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DIFFUSION PROCESSES AND DETERMINISTIC EVOLUTIONS

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ABSTRACT. *It is proved that Ito stochastic integral $\int X_s\,dB_s$ is zero with probability one on the set on which $\int X_s^2\,ds$ is zero. This result is used to show that, for Lipschitz continuous coefficients a and b, diffusive process $\{X_t\}$, defined as $dX_t = a(X_t)\,dB_t + b(X_t)\,dt$, and deterministic process $\{Y_t\}$, defined as $dY_t = b(Y_t)\,dt$, with the same initial values are indistinguishable up to the first exit from the domain where diffusive coefficient a is zero.*

Let $\{B_t, t \geq 0\}$ be a standard Braunian motion on the canonical flow of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ living in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{X_t, t \geq 0\}$ be adapted random process. As well known [1], [2], if process $\{X_t, t \geq 0\}$ is locally integrable relative to Doleane measure generated by Braunian motion, i.e. if for each $t > 0$

$$\int_0^t \mathbb{E}\{{X_s}^2\}\,ds < \infty,$$

then stochastic integral could be defined

$$\int_0^t X_s\,dB_s.$$

Properties of stochastic integral are globally determined by the Brownian motion, and the process being integrated. So local properties of underlying processes need not imply corresponding local properties of stochastic integral. Theorems that treat local properties of stochastic integral in general require very strong assumptions [3].

We shall demonstrate a Lemma of that kind, applicable on the widest class of Ito integrable processes.

Lemma. *Let $\{X_t, t \geq 0\}$ be locally integrable process relative to Doleane measure generated by Braunian motion. If on the set $A \in \mathcal{F}$, with probability one, $\int_0^t {X_s}^2\,ds = 0$, then, again with probability one on the set A we have $\int_0^t X_s\,dB_s = 0$.*

Proof. Define optional time τ as

$$\tau = \inf\{T \geq 0 : \int_0^T {X_s}^2\,ds > 0\}.$$

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For almost all $\omega \in A$ we have $t \leq \tau(\omega)$, hence $\mathbb{I}_A \leq \mathbb{I}\{t \leq \tau\}$. On the other side

$$\Big(\int_0^t X_s\, dB_s\Big)^2 \mathbb{I}\{t \leq \tau\} \leq \Big(\int_0^{t\wedge\tau} X_s\, dB_s\Big)^2,$$

where $t \wedge \tau = \min(t, \tau)$. As $t \wedge \tau$ is also optional time we can conclude that

$$\mathbb{E}\Big\{\Big(\int_0^t X_s\, dB_s\Big)^2 \mathbb{I}_A\Big\} \leq \mathbb{E}\Big\{\Big(\int_0^t X_s\, dB_s\Big)^2 \mathbb{I}\{t \leq \tau\}\Big\} \leq$$
$$\leq \mathbb{E}\Big\{\Big(\int_0^{t\wedge\tau} X_s\, dB_s\Big)^2\Big\} = \mathbb{E}\Big\{\int_0^{t\wedge\tau} {X_s}^2\, ds\Big\} = 0,$$

and the Lemma is proved. □

We shall use this approach to compare the solution of stochastic differential equation (SDE) to the corresponding Coushy deterministic equation.

Precisely, let $\{X_{t,x}, t \geq 0\}, x \in \mathbb{R}$, be a solution od SDE

$$X_{t,x} - x = \int_0^t a(X_{s,x})dB_s + \int_0^t b(X_{s,x})ds \tag{1}$$

with Lipschitz continuous coefficients a and b, and $\{Y_{t,x}, t \geq 0\}, x \in \mathbb{R}$ deterministic evolution defined through Coushy equation :

$$Y_{t,x} - x = \int_0^t b(Y_{s,x})\, ds. \tag{2}$$

Processes $\{X_{t,x}, t \geq 0\}$ and $\{Y_{t,x}, t \geq 0\}$ start from the same initial value so one can ask until what point random process X and deterministic process Y coincide. Also, it is interesting whether the first moment when differences occur, is deterministic or random. The answer to those questions is given by the next theorem.

Theorem. *For Lipschitz continuous coefficient a and b, diffusive process $\{X_{t,x}, t \geq 0\}$, $x \in \mathbb{R}$, is defined by stochastic differential equation (1), and deterministic process $\{Y_{t,x}, t \geq 0\}$ is defined by (2). Define*

$$\tau_x = \inf\{t \geq 0 : a(X_{t,x}) \neq 0\} \text{ and } D_x = \inf\{t \geq 0 : a(Y_{t,x}) \neq 0\}.$$

Then, with probability one, $\tau_x = D_x$ and $X_{t,x} = Y_{t,x}$ for $t \leq D_x$.

