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2-to-1 OPEN MAPPINGS OF COMPACT SPACES

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Abstract. *In this paper we consider the problem of existence of inverse branches of 2-to-1 open mappings of compact spaces.*

Introduction.

Since we plan to reduce this problem to the case when the spaces involved are 0-dimensional, we will use their Gleason spaces. Let us recall their construction and properties. Let X be a compact Hausdorff space. $RO(X)$ will denote the set of regular open subsets of X. It is a complete Boolean algebra for operations defined as $a \wedge b = a \cap b$ and $a'=\text{int}(X\backslash a)$. For $q \in X$ $\mathcal{F}_q = \{A \in RO(X) \mid q \in A\}$. It is obviously a filter of $RO(X)$. $G(X)$, the Gleason space of X is the Stone space of $RO(X)$. The mapping $\phi : G(X) \to X$ defined by $\phi(p) = \cap\{cl_X A \mid A \in p\}$ is an irreducible onto mapping. $F_q = \phi^{-1}(\{q\})$ is the closed set corresponding in Stone duality to $\mathcal{F}_q$. For $A \subset X, A \in RO(X)$ $\theta(A) = \{p \in G(X) \mid A \in p\}$ is a clopen set in $G(X)$.

1. 2-to-1 mappings

Definition. A mapping $f : X \to Y$ is $2-to-1$ if every element of Y has exactly two originals in X.

Proposition 1. *Let X, Y be 0-dimensional compact Hausdorff spaces and $f : X \to Y$ an open $2-to-1$ continious mapping. Then there exists a homeomorphism $g : Y \times 2 \to X$ such that $f \circ g = \pi$ where π is the natural projection.*

Proof. Let $q \in Y$ and $p_1, p_2 \in X$ so that $f(p_i) = q, i = 1, 2$. Let also a be a clopen neighbourhood of p_1 not containing p_2. Let $b_q = f[a] \cap f[a^c]$, and $c_q = f^{-1}[b_q] \cap a$. b_q is nonempty since it contains q. b_q, c_q are obviously clopen sets and $f \mid c_q$ is a 1-to-1 mapping (second original is in a^c). Collection $\{b_q \mid q \in Y\}$ obtained in this way is an open cover of Y. Let $F \subset Y$ be a finite set such that $\{b_q \mid q \in F\}$ is also a cover of Y. Without loss of generality we can suppose that they are disjoint. Then the appropriate $\{c_q \mid q \in F\}$ have as their union a clopen set c such that $f \mid c$ is a homeomorphism on Y. Hence the mapping g defined as $g(y, 0) = f^{-1}[\{y\}] \cap c$, and $g(y, 1) = f^{-1}[\{y\}] \cap c^c$ has the prescribed properties.

Lemma 1. *Let X, Y be compact spaces and f an open mapping from X onto Y. Then the mapping $\varphi : G(X) \to G(Y)$ defined by $\varphi(p) = \{f[A] \mid A \in p\}$ is open and continious, and if f is $2 - to - 1$, then φ is $2 - to - 1$ also.*

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PROOF. Since f is an open and closed mapping, for $A \in RO(X) f[A] \in RO(Y)$, and for $B \in RO(Y)$ $f^{-1}[B] \in RO(X)$ also. The mapping $\rho : RO(Y) \to RO(X)$ defined by $\rho(B) = f^{-1}[B]$ is an embedding of Boolean algebras. Hence, its dual mapping $S(\rho) : G(X) \to G(Y)$ defined by $S(\rho)(p) = \{B \in RO(Y) \mid \rho(B) \in p\} = \{B \in RO(Y) \mid f^{-1}[B] \in p\} = \{f[A] \mid A \in p\} = \varphi(p)$ is a continious onto mapping. It is open since for a base set $A \in RO(X) \varphi[A] = f[A] \in RO(Y)$. Let $t \in G(Y), \phi(t) = s$. Let $\{q_1, q_2\} = f^{-1}(s)$. Since $\mathcal{F}_s \subset t$ $f[\mathcal{F}_q] \subset \mathcal{F}_s \subset t$, each of them has an extension ultrafilter which maps into t . They are distinct as extensions of distinct $\mathcal{F}_q$. So each point from $G(Y)$ has at least two originals. Their number is exactly two since otherwise we would have three disjoint regular open sets in X whose images have nonempty intersection.

