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GENERAL ECART

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0. The notion of distance d(x,y) between points x, y is very old and is connected with measurments. Certainly, the notion is present in works of Thales (Milet $\leftarrow$ 624-$\leftarrow$ 546), one of the seven wise men of the Antic Greece, the first man who predicted the eclipse of the sun (for the year $\leftarrow$ 585); he performed various calculations with distances and angles. A perpetual monument to the old notion of distance is the Pythagora Theorem on triangles with an angle of 90°. The determination of the gravity center of various figures like: triangle, arc, part of a circle, sphere, part of ball,... is a further nice application of the distance, researches tied with the names of Archimedes, Euclid, Jordanus Nemorarius, Fermat, Guldin, Torricelli, Cavalieri, Descartes, Roberval, Pascal, Ceva, Newton, Leibniz, Clairaut.

1. The progress of the mankind is visible spectaculary in the extension of cases and situations in which, given two distinct objects A,B, a distance d(A,B) from A to B is determinable; in this recpect it is very instructive to be aware of some typical situations like distance between the Earth and the Moon, the problem of geodesics on a given surface or in a given manifold (example in the Theory of relativity). During the present 20-th Century the terrestrian human beings extended tremendously their capacity to determine $d(A,B)$ for bodies A, B belonging to the microscopic as well as to the megascopic world very far in the Universe and outside of the Solar and Galactic system.
In Mathematics one considered $d(A,B)$ for points A, B belonging to any Euclidian space R_n or to spherical space, hyperbolic space of n dimensions,... One considered also the case when A, B are given sets in a given space.

2. Maurice Fréchet in 1905 and in particular in his doctoral dissertation 1906, p.30 considered $d(x,y)$, distance between any two given general objects x, y of a given set E requesting that $d(x,y)$ is a real number ≥ 0 such that
M_1 $d(x,y) = 0$ if and only if $x = y$
M_2 $d(x,y) = d(y,x)$
M_3 $d(x,y) \leq d(x,z) + d(z,y)$ for any $(x,y,z) \in E^3$.
Thus one has a mapping
(1) $d : E^2 \to R[0,\cdot)$ where $R[0,\cdot)$ or $R_{\geq}$ denotes the set of nonnegative real numbers.

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2.1. The fundamental notion tied with a distance function (1) on E is the one on the ball: given any $c \in E$ and any real $r > 0$ the ball or spheroid with a center c and radius r is defined by

$$E(c,r) := \{x : x \in E, d(c,x) < r\}. \tag{2}$$

2.2. These notins allow us to associate to any subset S of E the corresponding closure or the adherence clS by
(3) $cl\,S := \{x : x \in E$ and $E(x,r) \cap S \neq v$, whenever $0 < r \in R\}$. R denotes the set of all real numbers; v denotes the vacuous or empty set$\}$.

If so, one speaks of a metric space (E,d), where the function (1) is called the metrics in the space.

2.3. A special feature in the preceding notion is the "triangular relation" M3 occuring in the elementary geometry and in many other cases.

At the same time, Fréchet considered instead of M3 the following, apparently weaker, regularity condition:
F There exists a selfmapping f of $R_{>} := \{x : x \in R, x > 0\}$ into itself such that $lim f(x) = 0$ for $x \to 0$ and that for any triple (a,b,c) of elements of E one has $d(a,b) < x$, $d(b,c) < x \Rightarrow d(a,c) < f(x)$.

Fréchet remarked that spaces (E, M_1, M_2, M_3) and spaces (E, M_1, M_2, F) have similar properties. In 1910 he asked whether this two classes of spaces should be the same. E.W.Chittenden in 1917 confirmed this conjecture; a simple proof was exibited by A.H.Frink in 1937.
Finally, we see that M.Fréchet in his definition of metric (distantiable) spaces starts from any given set E, forms the square $E \times E := E^2$ of all ordered pairs (x,y), where $x, y \in E$ and examines the existence of a mapping (1) satisfying M_1 , M_2 and M_3 or the regularity condition F (of course, M_3 is a pareticular case of F), in order to define the contiguity of $x \in E$ and $S \subset E$ in the sense that $x \in clS$ in the sense of the definition (3).

