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PREDICTABILITY AND STOCHASTIC REALIZATION PROBLEM

LJILJANA PETROVIĆ

ABSTRACT. *A family $H = (H_t), t \in R$ can be considered as a flow of information representing outputs of a stochastic dynamic system S. We consider the next problem (which follows directly from the realization problem) : how to find realizations $G = (G_t), t \in R$ of a stochastic dynamic system such that families H and G are equally predictable by each other.*

1. Preliminaries

Let $H = (H_t), t \in R$ be a family of Hilbert spaces. The space H_t will be thought of as information available at time t. Total information $H_{<\infty}$ carried by H will be defined by $H_{<\infty} = \bigvee_{t\in R} H_t$, while past and future information of H at t will be defined as $H_{\leq t} =_s \bigvee_{s\leq t} H_s$ and $H_{\geq t} = \bigvee_{s\geq t} H_s$, respectively. It should be clear that $H_{<t} = \bigvee_{s<t} H_s$ and $H_{>t} = \bigvee_{s>t} H_s$ does not have to be equal to $H_{\leq t}$ and $H_{\geq t}$ respectively; $H_{<t}$ and $H_{>t}$ are sometimes referred to as real past and real future of H at t.

If H_1 and H_2 are arbitrary subspaces of a Hilbert space $\mathcal{H}$, then $P(H_1|H_2)$ will denote the orthogonal projection of H_1 onto H_2 and $H_1 \ominus H_2$ will denote a Hilbert space generated by all elements $x - P(x|H_2)$ where $x \in H_1$ (compare with [3]). If $H_1 \subseteq H_2$ then $H_1 \ominus H_2$ coincides with $H_1 \cap H_2^\perp$, where $H_2^\perp$ is the orthogonal complement of H_2 in $\mathcal{H}$.

DEFINITION 1. It is said that H^1 is *submitted* to H^2 (and written as $H^1 \subseteq H^2$) if and only if $H^1_{\leq t} \subseteq H^2_{\leq t}$ for each t.

It will be said that families H^1 and H^2 are *equivalent* (and written as $H^1 = H^2$) if and only if $H^1 \subseteq H^2$ and $H^2 \subseteq H^1$.

DEFINITION 2. It is said that H^1 is *strictly submitted* to H^2 (and written as $H^1 \leq H^2$) if and only if $H^1_t \subseteq H^2_t$ for each t.

It is easy to see that strict submission implies submission and that the converse does not hold.

DEFINITION 3. It is said that families H^1 and H^2 are *equally predictable by each other* if $P(H^1_{\geq t}|H^2_{\leq t}) = P(H^2_{\geq t}|H^1_{\leq t})$ for each t.

The notion of minimality and maximality of families of Hilbert spaces are specified in the following definition.

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Definition 4. It will be said that H is a *minimal* (respectively, *strictly minimal*) family having a certain property if there is no family H^* having the same property which is submitted (respectively, strictly submitted) to H.

It will be said that H is a *maximal* (respectively, *strictly maximal*) family having a certain property if there is no family H^* having the same property such that family H is submitted (respectively, strictly submitted) to H.

It should be understood that a minimal (respectively, strictly minimal) and maximal (respectively, strictly maximal) family having a certain property are not necessarily unique.

Definition 5. (compare with [3] and conditional indenpendence from [2]) If H, H_1 and H_2 are arbitrary Hilbert spaces, then it is said that H *is splitting for* H_1 *and* H_2 (and written as $H_1 \perp H_2 \mid H$) if $H_1 \ominus H \perp H_2 \ominus H$. In the present framework, the following definition of Markovian property will be used.

Definition 6. (compare with [3]) Family H will be called *Markovian* if $P(H_{\geq t}|H_{\leq t}) = H_t$ for each t.

Definition 7. (compare with [2]) A *stohastic dynamic system* (s.d.s.) is a set of two families H (outputs) and G (states), such that satisfy the condition

$$H_{<t}VG_{<t} \perp H_{>t}VG_{>t}|G_t \tag{1}$$

For given family of outputs H, any family G satisfying (1) is called a *realization* of a s.d.s. with those outputs.(for detailed definition of a stochastic dynamic system and the realization problem see [2])

It is clear that realization of a s.d.s. is a Markovian family.

