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ON DIFFERENT TYPES OF DIVISIBILITY OF TOPOLOGICAL SPACES

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ABSTRACT. *We consider a general concept of divisibility of topological spaces and as special cases the following kinds: pointwise divisibility, H-divisibility, k-divisibility and (usual) divisibility. Among other, we give characterizations of pointwise divisibility and open pointwise H-divisibility. We also give some examples concerning divisibility.*

0. Introduction

Let X be a topological space and A a subset of X. A family $\mathcal{D}_A$ of subsets of X is called a *divisor for A* (resp. a *Hausdorff divisor for A*) if for every $x \in A$ and every $y \in X \setminus A$ there exists $D \in \mathcal{D}_A$ such that $x \in D$ and $y \notin D$ [2] (resp. there exist elements P and Q in $\mathcal{D}_A$ such that $x \in P$, $y \in Q$ and $P \cap Q = \emptyset$ [4]). If all members of $\mathcal{D}_A$ are closed (open, compact ...) in X, then we say that $\mathcal{D}_A$ is a closed (open, compact ...) divisor (resp. Hausdorff divisor) for A. In [2], A. Arhangel'skii defined divisible and strictly divisible spaces: a space X is said to be *divisible* (resp. *strictly divisible*) if for every $A \subset X$ there is a countable closed divisor for A (resp. a countable divisor for A consisting of closed G_δ-sets). Here for a space X we will use the term "*H-divisible*" when every subset of X has a countable closed Hausdorff divisor.

Let us note that from these definitions we have the following facts:

0.1. FACT. If $\mathcal{D}$ is a divisor for $A \subset X$, then $\{X \setminus D : D \in \mathcal{D}\}$ is a divisor for $X \setminus A$. ■

0.2. FACT. If $\mathcal{H}$ is a Hausdorff divisor for $A \subset X$, then $\{X \setminus H : H \in \mathcal{H}\}$ is a divisor for A and $X \setminus A$. In particular, open H-divisible spaces are divisible. ■

0.3. FACT. Every open H-divisible space is Hausdorff. ■

In [11] (see also [12], [13]) the *divisibility degree* $\mathrm{dvs}(X)$ of a space X was defined as follows. If A is a subset of X one defines

$$\mathrm{dvs}(A, X) = \min\{\tau : \text{ there is a closed divisor } \mathcal{D} \text{ for } A \text{ of cardinality} \leq \tau\}$$

and then

$$\mathrm{dvs}(X) = \sup\{\mathrm{dvs}(A) : A \subset X\}.$$

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In [12] and [13], a few cardinal inequalities involving the divisibility degree were shown.

Here we define a general concept of divisibility, i.e. we define different types of divisibility as follows.

0.3. DEFINITION. Let τ be a cardinal and let $\mathcal{A}$ and $\mathcal{P}$ be two families of subsets of a space X. We say that X is *$(\mathcal{A},\mathcal{P}) - \tau$ -divisible* if for every $A \in \mathcal{A}$ there exists a divisor for A of cardinality $\leq \tau$ consisting of members from $\mathcal{P}$. When $\mathcal{P}$ is the family of closed subsets of X we say that X is *$\mathcal{A}$–τ-divisible*, and if $\mathcal{A} = \{A : A \subset X\}$ (resp. $\mathcal{A} = \{\{x\} : x \in X\}$; $\mathcal{A} = \{X \setminus \{x\} : x \in X\}$) we use the term "*$\tau$-divisible*" (resp. "*pointwise τ-divisible*"; "*co-pointwise τ-divisible*") instead of "$\mathcal{A} - \tau$-divisible". *Divisibility* (*pointwise divisibility*; *co-pointwise divisibility*) means ω-divisibility (pointwise ω-divisibility; co-pointwise ω-divisibility). ■

It is understood now how we should define the corresponding types of H-divisibility.

It should be mentioned that divisibility is closely connected with cleavability. Recall that a space X is called *cleavable* (resp. *pointwise cleavable*) *over a space* Y if for every $A \subset X$ (resp. for every $x \in X$) there exists a continuous mapping f from X into Y for which $f^{-1}f(A) = A$ (resp. $f^{-1}f(x) = \{x\}$). Cleavability over $\mathbb{R}^\omega$ is named briefly by "cleavability".