3. I focused my attention on the range or codomain of the mapping (1) in 2

3.1. The role of the set $R_{\geq}$ in (1) in 2. could be played equivalently by any totally ordered set M having a first element, denoted say by 0 and which is not right isolated, i.e. 0 has at least one successor but not an immediate one. If the set $M \setminus \{0\} := M'$ is cofinal in ω_α^* for a regular initial ordinal ω_α, then instead of (1) in 3.1 one considers mappings

$$d : E^2 \to M \tag{3.1}$$

such that M_1, M_2 and F are satisfied; spheroids are analogously defined with centers $c \in E$ and radii $0 \neq r \in M' := M \setminus \{0\}$; the set E is topologized on considering each spheroid as the neighborhood of each point it contains. The class of spaces obtained in this way for a given ordinal α was denoted as (Δ^α) or (D_α) and was introduced as the class of pseudo-metric spaces in Kurepa 1934(3). The case $\alpha = 0$ coincides with the class of metric spaces (see Kurepa 1936(3), 1937(1)). But it was only in 1963 that I succeeded to prove the existence of a space of the class (D_1) which is not totally orderable (see Kurepa 1963(5)): analogously for D_α for any regular uncountable ω_α.

3.2. Instead to consider for M in (3.1) any ordered chain $(L, \leq)$ of type $1+\omega_\alpha^*$ one could take for M equivalently any topological space having just one accumulation point, say 0, which is the only point of an ω_α-sequence of strictly decreasing sets W_n $(n < \omega_\alpha)$ (see Kurepa 1936(3),1937(1)).

3.3. General approach: General ecart (see Kurepa 1936(4))

3.3.1. Instead to be an ordered chain, codomain or the parametrization set M of the mapping (1) in (2.1) could be any non-empty space or any structure. Given a set E and a space or structure M, one speaks of an M-structure E in the sense that there exists a mapping

$$e : E^2 \to M \tag{1}$$

satisfying the following conditions O^1,O^2, O^3, O^4:
O^1 (Identification axiom) $e(a,b) = e(a,a)$ implies $a = 0$.
The converse is obvious.
O^2 (Symmetry axiom) $e(a,b) = e(b,a)$.
O^3 (Transfer axiom) Let $a \in E, F \subset E$ and $(a,F) := \{(a,f) : f \in F\}$.If $e(a,a) \in Cl(a,F)$ in M, then $a \in ClF$ in E, and vice versa.
If one has O^1,O^2, O^3 one says that e is an **M-ecart in** E and that E is an **ecart space** with ecart in M or that E is an $\mathcal{E}(M)$ space; one writes $E \in \mathcal{E}(M)$; $\mathcal{E}$ the script letter E is the initial letter of the word ecart.

3.3.2. Balls or spheroids with the set radius. Ecart spaces

Already in Kurepa 1936(4) p.1051_{10-9}, 1937(1) p.60^{13} occured a definition of a spheroid or ball in E for a given point $c \in E$ as "center" and a given nonempty set r as "radius":

$$S(c,r) := E(c,r) := \{x : x \in E \text{ and } e(c,x) \in r\}.$$

This definition is obtained from the elementary definition of a ball $S(c,r)$ substituting the order relation $<$ and number $r > 0$ by the membership relation $\in$ and appropriate non-empty set r, respectively. It is unbeilivable how these naive substitutions:

$$< | \in \text{ and } r > 0 \mid r \text{ is some set} \neq v$$

are far reaching.

