$ is the interpretation of $\underline{f}$ as coherent map ([ ] denotes the homotopy class).

First we have to proof that $\mathcal{F}$ is well defined. Let $\underline{f} \cong f'$ with a special coherent homotopy $\underline{F}$, and let f, f' and F respectively, be their interpretations as coherent maps.

Let $H : I \times \mathcal{X} \to \mathcal{Y}$ be a coherent map defined by function Φ and by maps $H^{\underline{j}}_{\underline{b}} : I^{\underline{j}} \times I \times X_{\Phi(bn)} \to Y_{b_0}$ defined for $(t_1, \dots, t_n) \in K^{\underline{j}}_{j_m}$ by

$H^{\underline{j}}_{\underline{b}}(t_1, \dots, t_n, s, x) =$

$$F^{j_0 \dots j_m}_{b_0 \dots b_{j_m}}(\sum_{i=0}^{m-1}(0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 2t_{j_{i+1}} - 1, 0, \dots, 0), s, p_{\Phi(bj_m bn)}(x)).$$

To check that this is a homotopy between coherent maps f, f', first we mention that the following equality hold for the coherent map over B, $p[\phi, \Phi]$, for all $(t_1, \dots, t_n) \in I^{\underline{j}}$ and all $\underline{j} = (j_0, \dots, j_k)$, $0 = j_0 < \dots < j_k = n$

$$p^{j_0 \dots j_k}_{\phi(b_0 \dots b_n)}(t_1, \dots, t_n, x) = p_{\phi(b_0)\Phi(b_n)}(x). \tag{19}$$

Using the equality (19) in the final step,for $(t_1, \dots, t_n) \in K^{\underline{j}}_{j_m}$ we have

$H^{\underline{j}}_{\underline{b}}(t_1, \dots, t_n, 0, x) = F_{b_0 \dots b_{j_m}}(0, \dots, 0.2t_{j_1} - 1, 0, \dots, 0, 2t_{j_m} - 1), 0, p_{\Phi(b_{j_m} b_n)}(x)) =$
$= \underline{f}_{b_0 \dots bj_m}(0, \dots, 0.2t_{j_1} - 1, 0, \dots, 0, 2t_{j_m} - 1, p_{\phi(bj_m)\Phi(b_{j_m})} p_{\Phi(b_{j_m} b_n)}(x))$
$= f^{j_0 \dots j_m}_{b_0 \dots b_{j_m}}(\sum_{i=0}^{m-1}(0, \dots, 0_{j_i}, t_{j_i+1}, \dots, t_{j_{i+1}-1}, 2t_{j_{i+1}} - 1, 0, \dots, 0), p_{\phi(b_{j_m})\Phi(b_n)}(x))$
$= (f \circ p[\phi, \Phi])^{\underline{j}}_{\underline{b}}(t_1, \dots, t_n, x).$

Similiarly, $H^{\underline{j}}_{\underline{b}}(t_1, \dots, t_n, 1, x) = (f' \circ p[\phi', \Phi])^{\underline{j}}_{\underline{b}}(t_1, \dots, t_n, x)$ It follows that $f \cong f'$.

To show that $\mathcal{F}$ is a functor, if $\mathcal{F}(\underline{f}) = f$, $\mathcal{F}(\underline{g}) = g$, and $h = \mathcal{F}(\underline{gf})$ for $(t_1, \dots, t_n) \in K^{\underline{j}}_{j_m}$ we have

$h^{\underline{j}}_{\underline{c}}(t_1, \dots, t_n, x) = (\underline{gf})_{\underline{c}}(0, \dots, 0, t_{j_1}, 0, \dots, 0, t_{j_i}, 0, \dots, 0, t_{j_k}, x)$
$= \underline{g}_{c_0 \dots c_{j_m}}(0, \dots, 0, 2t_{j_1} - 1, 0, \dots, 0, 2t_{j_i} - 1, 0, \dots, 0, 2t_{j_m} - 1,$
$\underline{f}_{\psi(c_{j_m} \dots n)}(0, \dots, 0, 2t_{j_{m+1}}, 0, \dots, 0, 2t_n, x) == (gf)^{\underline{j}}_{\underline{c}}(t_1, \dots, t_n, x)$
i.e. $\mathcal{F}(\underline{g})\mathcal{F}(\underline{f}) = \mathcal{F}(\underline{gf})$. ■

QUESTION: Is $\mathcal{F}$(CPHTop) a full subcategory of *Coh*?

Of course, for a cofinite set E we may consider the category CPHTop(E), a functor $\mathcal{F}$:CPHTop(E) $\to$ *Coh*(E) and state the question: Is $\mathcal{F}$(CPHTop(E)) a full subcategory of *Coh*(E)?

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