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REPRESENTATION OF EQUIVALENCE TRANSFORMATIONS OF HILBERT SPACE VALUED WIENER PROCESS

LJILJANA PETRUŠEVSKI

ABSTRACT. *This paper deals with the Hilbert space valued processes equivalent to a Wiener process in the sense of being equivalence transformations of the Wiener process. It is being proved that each equivalence transformation of the Wiener process can be presented in the following form* $X(t) = W(t) - \int_0^t Y(s)\,ds$ *and separately* $X(t) = W(t) - \int_0^t \int_0^T l(s,x)\,dW(x)\,ds$ *The obtained results are different forms of representation and they make a very significant step in solving the problem of canonical representation (Hida-Cramer) of Hilbert space valued processes equivalent to the Wiener process without the use of the Factoriza- tion theorem.*

1. Introduction. This paper represents a continuation of the study of Hilbert space valued processes equivalent to the Wiener process initiated in the previous papers [3], [4], [5]. In contrast to previous papers, which deals with sufficient conditions of equivalence, this paper proves that some of these conditions are necessary (Theorems 1 and 4). The obtained results represent different forms of representa- tions and they make a significant step in solving the problem of a canonical representation (without using the Factorization theorem) of Hilbert space valued (infinite-dimensional) processes equivalent to the Wiener process. Its final version will represent the generalization of the well known facts for Gaussian real [1] and finite-dimensional processes [2]. The equivalence is defined as in [5] over equivalence operators. The process is considered as the equivalence transforma- tion of the process equivalent to it. The significance of the results is in their simplicity in comparison with the well known facts for Gaussian finite-dimensional processes where the equiva- lence is regarded as the equivalence of Gaussian measures in the corresponding probability spaces.

2. On some representation of equivalence transformations of the Wiener process. This section comprises some definitions and well known facts necessary for further deliberations. The obtained result is contained in the Theorem 1 and it is basically a form of representing equivalence transformations of the Wiener process.

Definition 1. An operator from Hilbert space H_1 to Hilbert space H_2 will be called the equivalence operator if

(1) A is one-to-one onto, and has a bounded inverse A^{-1}

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(2) $I - \sqrt{A * A}$ is Hilbert-Shmidt operator.

Lemma 1. *[5] The operator A from Hilbert space H_1 to Hilbert space H_2 is an equivalence operator if and only if*

(1) A is one-to-one onto, and has a bounded inverse A^{-1}

(2) $I - A * A$ is Hilbert-Shmidt operator.

Lemma 2. *The operator $A : H_1 \rightarrow H_2$ is the equivalence operator if and only if $A^{-1} : H_2 \rightarrow H_1$ is the equivalence operator.*

Proof. The statement is a direct result of the Lemma 1 and the equation $I-(A^{-1})* A^{-1} = -(A^{-1}) * (I - A * A)A^{-1}$.

Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space and H an arbitrary real separable Hilbert space with a scalar product $(\cdot, \cdot)$ and a norm $\|\cdot\|$.

Definition 2. The mapping $z : \Omega \rightarrow H$ is a Hilbert space valued random variable if and only if (z, u) is a real random variable for each $u \in H$.

Definition 3. The Hilbert space valued random process $Z = Z(t), 0 < t < T$ is the family of Hilbert space valued random variables $Z(t), 0 < t < T$.

Let $\mathcal{L}_t(Z)$ be a linear vector space generated by random variables $(u, Z(s)), u \in H, s \leq t$, $H_t(Z) = \bar{\mathcal{L}}_t(Z)$ its closure in quadratic mean and $H(z) = \cup_{0<t<T} H_t(Z)$. Spaces $H_t(Z)$ and $H(Z)$ are observed as Hilbert space subspaces of all real random variables $x, Ex^2 < \infty$ with the scalar product $\langle x_1, x_2 \rangle$ and the norm $\|x\| = Ex^2$.

Definition 4. A Hilbert space valued process $Y(t)$ will be considered as equivalent to $X(t)$ process on finite or infinite interval $(0, T)$ if mapping

$$B : (X(t), u) \rightarrow (Y(t), u), \quad u \in H, \quad 0 < t < T$$

can be extended to an equivalence operator from Hilbert space $H(X)$ to Hilbert space $H(Y)$. The random process $Y(t)$, ($(Y(t), u) = B(X(t), u), \quad u \in H, \quad 0 < t < T$) is also said to be an equivalence transformation of Hilbert space valued process $X(t)$.