3.3.3. Example. Let E be a topological neighbourhood space and $E^2 := E \times E$ its square; then the second projection $p_2(x,y) := y$, $(x,y) \in E^2$ is an E-ecart in E. For each $c \in E$ and each neighbourhood $V(c)$ of c in E as "set-radius" one has the corresponding ball $E(c, V(c))$ and one sees that $E(c, V(c)) = V(c)$; thus, ball coincides with its radius; each open set G in E is at the same time a spheroid $E(g,G)$ whenever $g \in G$.

3.3.4. Appearence of O. Case of ecart space in which the ecart is constant on the diagonal. The ecarts (distances) of metric and pseudometric spaces E

have the property that $d(a,a)$ has the same value for all $a \in E$: the function d in (3.1) is a determined constant, say $O \in M$, on the diagonal $\Delta E^2 := \{(x,x) : x \in E\}$; thus $d(E^2) = \{O\}$ for some $O \in M$.
Assume O to be an accumulation point in M: if M is a topological space , let V_0 be a neighbourhood base of O in M; then like in metric and pseudometric spaces one formulates a regularity axiom in this new situation: members $r \in V_0$ serve as radii of balls $E(c,r)$ for each $c \in E$ (just to avoid the complication that radii r are defined on c). We formulate the following:

3.3.5. O^4 (Regularity of ecart). There is $O \in M$ such that $d(x,x) = O$, $(x \in E)$. If V_0 is a neighbourhood base of O in M, then there are selfmappings g, h of V_0 such that $b \in E(a, gr)$ implies $E(b, hr) \subset E(a,r)$ for each $r \in V_0$. Equivalently, there is a selfmapping f of V_0 such that $b \in E(a, fr)$ and $c \in E(b, fr)$ implies $c \in E(a,r)$, $(r \in V_0)$. (cf. Kurepa 1036(4) p.1051_{15-9}, 1937(1), 4', 1956(1)n.7.2; 1963(7) n.5.3).

3.3.6. Separation requirement. For simplicity, we assume that O is a unique element of M belonging to all members of V_0, i.e. $\{O\} = \bigcap V_0$.
In order to satisfy this requirement it is sufficient that the space M satisfies the Fréchet axiom of separation T_1 for the element O: there is no $b \in M$ such that $O \in Der\{b\}$.

3.3.7. Definition of general distance (metric). If the ordered pair (E, M) satisfies O^1, O^2, O^3, O^4 one says that $d : E^2 \to M$ is an M-metric or M-distance. E is quoted as an M-metric space with O as the zero value of the distance d. One could speak of the structure (E, M, d, O). Usually, one assumes that $\{O\} = \bigcap V_0$.

3.3.8. A natural step. Let us remark that the above apparatus (E, M, d, O) was contained in my paper 1936(4), where, following Fréchet, the ecart from a to b was denoted by (a, b).
What was more natural then, than athomatically to use the identity mapping of E^2 to get the whole M as E^2 followed by the identification of (x,y) and (y,x) in order to satisfy O^2 and the identification of (x,x), (y,y), ... with O for all $x \in E$ in order to get a unique zero O with all representations (a,a) $(a \in E)$ of O; they form the diagonal ΔE^2 of the square E^2. Then each $V \in V_O$ contains each (x,x) $(x \in E)$ and E^2 as its element and as its subset, respectively. V_O is any neighborhood base of O in M such that O is the unique non-isolated point in M.

3.3.9. Appearance of uniform spaces. This trivial performance of machinery from 1936(4) yields exactly the spaces called uniform spaces in A.Weil 1938; varying V_0 on M_E one gets all uniform spaces on E.
Each V_0 is a uniformity base over E; and vice versa: each neighbourhood base of O in M_E generates a corresponding uniform space over E. For each $c \in E$ and $r \in V_0$, the corresponding spheroid (ball) $E(c,r)$ is precisely the neighborhood of c in the uniform space E; $r(c)$ is called also the section (fr. coule) of r relative to the point $c \in E$. For every $r \in V_0$ in M_E the corresponding preimage

$$d^{-1}r := \{(x,y) \mid (x,y) \in E^2, d(x,y) = r\}$$

is an entourage of sE^2 in E^2; on varying $r \in V_0$ one gets the system $d^{-1}V_0$ which is an uniformity base in E.