Proof. At the beginning, let us assume that $0 < D_x < \infty$, and $0 < t < D_x$. Then $a(Y_{s,x}) = 0$, hence we have

$$X_{t,x} - Y_{t,x} = \int_0^t \big(a(X_{s,x}) - a(Y_{s,x})\big)\, dB_s + \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)\, ds, \quad \text{and}$$

$$\mathbb{E}\{(X_{t,x}-Y_{t,x})^2\}^{1/2} \leq$$
$$\leq \mathbb{E}\Big\{\int_0^t \big(a(X_{s,x})-a(Y_{s,x})\big)\,dB_s{}^2\Big\}^{1/2} + \mathbb{E}\Big\{\int_0^t \big(b(X_{s,x})-b(Y_{s,x})\big)\,ds^2\Big\}^{1/2}$$
$$\leq \mathbb{E}\Big\{\int_0^t \big(a(X_{s,x})-a(Y_{s,x})\big)^2 ds\Big\}^{1/2} + t^{1/2}\mathbb{E}\Big\{\int_0^t \big(b(X_{s,x})-b(Y_{s,x})\big)^2 ds\Big\}^{1/2}$$
$$\leq K\Big(\int_0^t \mathbb{E}\big\{(X_{s,x}-Y_{s,x})^2\big\}ds\Big)^{1/2} + t^{1/2}K\Big(\int_0^t \mathbb{E}\big\{(X_{s,x}-Y_{s,x})^2\big\}ds\Big)^{1/2}$$
$$\leq K(1+D_x^{1/2})\Big(\int_0^t \mathbb{E}\big\{(X_{s,x}-Y_{s,x})^2\big\}ds\Big)^{1/2},$$

where K is Lipschitz constant for functions a and b. Using Gronwall inequality, [2], one can conclude that, almost sure, $X \equiv Y$ on $[0, D_x]$, and consequently with probability one

$$D_x \leq \tau_x \tag{3}$$

On the other side, assume now that $0 < \tau_x < \infty$, and $0 < t < \tau_x$. As τ_x is optional time, the set $A = \{\omega : t < \tau_x(\omega)\}$ is $\mathcal{F}_t$ measurable. From the assumptions of theorem it follows that $\mathbb{E}\{\int_0^t a(X_{s,x})^2 ds \mathbb{I}_A\} = 0$.

We shall demonstrate the same procedure as in **Lemma**. In that way

$$\Big(\int_0^t a(X_{s,x})\,dB_s\Big)^2 \mathbb{I}\{t < \tau_x\} \leq \Big(\int_0^{t\wedge\tau_x} a(X_{s,x})\,dB_s\Big)^2.$$

Having in mind that $t \wedge \tau_x$ is also optional time, one can conclude that

$$\mathbb{E}\Big\{\Big(\int_0^t a(X_{s,x})\,dB_s\Big)^2 \mathbb{I}\{t < \tau_x\}\Big\} \leq \mathbb{E}\Big\{\Big(\int_0^{t\wedge\tau_x} a(X_{s,x})\,dB_s\Big)^2\Big\} =$$
$$= \mathbb{E}\Big\{\int_0^{t\wedge\tau_x} a(X_{s,x})^2\,dB_s\Big\} = 0,$$

hence $\int_0^t a(X_{s,x})dB_s = 0$ almost sure on the set A. Then for each $0 < t < \tau_x(\omega)$ we have

$$X_{t,x} - Y_{t,x} = \int_0^t \big(b(X_{s,x}) - b(Y_{s,x})\big)ds, \qquad \text{and}$$
$$\mathbb{E}\{(X_{t,x}-Y_{t,x})^2\mathbb{I}\{t \leq \tau_x\}\} = \mathbb{E}\Big\{\int_0^t \big(b(X_{s,x})-b(Y_{s,x})\big)ds^2\mathbb{I}\{t \leq \tau_x\}\Big\} \leq$$
$$\leq tK^2\mathbb{E}\Big\{\int_0^t \big(b(X_{s,x})-b(Y_{s,x})\big)^2 ds\mathbb{I}\{t \leq \tau_x\}\Big\}$$
$$\leq tK^2\int_0^t \mathbb{E}\big\{(X_{s,x}-Y_{s,x})^2\mathbb{I}\{s \leq \tau_x\}\big\}ds.$$

Let us mark $\Delta_t = \mathbb{E}\{(X_{t,x} - Y_{t,x})^2 \mathbb{I}\{t \leq \tau_x\}\}$. Then previous relation could be written as

$$\Delta_t \leq tK^2 \int_0^t \Delta_s \, ds, \quad \text{i.e.}$$
$$\left(e^{-K^2t^2/2} \int_0^t \Delta_s \, ds\right)' \leq 0$$

from where it follows that $\Delta_t = 0$ for each $t > 0$, hence we can conclude that

$$\mathbb{P}\{\omega : t < \tau_x(\omega), X_{t,x} \neq Y_{t,x}\} = 0.$$

We can extrapolate the previous relation on the set of positive rationals ($\mathbb{Q}_+$),

$$\mathbb{P}\{\omega : (\exists t \in \mathbb{Q}_+)(t < \tau_x(\omega), X_{t,x} \neq Y_{t,x})\} = 0.$$

Having in mind that processes X and Y are almost sure continuous we conclude that $\mathbb{P}\{\omega : (\exists t > 0)(t < \tau_x(\omega), X_{t,x} \neq Y_{t,x})\} = 0$. Then, almost sure $X \equiv Y$ na $[0, \tau_x]$, i.e. with probability one,

$$\tau_x \leq D_x. \tag{4}$$

Relations (3) and (4) imply the theorem. □

Similar results are not totally unknown. Up to date results are proved using stochastic calculus in sense of Malliavin, [3], [4], [5]. But strict Malliavin calculus approach requires the coefficients a and b to be continuously differentiable, Also neither the first moment from which stochastic process becomes really stochastic nor the type of the distribution up to that moment were not defined. The main benefit of our approach is that assuming only conditions needed for existence and uniqueness of solution of SDE we are able to relate solutions of SDE and Couchy determinisic processes.

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