Our notation and terminology are as in [6] (for general concepts), [1], [9], [10] (for cardinal functions). All spaces are at least T_1.

1. Pointwise divisibility

1.1. PROPOSITION. *A T_1-space X is co-pointwise divisible if and only if its pseudocharacter is countable.*

PROOF. ($\Rightarrow$) Let $x \in X$ and let $\mathcal{D}_x$ be a countable closed divisor for $X \setminus \{x\}$. Then $\mathcal{U} = \{X \setminus D : D \in \mathcal{D}_x\}$ is a countable collection of open subsets of X for which $\cap\mathcal{U} = \{x\}$, i.e. $\psi(X) \leq \omega$.

($\Leftarrow$) Let $\{x\} = \cap\{U_i : i \in \omega\}$. Then as can easily be verified the family $\{X \setminus U_i : i \in \omega\}$ is a countable closed divisor for $X \setminus \{x\}$. ■

1.2. PROPOSITION. *A regular space is pointwise divisible iff it is co- pointwise divisible iff its pseudocharacter is countable.* ■

For Tychonoff spaces we have the following result.

1.3. THEOREM. *For a Tychonoff space X the following statements are equivalent:*

(1) *For every $x \in X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $\{x\} = f^{-1}f(x)$ (i.e. X is pointwise cleavable over $\mathbb{R}$);*

(2) *X is pointwise (and co-pointwise) divisible;*

(3) $\psi(X) \leq \omega$.

PROOF. (1) $\Rightarrow$ (2). Let $x \in$ X. Choose a continuous function $f : X \to \mathbb{R}$ such that $\{x\} = f^{-1}f(x)$. Let $\{V_i : i \in \omega\}$ be a countable family of neighbourhoods of $f(x)$ such that $\{f(x)\} = \cap\{\overline{V}_i : i \in \omega\}$. Then the families $\{f^{-1}(\overline{V}_i) : i \in \omega\}$ and $\{f^{-1}(\mathbb{R} \setminus V_i) : i \in \omega\}$ are countable closed divisors for $\{x\}$ and $X \setminus \{x\}$, respectively, i.e. X is both pointwise and co-pointwise divisible.

(2) $\Rightarrow$ (3) See the previous propositions.

(3) ⇒ (1) Let $x \in X$ be any point. Since x is a G_δ point and X is a Tychonoff space, there exists a continuous function $f : X \to \mathbb{R}$ such that $\{x\} = f^{-1}(0)$, i.e. $\{x\} = f^{-1}f(x)$ and X is pointwise cleavable over $\mathbb{R}$. ■

1.4. EXAMPLES. (1) Every ccc LOTS is pointwise divisible. (This follows from the fact that every ccc LOTS has countable pseudocharacter.)

(2) The Alexandroff double circle [6] is a pointwise divisible non-divisible compact space (being compact non-metrizable; see Example 3.6). ■

For the definitions of the notions below see, for example, [14].

1.5. COROLLARY. *A radial (pseudoradial, biradial) co-pointwise divisible space is Fréchet-Urysohn (sequential, bisequential).*

(It follows from the fact that every radial (pseudoradial, biradial) space of countable pseudocharacter is Fréchet-Urysohn (sequential, bisequential); see, for instance, [14].)

1.6. COROLLARY. *A linearly uniformizable non-metrizable topological group cannot be pointwise divisible.*

(It follows from these facts: (1) Every non-metrizable linearly uniformizable topological group is orderable [14]; (2) for every orderable space X, $\psi(X) = \chi(X)$ [6]; (3) for a pointwise divisible (Tychonoff) space X, $\psi(X) \leq \omega$; (4) every first countable topological group is metrizable [6].) ■

1.7. COROLLARY. *Every pointwise divisible pseudocompact topological group is metrizable.*

(It follows from: (1) $\psi(G) \leq \omega$; (2) a pseudocompact topological group of countable pseudocharacter is compact [Arhangel´skii]; (3) a compact divisible space is metrizable [11].) ■

2. H-divisibility

Recall that the *H-pseudocharacter* of a Hausdorff space X is the smallest cardinal τ with the property: for every point $x \in X$ there exists a family $\mathcal{U}$ of neighbourhoods of x such that $|\mathcal{U}| \leq \tau$ and $\{x\} = \cap\{\overline{U} : U \in \mathcal{U}\}$.