LEMMA 2. *Let the notation be as in the preceeding lemma. Let $q \in X$ and $f(q) = z$. Then $\varphi(F_q) = F_z$.*

PROOF. Let us note first that $\rho[\mathcal{F}_z] \subset \mathcal{F}_q$ since inverse image of a regular open set containing z is a regular open set containing q. Let $p \in F_q$. Then $\mathcal{F}_q \subset p$. $\varphi(p) = \{f[a] \mid a \in p\} \supset \{f[a] \mid a \in \mathcal{F}_q\} = \mathcal{F}_z$, hence $\varphi(p) \in F_z$. Similarly, let $u \in F_z$. Then $\mathcal{F}_z \subset u$ hence $\rho[u] \cup \mathcal{F}_q$ has the finite intersection property. Hence it could be extended to an ultrafilter p on $RO(X)$. Then $p \in F_q$ and $\varphi(p) = u$, hence $u \in \varphi[F_q]$.

THEOREM 1. *Let $f : X \to Y$ be an open $2-to-1$ mapping of compact spaces. Then there exists an open dense subset $C \subset Y$ and a homeomorphism $g : C \times 2 \to X$ such that $f \circ g = \pi$ where π is the natural projection.*

PROOF. Let $G(X), G(Y)$ and φ be as in the preceeding lemma. By proposition 1 there exists a set $a \subset G(X)$ such that $\varphi \mid a$ and $\varphi \mid a^c$ are homeomorphisms. For $q \in X$ we say that F_q is bad if it intersects both a and a^c. We claim that the union of bad sets has empty interior. Suppose not. Then there exists an open set U consisting of points of bad sets. Since $U \cap a$ and $U \cap a^c$ are both open and one of them is nonempty, without loss of generality I can suppose $U \subset a$. For each $q \in X$, if F_q is good then it is whole in $a \backslash U$ or in a^c , anyway in U^c, hence q is in the image of U^c. If F_q is bad then there is a part of F_q in a^c hence in U^c which maps in q. Hence U^c is a closed set which maps onto X . It contradicts the fact that φ is irreducible. Let R be the union of good $F_q's$ which are contained in a and B similarly for a^c . Let A, A' be regular open sets in X such that $\theta(A) = a$ and $\theta(A') = a^c$. We claim that $\phi[R] = A$ and $\phi[B] = A'$. Let $q \in A$. Then $A \in \mathcal{F}_q$ hence there exist no ultrafilter p on $RO(X)$ containg A' and extending $\mathcal{F}_q$. Henceforth $F_q \cap a^c = \emptyset$, so $F_q \subset a$, and F_q is good and therefore $F_q \subset R$. Now $\phi[F_q] = \{q\}$, so $q \in \phi[R]$. On the other hand, if $q \in \phi[R], F_q \subset R$, so $F_q \cap a^c = \emptyset$. Hence, $\mathcal{F}_q \cup \{A'\}$ does not have the finite intersection property, so there exists $U \in \mathcal{F}_q$ such that $U \cap A' = \emptyset$ Therefore $U \subset A$, and $q \in$A. Similarly we prove that $\phi[B]$=A'. Let now $C = f[A].C \in RO(Y)$. We will prove that $C = f[A']$ also. So let $z \in C$ and $q \in A$ so that $f(q)$=z. Now we have that $F_q \subset a$ and $\varphi[F_q] = F_z$. Since $z \in C \in RO(Y)$, $C \in \mathcal{F}_z$ hence $F_z \subset \theta(C)$. Let now $t \in X$ be the second point such that $f(q') = z$. Then $\varphi[F_t] = F_z$ also. Now we have that F_t and F_q are disjoint, have the same φ image, $F_q \subset a$ and $\varphi \mid a$ is $1-1$, $F_t \cap a = \emptyset$ i.e. $F_t \subset a^c$, so it is good and therefore contained in B. Now,finally $t \in \phi[B] = A'$ and $z \in f[A']$, which proves our claim. Now since $A \cup A'$ is dense in X , $C = f[A \cup A']$ is dense in Y. Mapping $g : C \times 2 \to X$ defined by $g(c, 0) = f^{-1}[\{c\}] \cap A$ and $g(c, 1) = f^{-1}[\{c\}] \cap A'$ obviously has the required properties.

The following example shows that the theorem we obtained is the best possible.