Reciprocally, for each uniformity base U in E the d-image $dU := \{dX : X \in U\}$ is a neighborhood base V_0 for O in the space M_E.

3.3.10. Where, in this beautiful panorama are metric spaces? The answer is the following

THEOREM. *A uniform space is metric if and only if it is definable by a countable uniformity base (Kurepa 1936(4) p. 1051 Th. V, 1937(1) p. 59 Th. D_0, and A. Weil 1938 p. 16^{4-7}).*

4. Scope of uniform spaces

4.1. It should be indicated that the class of uniform spaces coincides with each of the following

(T) The class of Tychonov spaces (= topological completely regular spaces or T-space) introduced by Tychonov in 1930.

(Ef) The class of Efremovič's proximity spaces (1951) is just the class of uniform spaces.

(ABS) Class of spaces defined by distances d, the values of which are in topological semifields (M. J. Antonovski, V. G. Boltjanski, T. A. Sarymsakov 1960).

4.2. As very important classes of uniform spaces one has: compact spaces, topological groups (theorems due to A . Weil 1938).

4.3. It is interesting to mention the following result. Each well-ordered space W is a uniform space (because W is Tychonov; and Tychonov spaces are characterized by the property to admit a compactification; and obviously, each W is compactiable).

Since each well-ordered space W is homeomorphic to the space of all ordinals $<$ type W, one could consider instead of W the segment $I(z)$ of all ordinals $< z :=$ ord. type of W. It is easy to see that Iz is an ecart space with ecarts in Iz (cf. Kurepa 1936(4)) it suffices to put $e(a,b) = \inf\{a,b\} + 1$ for $a \neq b$ and

$e(a,a) = 0$ if a is isolated.

$e(a,a) = a$ for each $\lim(a)$ (without restriction, one can assume $\lim(z)$).

But here the relation $e(a,a) = a$ is satisfied pz times for each regular infinite ordinal z.

Therefore it is an important result that $I(z)$ is uniformizable by a distance allowing a unique zero!

It is a particular job to perform a distance, e. g. if z is regular ω_a with $a > 0$.

4.4. A crucial point in all this matter is the set $M_p := \{e(x,x) : x \in E\}$.

5. A historical fact

The bibliography concerning uniform spaces is very great (cf. Bushaw). My contribution was appreciated in particular by Collatz (1968) and Nagata (1984). My work 1963(4) is very important. It was the date 1936.10.10.6. that from Glina, Yugoslavia, I sent to M. Fréchet (Paris) my manuscript of 1936(4) jointly with a long letter, in which I proposed that the Note should be presented in The Academy of Sciences, Paris, the overnext Monday 1936.10.19.1. I got the answer already

1936.10.17.6 reproaching me my "ultimatum"; the same day I sent my answer (2 pages) in which one reads (in my concept-letter):

"En ce moment je suis train de terminer un Mémoire pour l'Académie de Zagreb portant sur la structure des relations d'ordre; il comprendra a peu preš deux feuilles dont une demi-fuelle sera écrite en français; après, je devrais terminer l'expose sur dimension, mais paralèllement, j'éorizai un Mémoire sur l'uniformité dans la théorie des espaces abstraits se rettachant aux conditions φ_0, Φ_0 que je vous ai indiquées dernière fois et qui se rattachent à vos conditions 5^0 et 5^0bis (v. Espaces abstraits, p. 184). A ce propos j'exprime mon regret que vous n'ayiez pas de temps libre pour voir si la condition Φ_0 est aussi suffisante (et non seulement nécessaire) pour la distanciabilité d'un $(\mathcal{E})$".

The quoted day 1936.10.19.1 was just the day of 11 months after the defense in Paris of my doctoral dissertation.

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