If Hilbert space valued process $Y(t)$ is equivalent to process $X(t)$, then , according to the Lemma 2, the process $X(t)$ is also equivalent to process $Y(t)$, so these processes are simply said to be equivalent. Each of these two mutually equivalent processes can be observed as the equivalence transformation of the other process.

Theorem 1. *For each process $X(t)$ equivalent to a Wiener process there is a Wiener process $W(t)$ and a process $Y(t)$, $\int_0^T E\|Y(t)\|^2 < \infty$ such that*

$$X(t) = W(t) - \int_0^t Y(s)\, ds.$$

Proof. On the basis of the equivalence definition for the random process $X(t)$ and the Wiener process $W(t)$ the mapping

$$B : (W(t), u) \rightarrow (X(t), u), \quad u \in H, \quad 0 < t < T \tag{1}$$

can be extended to a linear bounded operator for which $I - \sqrt{B * B}$ is a Hilbert-Shmidt operator. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal base of the real separable Hilbert space H. It is well known that the Hilbert-Shmidt operator can be presented in the form [6]

$$I - \sqrt{B * B} = \int_{\Omega} \eta(\omega, \omega') \, P(d\omega) \tag{2}$$

where

$$\eta(\omega, \omega') = \sum_{j,k=1}^{\infty} \lambda_{j,k} e_j e_k \tag{3}$$

and

$$EE' |\eta(\omega, \omega')|^2 = \int_{\Omega} \int_{\Omega'} |\eta(\omega, \omega')|^2 \, P(d\omega) \, P(d\omega') < \infty \tag{4}$$

From these equations ((3) and (4)) it follows that it is

$$\eta(\omega, \omega') \in H(W) \tag{5}$$

with probability one and particularly for almost each $\omega \in \Omega$

$$\eta(\omega, \omega') = \sum_{j=1}^{\infty} \int_0^T c_j(\omega, s) \, d(W(x, \omega'), e_j). \tag{6}$$

Let it be noted that correlation function of the Wiener process is $B_W(s,t) = \min\{s,t\} \cdot I$ which results in the equation

$$\langle (W(s), e_j), (W(t), e_k \rangle = \min\{s,t\}(e_j, e_k), \quad j,k = 1,2,\ldots, \quad 0 < s,t < T. \tag{7}$$

The processes $(W(t), e_k), k = 1, 2, \ldots$ are mutually orthogonal real Wiener processes.

The equation (2) proves that

$$(I - \sqrt{B * B})(W(t), e_k) = \int_{\Omega} \eta(\omega, \omega')(W(t, \omega'), e_k) \, P(d\omega') \tag{8}$$

which on the basis of the relation (5) can be written in the following form:

$$(I - \sqrt{B * B})(W(t), e_k) = \langle \eta(\omega, \omega'), (W(t, \omega'), e_k) \rangle \tag{9}$$

and on the basis of the equations (6) i (7) in the form

$$(I - \sqrt{B * B})(W(t), e_k) = \int_0^t c_k(\omega, s) \, ds \tag{10}$$

Further on, from this equation it simply follows that

$$\sqrt{B*B}(W(t),e_k) = (W(t),e_k) - \int_0^t c_k(s)\,ds. \tag{11}$$

Let V be an isometric operator so that $V\sqrt{B*B} = B$. Let also $Vc_k(s) = y_k(s)$ and real mutually orthogonal Wiener processes $W_k(t) = V(W(t),e_k)$, $k = 1,2,\ldots$. If operator V is applied to the left-hand and right-hand side of the equation (11) it follows that

$$(X(t),e_k) = W_k(t) - \int_0^t y_k(s)\,ds, \quad k = 1,2,\ldots \tag{12}$$

and with respect to that it also follows that

$$(X(t),e_k)e_k = W_k(t)e_k - \int_0^t y_k(s)e_k\,ds, \quad k = 1,2,\ldots \tag{13}$$

Therefore it is simple to establish the fact that

$$\sum_{k=1}^{\infty}\int_0^t y_k(s)e_k\,ds = \int_0^t Y(s)\,ds \tag{14}$$

where

$$Y(s) = \sum_{k=1}^{\infty} y_k(s)e_k = \sum_{k=1}^{\infty} Vc_k(s)e_k\,ds$$

is the Hilbert space valued process for which

$$\int_0^T E\|Y(s)\|^2\,ds = EE'|\eta(\omega,\omega')|^2 = \int_\Omega\int_{\Omega'}|\eta(\omega,\omega')|^2\,P(d\omega)\,P(d\omega') \tag{15}$$