Obviously we have:

2.1. PROPOSITION. *If the H-pseudocharacter of a Hausdorff space X is countable, then X is pointwise divisible.* ■

In fact, the following holds:

2.2. THEOREM. *A Hausdorff space X is open pointwise H-divisible if and only if $H\psi(X) \leq \omega$.*

PROOF. (⇔) Let x be any element in X and let $\mathcal{H}_x = \{V_i : i \in \omega\}$ be a countable open Hausdorff divisor for $\{x\}$. Then by the definition we have $\{x\} = \cap\{\overline{V}_i : i \in \omega\}$, i.e. $H\psi(X) \leq \omega$.

(⇐) Let $x \in$ X. Choose a countable family $\{U_i : i \in \omega\}$ of open neighbourhoods of x such that $\{x\} = \cap\{\overline{U}_i : i \in \omega\}$. It is very easy to see that the family $\mathcal{H} = \{U_i : i \in \omega\} \cup \{X \setminus \overline{U}_i : i \in \omega\}$ is an open (countable) Hausdorff divisor for $\{x\}$. ■

2.3. THEOREM. *Every H-closed open H-divisible space X has a countable pseudobase.*

PROOF. According to Theorem 2.2 we have $H\psi(X) \leq \omega$. Using the following result due to Gryzlov [8]: for every H-closed space Y of countable H-pseudocharacter we have $|Y| \leq 2^\omega$, one obtains $|X| \leq 2^\omega$. There exists a countable point separating family $\gamma = \{A_i : i \in \omega\}$ of subsets of X(see, for example, Lemma in [12]). By Fact 0.2X is divisible. For every $i \in \omega$ choose a countable closed divisor $\mathcal{D}_i$ for A_i and let $\mathcal{D} = \cup\{\mathcal{D}_i : i \in \omega\}$. Then the family $\mathcal{B} = \{X \setminus D : D \in \mathcal{D}\}$ is a countable pseudobase for X. ■

3. Divisibility

Note first that all spaces of countable pseudoweight are divisible. For ω_1-compact [9] perfect spaces the converse is also true [13].

Every cleavable space is strictly divisible. It is known [12] that for some classes of spaces cleavability and divisibility coincide:

3.1. THEOREM.([12]) *A perfectly normal space is divisible if and only if it is cleavable.* ■

Using the fact that every closed G_δ-set in a normal space is a zero-set we have

3.2. THEOREM. *A normal space is strictly divisible if and only if it is cleavable.* ■

The previous two results are only special cases of more general results shown in [12] (for normal spaces) and [5] (for the class of D-normal spaces which is larger than the class of normal spaces).

3.3. EXAMPLE. There are Tychonoff (pseudocompact) strictly divisible spaces which are not cleavable. Such a space is the well knowm Mrovka´s space $\Psi(N, \mathcal{A})$ [7;5I] (denoted by Ψ in [7]). This space has a countable pseudobase and so it is divisible [12]; it is strictly divisible because all its subsets are G_δ. But this space is not metrizable and thus it cannot be cleavable, since all cleavable pseudocompact spaces are metrizable [4]. ■

However, using the well known fact that every compact G_δ-set in a Tychonoff space is a zero-set [7] we have:

3.4. THEOREM. *A Tychonoff strictly k-divisible space X is cleavable.* ■

Here "k-divisibility" means that all members of divisors are compact sets.

Another result which gives connection between divisibility and cleavability (over a class) was established in [5].

3.5. THEOREM. *A perfect space X is divisible if and only if X is cleavable over the class of all developable T_1-spaces with a countable base.* ■

We will now discuss the behaviour of divisibility under basic operations with topological spaces. Of course, divisibility is a hereditary property.

The union of two divisible spaces need not be divisible as the following example shows.