EXAMPLE. Let X be the disjoint union of $A = \omega_1 + 1 + \omega_1^*$, where ω_1^* is ω_1 in the reversed order having $\{\omega_1\}$ as a limit, and $B = \omega + 1 + \omega^*$. Let Y be $\omega_1 + 1 + \omega^*$, and f the mapping defined on A to be identity on $\omega_1 + 1$ and to reverse order on ω_1^* , and similarly on B . There is no inverse branch from the whole Y on X, since there is no point in X having characters ω and ω_1 .

QUESTION 1. Is it true that Proposition 1 holds just for 0-dimensional spaces i.e. if a space X has the property that for any compact space Y and every open $2 - to - 1$ mapping $f : X \to Y$ there exists a homeomorphism $g : Y \times 2 \to X$ so that $g \circ f = \pi$, then X is 0-dimensional?

QUESTION 2. What are possible $2 - to - 1$ (open) continious images of a given space?

2. LE2-to-1 mappings

DEFINITION. *A mapping $f : X \to Y$ is $LE2 - to - 1$ if every point from Y has one or two originals.*

The following proposition is a special case of a known theorem of Arhangelski, but we prove it here for the sake of completeness, and it will not take long.

PROPOSITION 1. *Let $f : X \to Y$ be an open $LE2 - to - 1$ mapping and let $S = \{p \in Y \mid | f^{-1}(p) |= 2\}$. S is an open set. We call it the set of splitting points.*

PROOF./quad Let $q \in S$ and $\{p_1, p_2\} = f^{-1}[\{q\}]$, and G_1, G_2 be their disjoint neighbourhoods. $f[G_1] \cap f[G_2]$ is an open set containing q and contained in S. Hence S is an open set.

PROPOSITION 2. *Let the notation be as in the preceeding proposition. If S is a clopen set then there exists an open dense set $G \subset Y$ and $C \subset S$, open and dense in S, and a homeomorphism $g : G + C \to X$ so that $f \circ g$ is identity on G, and also on C.*

PROOF. Let $A = f^{-1}[S]$. The mapping $f \mid A : A \to S$ is a $2 - to - 1$ mapping of compact spaces, hence by Theorem 1.1 there exists $C \subset S$, dense in S and open and a mapping $h : C \times 2 \to A$ such that $f \circ h$ is a natural projection. Since $f \mid X \backslash A$ is a homeomorphism onto $Y \backslash S$, we have that for $G = (X \backslash S) \cup C$ the mapping g defined so that $g \mid X \backslash S = f^{-1}$, $g \mid C = h \mid C \times \{0\}$ and on the disjoint copy of C to be $h \mid C \times \{1\}$, has the desired properties.

COROLLARY. *Let $f : X \to Y$ be an $LE2 - to - 1$ open mapping of compact spaces such that $S = \{p \in Y \mid | f^{-1}(p) |= 2\}$ is clopen. Then there exists an open dense subset $G \subset Y$ and a mapping $g : G \times 2 \to X$ so that $f \circ g = \pi$.*

PROOF. It is G from the preceeding proposition.

THEOREM 1. *Let $f : X \to Y$ be an open onto $LE2 - to - 1$ mapping. There exists a dense open subset $G \subset Y$ and a mapping $g : G \times 2 \to X$ so that $f \circ g = \pi$.*

PROOF. Let us suppose first that S is dense in Y. Let us consider their Gleason spaces $G(X)$ and $G(Y)$. Then the mapping φ is also open $LE2 - to - 1$ mapping. $\phi^{-1}[S]$ is an open dense set in Y consisting of splitting points of φ. Using Stone duality on Theorem 2 from [2], we get that the set of splitting points in $G(Y)$ is clopen, hence the whole $G(Y)$. Hence φ is a $2 - to - 1$ mapping. Now we proceed exactly like in the proof of Theorem 1.1 up to the point where we prove that $f[A] = C = f[A']$. Here we proceed in a bit different way, we prove $f[A] \cap S = f[A'] \cap S$, again in the same way as in Theorem 1.1. Taking $C = f[A] \cap S$ we get a dense open subset $C \subset Y$ and a homeomorphism h.

In the general case we use this special case theorem on $cl(f^{-1}[S])$ to get C and h. Then we define $G = C \cup (Y \setminus \mathrm{clG})$ and g so that $g \mid C \times 2 = h$ and for $y \in Y \setminus \mathrm{clS}$ we define $g(y, 0) = g(y, 1) = f^{-1}(y)$. It obviously has the prescribed properties.

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