2. Main results

Let $H = (H_t), t \in R$ be a given family of Hilbert spaces which represents output of a stochastic dynamic system S. We consider the next problem: how to find the minimal (respectively, maximal) realization G (understood as a family of Hilbert spaces) that contains (respectively, is contained in) given outputs of a system S, and is such that each of these two families are equally predictable by each other. This problem is, as we shall see, closely related to the problem of determining a Hilbert spaces splitting for two Hilbert spaces.

The solution of the problem of finding minimal realization G of a system S, such hat $H_t \subseteq G_t$ and H and G are equally predictable by each other, is given by the following result.

Theorem 1. Let family $H = (H_t), t \in R$ be an output of a stochastic dynamic system. Family $G = (G_t), t \in R$ is the strictly minimal realization of this system satisfying conditions $H_t \subseteq G_t \subseteq H_{\leq t}$ and H and G are equally predictable by each other if and only if is defined by

$$G_t = P(H_{\geq t}|H_{\leq t}), t \in R.$$

Proof. It is easy to check that G is a Markovian family, which, because $H_t \subseteq G_t$, implies $H_{<t}VG_{<t} \perp H_{>t}VG_{>t}|G_t$, i.e. G is a realization of a s.d.s. with outputs H.

From $H_s \subseteq G_s \subseteq H_{\leq s} \subseteq H_{\leq t}$ for $s \leq t$, it follows $G_{\leq t} = H_{\leq t}$, which gives $P(H_{\geq t}|H_{\leq t}) = P(H_{\geq t}|G_{\leq t})$ and, because G is Markovian, $P(G_{\geq t}|H_{\leq t}) = P(G_{\geq t}|G_{\leq t}) = G_t = P(H_{\geq t}|H_{\leq t})$, which proves that H and G are equally predictable by each other. If G^* is some other realization of a s.d.s. with outputs H such that $H_t \subseteq G_t^* \subseteq H_{\leq t}$ and H and G^* are equally predictable by each other, then it follows $G_t^* = P(G_{\geq t}^*|G_{\leq t}^*) \supseteq P(H_{\geq t}|G_{\leq t}^*) = P(H_{\geq t}|H_{\leq t}) = G_t$ which proves the minimality of G.

The other half of the proof is obvious.

Next result gives realization of a system S such that $H_t \subseteq G_t$ and H and G are equally predictable by each other.

THEOREM 2. Let A be a Hilbert space from $H_{<\infty}$ such that $A \perp P(H_{\geq t}|H_{\leq t} \ominus H_t)$ and $H_{>t} \perp H_{<t} \ominus P(A|H_t)$. Then family $G = (G_t), t \in R$, defined by

$$G_t = P(A|H_t), t \in R, \tag{2}$$

is a realization of a s.d.s. with outputs H such that families H and G are equally predictable by each other. If A from (2) is replaced by

$$A^M = \bigcap_{t \in R} H_{<\infty} \ominus P(H_{\geq t}|H_{\leq t} \ominus H_t), \tag{3}$$

then family G is the strictly maximal such realization of that system.

Proof.From assumption $H_{>t} \perp H_{<t} \ominus P(A|H_t)$ it follows $H_{<t} \perp H_{>t}|G_t$ which, because $G_t \subseteq H_t$(follows immediately from (2)) means that G is a realization of a s.d.s. with outputs H.

For $s > t$ we get $P(H_{\geq t} \mid G_{\leq t}) \subseteq G_t$ which implies

$$P(H_{\geq t}|G_{\leq t}) = G_t. \tag{4}$$

Also, we have $P(G_s|H_{\leq t}) = P(G_s|H_t) \subseteq P(A|H_t) = G_t$, which, together with (4) prove that H and G are equally predictable by each other.

Let us prove that $G_t^M = P(A^M|H_t)$, where A^M is defined by (3), is strictly maximal such that realization. Let G^1 be any realization of a s.d.s. with outputs H such that G^1 and H are equally predictable by each other. From Corollary 1 ([1], p.527) it follows $G_t^1 \subseteq A^M$, which implies $G_t \subseteq P(H_{\geq t}|H_{\leq t} \ominus H_t)$ for all t, u, that is $G_t^1 \subseteq A^M$, which implies $G_t \subseteq P(A^M|H_t) = G_t^M$, so strictly maximality is proved.

REMARK. The problem of finding all realizations G of a s.d.s. with a given outputs H, such that H and G are equally predictable by each other, is still open.

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KRAGUJEVAC, YUGOSLAVIA