The random process

$$\sum_{k=1}^{\infty} W_k(t)e_k$$

is a Hilbert valued Wiener process which, for simplicity, we will denote again by $W(t)$:

$$W(t) = \sum_{k=1}^{\infty} W_k(t)e_k. \tag{16}$$

Considering the equations (14) and (16), by summing up over k, it finally follows from equation (13) that

$$X(t) = W(t) - \int_0^t Y(s)\,ds \tag{17}$$

for process $Y(t)$ which is integrable on each finite interval, and for which, considering (4) and (15) it follows that

$$\int_0^T E\|Y(s)\|^2\,ds < \infty. \tag{18}$$

This completes the proof. □

3. Analysis of the result. If we examine carefully the proof of the Theorem 1 we will notice that only the condition that $I - \sqrt{B * B}$ is a Hilbert-Shmidt operator, (in which case B is bounded) was taken into consideration, while the condition that B has a bounded inverse was not taken into consideration. The statement holds:

Theorem 2. *The operator $I - \sqrt{B * B}$ is a Hilbert-Shmidt operator if and only if there exist a Wiener process $W(t)$ and Hilbert space valued process $Y(t)$ for which $\int_0^T E\|Y(s)\|^2\,ds < \infty$ such that*

$$X(t) = W(t) - \int_0^t Y(s)\,ds.$$

Proof. If $I - \sqrt{B * B}$ is a Hilbert-Shmidt operator, then by applying the procedure realized in the proof of the Theorem 1, we come to the conclusion that there exist a Wiener process $W(t)$ and a process $Y(t)$ integrable on each finite interval

$$\int_0^T E\|Y(s)\|^2\,ds < \infty \tag{19}$$

such that

$$X(t) = W(t) - \int_0^t Y(s)\,ds \tag{20}$$

Conversely, for the process $X(t)$, given by equation (20), under the condition (19), on the basis of the considerations in [5], $I - B * B$ is a Hilbert-Shmidt operator, which according to the Lemma 1 means that $I - \sqrt{B * B}$ is a Hilbert-Shmidt operator as well. □

4. Conclusion. Let $l(s,x)$, $s, x \in (0,T)$ be a Hilbert-Shmidt operator in H with Hilbert-Shmidt norm $|l(s,x)|$for which $\int_0^T \int_0^T |l(s,x)|\,dx\,ds < \infty$ and $l(s,x) = 0$ for $x > s$.

Theorem 3. *[5] The random process*

$$x(t) = W(t) - \int_0^t \int_0^s l(s,x)\,dW(x)\,ds \tag{21}$$

is an equivalent transformation of a Wiener process.

It is obvious that this statement refers exactly to the process of the same form as in (20) where

$$Y(s) = \int_0^s l(s,x)\,dW(x). \tag{22}$$

With respect to the well known statements for real and finite- dimensional processes [1], [2], it would be realistic to expect that it is possible to prove the statement contrary to the one in the Theorem 3. Basically it means that in the proof of the Theorem 1 one should use the condition that has an inverse bounded operator and consequently that $Y(s) \in H_s(W)$, i.e. that $Y(t)$ has the form (22). In this way it would be prove that each of the equivalence transformations of the Hilbert space valued Wiener processes can be presented in the form (21) and this form would represent its canonical representation (Hida-Cramer) obtained without the Factorization Theorem.

From the equation (10) it follows directly that $c_k(s) \in H(W)$ for almost each $s \in (0,T)$. Further deliberation of that fact shows that $Y(s) = \int_0^T l(s,x)\,dW(x)$ which we will (without detailed of proof) formulate as the following statement.

Theorem 4. *For each process $X(t)$ equivalent to a Wiener process there exists a Wiener process $W(t)$ such that*

$$X(t) = W(t) - \int_0^t \int_0^T l(s,x)\,dW(x)\,ds.$$

Finally, let it be noted that in the proof of this statement the condition that B has an inverse bounded operator is not used, therefore, it is obvious that the expected statement is not obtained.

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Faculty of Architecture,
Belgrade, Yugoslavia