3.6. EXAMPLE. Every divisible compact Hausdorff space is metrizable [11] (see also [12]). Therefore, the two arrows space [6] is not divisible (as it is a compact non-metrizable space). On the other hand, this space contains two divisible subspaces (homeomorphic to the Sorgengrey line, which is divisible since its pseudoweight is countable). Note that being first countable this space is pointwise divisible. ■

3.7. EXAMPLE. The product of two divisible spaces need not be divisible. Let $X = D((2^\omega)^+) \oplus [0,1]$ be the topological summ of the discrete space of cardinality $(2^\omega)^+$ and the unit segment. In [4], it was proved that this space is cleavable, so that X is divisible by Theorem 3.1. On the other hand, X^2 is not divisible since it contains the space $D((2^\omega)^+) \times [0,1]$ which is not divisible (the last space is not cleavable [4] and as it is perfectly normal, it is not divisible). ■

3.8. PROPOSITION. *Divisibility is an invariant of (continuous) open finite-to-one mappings.*

PROOF. Let X be divisible and let $f : X \to Y$ be an open finite-to-one mapping from X onto a space Y. Take any subset $B \subset Y$, $p \in B$, $q \in Y \setminus B$. Let $A = f^{-1}(B)$, $f^{-1}(p) = \{x_1, \ldots, x_k\}$ and let x be a point in X such that $f(x) = q$. Choose a countable closed divisor $\mathcal{D}_A$ for A and denote by $\mathcal{F}$ the family of all finite unions of members of $\mathcal{D}_A$. Obviously, $\mathcal{F}$ is a countable collection of closed subsets of X. Since $x \in X \setminus A$, for every $i \leq k$ there exists a member D_i in $\mathcal{D}_A$ for which $x_i \in D_i$ and $x \notin D_i$. Put $F = \cup\{D_i : i \leq k\}$. Then $F \in \mathcal{F}$. Consider the set $f^\#(F) = \{y \in Y : f^{-1}(y) \subset F\} = Y \setminus f(X \setminus F)$. Since f is open this set is closed in Y, $p \in f^\#(F)$ and $q \notin f^\#(F)$. Hence, the family $f^\#(\mathcal{F}) = \{f^\#(F) : F \in \mathcal{F}\}$ is a countable closed divisor for B. As B was an arbitrary subset of Y this means that Y is divisible. ■

3.9. EXAMPLE. Divisibility is not preserved by continuous open mappings. The Alexandroff double circle D [6] is a first countable compact non-metrizable and so non-divisible space with $w(D) = 2^\omega$. But every first countable space is an open image of a metric space of the same weight [6]. On the other hand, every metric space of weight $\leq 2^\omega$ is divisible (being cleavable [4]). ■

Let us note that this more general result holds: every space is a continuous open image of a (strictly) divisible space [4;Th.6.6].

The following simple result holds:

3.10. PROPOSITION. *Divisibility is an inverse invariant of continuous bijections.* ■

3.11. EXAMPLE. Divisibility is not an inverse invariant of continuous 2-to-1 mappings. The two arrows space [6] is not divisible but it admits a continuous (at most) 2-to-1 mapping onto the unit segment [0,1] which is divisible.

3.12. EXAMPLE. Divisibility is not an inverse invariant of perfect mappings. Let X be the unit segment $I = [0,1] \subset \mathbb{R}$ and let Y be the space I^2 with the lexicographic order topology. The projection $p : Y \to X$ is a perfect mapping. X is a (strictly) divisible space and Y is not divisible as it is a compact non-metrizable space. ■

3.13. PROPOSITION. (i)*If for every $x \in X$ the set $X \setminus \{x\}$ has a countable divisor consisting of closed Lindelöf subspaces of X, then X is a Lindelöf space.*

(ii) *If for every subset A of a space X there exists a countable divisor consisting of closed Lindelöf sets, then $hL(X) \leq 2^\omega$.*

PROOF. (i) Let x be any element in X. Let $\mathcal{D}_x = \{L_i : i \in \omega\}$ be a countable divisor for $X \setminus \{x\}$ consisting of closed Lindelöf subsets of X. By the definition of a divisor we have $X \setminus \{x\} = \cup\{L_i : i \in \omega\}$ so that $X \setminus \{x\}$ is a Lindelöf space. Thus X is also Lindelöf.

(ii) Let A be a subset of X and let $\mathcal{D}_A$ be a countable divisor for A consisting of closed Lindelöf subspaces of X. By the definition of a divisor it follows that A can be written in the form $A = \cup\{\cap\mathcal{C} : \mathcal{C} \subset \mathcal{D}_A$ and $\cup\mathcal{C} \subset A\}$ so that A is the union of $\leq 2^\omega$ many (closed) Lindelöf subsets of X. Therefore, $L(A) \leq 2^\omega$, i.e. $hL(X) \leq 2^\omega$. ■

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